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SOLVING DIRECT AND INVERSE PROBLEMS OF PLATE VIBRATION BY USING THE TREFFTZ FUNCTIONS

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The paper presents an approximate method of solving direct and inverse problems which are described by a non-homogenous plate vibration equation. The key idea of the presented approach is to use solving polynomials that satisfy the considered homogenous differential equation identically. Inhomogeneity is expanded into the Taylor series and then, for each monomial, the inverse operator is calculated. In the paper, the properties of solving functions are investigated – a theorem concerning their linear independence is formulated and proved. The method of identification of the load (source) is described. It belongs to the group of inverse problems. The paper includes examples which illustrate the usefulness of the method.

Key words: plate vibration, Trefftz method, inverse problem, source identification

1. Introduction

The term Trefftz methods means a wide class of approaches of solving partial differential equations. The solution is approximated by a linear combination of functions satisfying the equation identically. In general, we can distinguish two classes of Trefftz functions. The first one are F-functions called fundamental solutions. These functions have singularity in certain points. Usually, these points are chosen outside the domain. The second one are H-functions (Herrera functions). Good example are here harmonic or heat polynomials. Plate polynomials (Trefftz functions for plate vibration equations) presented in this paper can be classified as H-functions. The method originates from the paper by Trefftz (1926) and were developed by various authors, including: Herrera, Sabina, Kupradze, Jirousek, Leon, Zieliński and Zienkiewicz, who considered mostly stationary problems. Non-stationary problems were solved usually by time discretization. The time as a continuous variable, first appeared in Rosenbloom and Widde (1956) (1D heat polynomials). Next this aspect of the method was developed for the wave equation and thermoelasticity problems in Grysa and Maciąg (2011), Maciąg (2004, 2005, 2007, 2011a), Maciąg and Wauer (2005a,b). The Trefftz functions for the equation of beam vibration are presented in Al-Khatib *et al.* (2008). Also comprehensive monographs exist concerning the Trefftz functions method (Ciałkowski and Frąckowiak, 2000; Grysa, 2010, Kołodziej and Zieliński, 2009; Li *et al.*, 2008; Maciąg, 2009; Qin, 2000). Problems of plates vibrations can be solved by means of the Trefftz method. For example, F-functions were used in Reutskiy and Yu (2007), Wu *et al.* (2011). H-functions were used in Blanc *et al.* (2007), Vanmaele *et al.* (2007). Each method has advantages and disadvantages. For example, in four papers mentioned above, the authors assumed harmonic form of the vibrations. This approach is very effective for analyzing natural frequencies and solving direct problems. Trefftz functions with time as a continuous variable seems to be more effective for solving inverse problems. Plate polynomials used here are described in Maciąg (2011b).

The presented article can be considered as a remarkable step forward in comparison with the work by Maciąg (2011b). Firstly, the properties of the plate polynomials are considered here. Secondly, the method of source identification (inverse problem) is described. As a rule, inverse problems are difficult to solve. Moreover, the solution can be sensitive to disturbances in input data. The examples presented in this paper show high effectiveness of the Trefftz functions method for solving direct and inverse problems.

2. Stating the problem

Let us consider the inhomogeneous equation of vibrations of a plate in the dimensionless coordinates

$$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}+\frac{\partial^2 u}{\partial t^2}=Q(x,y,t) \qquad (x,y)\in(0,1)\times(0,1) \qquad t>0 \tag{2.1}$$

where $Q(x,y,t)$ is the load of the plate. In the case of a direct problem, equation (2.1) should be completed by suitable initial and boundary conditions. Initial conditions describe the deflection and velocity at the time $t=0$. Boundary conditions indicate the support of the plate. In the case of identification of the load, we additionally assume that deflections of the plate in chosen internal points are known (internal responses).

3. Linear independence of the Trefftz functions

Trefftz functions for the equation of plate vibration(plate polynomials) are presented in Maciąg (2011b). In this work, recurrent formulas for the plate polynomials and their derivatives are described. Table 1 includes the solving polynomials from degree 0 to 4.

Table 1. The solving polynomials from degree 0 to 4

Degree	Number of polynomials	Polynomials
0	1	1
1	3	x, y, t
2	5	$\frac{x^2}{2}, xy, \frac{y^2}{2}, tx, yt$
3	7	$\frac{x^2t}{2}, xyt, \frac{y^2t}{2}, \frac{x^3}{6}, \frac{x^2y}{2}, \frac{xy^2}{2}, \frac{y^3}{6}$
4	9	$\frac{x^3t}{6}, \frac{x^2yt}{2}, \frac{xy^2t}{2}, \frac{y^3t}{6}, \frac{x^4}{24}-\frac{t^2}{2}\frac{yx^3}{6}, \frac{x^2y^2}{4}-t^2, \frac{xy^3}{6}, \frac{y^4}{24}-\frac{t^2}{2}$

According to Table 1, we can infer that exactly $2n+1$ plate polynomials of degree n $(n\geqslant 0)$ exist. A particularly important property of them is their linear independence, which is the content of Theorem 1.

Theorem 1. *Accurate to the polynomial of third degree,* $2n+1$ *linearly independent plate polynomials of degree* n, $n\geqslant 0$ *exist.*

Proof

Let us denote u_n as a linear combination of the monomials of degree n

$$\begin{aligned}u_n&=\alpha_{n00}x^n+(\alpha_{(n-1)10}y+\alpha_{(n-1)01}t)x^{n-1}+(\alpha_{(n-2)20}y^2+\alpha_{(n-2)11}yt+\alpha_{(n-2)02}t^2)x^{n-2}\\&\quad+\ldots+(\alpha_{1(n-1)0}y^{n-1}+\alpha_{1(n-2)1}y^{n-2}t+\ldots+\alpha_{10(n-1)}t^{n-1})x\\&\quad+\alpha_{0n0}y^n+\alpha_{0(n-1)1}y^{n-1}t+\ldots+\alpha_{00n}t^n+R\end{aligned} \tag{3.1}$$

where α_{pqr} – coefficients, R – polynomial of three variables of degree less than n. Obviously, coefficients α_{pqr} might be expressed in the form

$$\alpha_{pqr} = \frac{1}{p!q!r!}\frac{\partial^n u_n}{\partial x^p \partial y^q \partial t^r} \tag{3.2}$$

We will show that if u_n is a linear combination of plate polynomials, the coefficients α_{pqr} for $r \geqslant 2$ equal zero. Using formula (3.2) and the equation of homogenous plate vibration we obtain

$$\begin{aligned}\alpha_{pqr} &= \frac{1}{p!q!r!}\frac{\partial^{n-2}}{\partial x^p \partial y^q \partial t^{r-2}}\frac{\partial^2 u_n}{\partial t^2} = \frac{1}{p!q!r!}\frac{\partial^{n-2}}{\partial x^p \partial y^q \partial t^{r-2}}\Big(-\frac{\partial^4 u_n}{\partial x^4} - 2\frac{\partial^4 u_n}{\partial x^2 \partial y^2} - \frac{\partial^4 u_n}{\partial y^4}\Big) \\ &= -\frac{1}{p!q!r!}\Big(\frac{\partial^{n+2} u_n}{\partial x^{p+4} \partial y^q \partial t^{r-2}} - 2\frac{\partial^{n+2} u_n}{\partial x^{p+2} \partial y^{q+2} \partial t^{r-2}} - \frac{\partial^{n+2} u_n}{\partial x^p \partial y^{q+4} \partial t^{r-2}}\Big) = 0\end{aligned} \tag{3.3}$$

Basing on equality (3.3) the linear combination takes the form

$$\begin{aligned}u_n &= \alpha_{n00}x^n + (\alpha_{(n-1)10}y + \alpha_{(n-1)01}t)x^{n-1} + (\alpha_{(n-2)20}y^2 + \alpha_{(n-2)11}yt)x^{n-2} \\ &\quad + \ldots + (\alpha_{1(n-1)0}y^{n-1} + \alpha_{1(n-2)1}y^{n-2}t)x + \alpha_{0n0}y^n + \alpha_{0(n-1)1}y^{n-1}t + R\end{aligned} \tag{3.4}$$

where R is a properly chosen polynomial. It is worth mentioning that in formula (3.4) there occur only coefficients of the type α_{pq0}, α_{pq1}, in the number correspondingly to $n+1$ and n. To simplify the notation, we denote

$$w = u_n - R$$

In order to obtain the polynomial R, the operator

$$L = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} + \frac{\partial^2}{\partial t^2} \tag{3.5}$$

should be applied to equation (3.4). Then we get: $L(w) + L(R) = 0$, hence $L(R) = -L(w)$, and finally $R = L^{-1}(-L(w)) + L^{-1}(0)$. It follows that there are exactly $2n+1$ linearly independent plate polynomials of degree n.

4. The Trefftz function method

The solution to equation (2.1) with suitable initial and boundary conditions is approximated by the linear combination: $u(x,y,t) \approx w(x,y,t) = \sum_{n=1}^{N} c_n V_n + w_p$, where c_n are coefficients, V_n – Trefftz functions, w_p – particular solution. In order to obtain c_n, we minimize the functional describing the fitting of the approximate solution to the given initial and boundary conditions. The approximation of the particular solution w_p is determined by the inverse operator $L^{-1}(Q)$. To obtain approximation of the inverse operator for the function $Q(x,y,t)$, the function should be expanded into the Taylor series. Then we calculate inverse operators for monomials by using recurrent formulas presented in Maciąg (2011b).

5. Examples

The plate polynomials will be used for solving a direct and an inverse problem for plate vibrations. In both cases, the exact solution will be known. It allows one to check the quality of the approximation. Two kinds of the load are considered. The first has a form of a polynomial. The second is a trigonometric function. The approximation will be calculated for the two dimensionless time intervals: $(0, 1/16)$ and $(0, 1/4)$. Additionally, the sensitivity of the method according to the noisy data is checked.

5.1. The solution of a direct problem

As the first example let us consider the non-homogeneous equation of plate vibrations in dimensionless coordinates:

$$
\begin{aligned}
&\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}+\frac{\partial^2 u}{\partial t^2}=\frac{24t^2(y^4-2y^3+y)+2t^2(12x^2-12x)(12y^2-12y)}{5}\\
&\qquad+\frac{24t^2(x^4-2x^3+x)+2(x^4-2x^3+x)(y^4-2y^3+y)}{5} \qquad \begin{array}{l}(x,y)\in(0,1)\times(0,1)\\ t>0\end{array}
\end{aligned}
\tag{5.1}
$$

Equation (5.1) has been completed by initial and boundary conditions

$$
\begin{aligned}
&u(x,y,0)=\frac{\sin(\pi x)\sin(\pi y)}{1000} \qquad \frac{\partial u(x,y,0)}{\partial t}=0\\
&u(0,y,t)=u(1,y,t)=u(x,0,t)=u(x,1,t)=0\\
&\frac{\partial^2 u(0,y,t)}{\partial x^2}=\frac{\partial^2 u(1,y,t)}{\partial x^2}=\frac{\partial^2 u(x,0,t)}{\partial y^2}=\frac{\partial^2 u(x,1,t)}{\partial y^2}=0
\end{aligned}
\tag{5.2}
$$

The exact solution to problem (5.1) and (5.2) is given by the following formula

$$
u(x,y,t)=\frac{\sin(\pi x)\sin(\pi y)\cos(2\pi^2 t)}{1000}+\frac{t^2(x^4-2x^3+x)(y^4-2y^3+y)}{5}
$$

In order to determine the approximation of the solution, a linear combination of polynomials was used, together with the function being a particular solution to (5.1), i.e.

$$
u(x,y,t)\approx w(x,y,t)=\sum_{n=1}^{N}c_nV_n+\frac{t^2(x^4-2x^3+x)(y^4-2y^3+y)}{5}
\tag{5.3}
$$

The approximate solution is calculated in the time interval $(0,\varDelta t)$. The coefficients c_n are chosen so that the functional described by formula (5.4) is minimized

$$
\begin{aligned}
I=&\underbrace{\int_0^1 dx\int_0^1\Big(w(x,y,0)-\frac{\sin(\pi x)\sin(\pi y)}{1000}\Big)^2dy}_{\text{condition }(5.2)_1}+\underbrace{\int_0^1 dx\int_0^1\Big(\frac{\partial w(x,y,0)}{\partial t}\Big)^2dy}_{\text{condition }(5.2)_1}\\
&+\underbrace{\int_0^{\varDelta t} dt\int_0^1\Big([w(0,y,t)]^2+[w(1,y,t)]^2+\Big(\frac{\partial^2 w(0,y,t)}{\partial x^2}\Big)^2+\Big(\frac{\partial^2 w(1,y,t)}{\partial x^2}\Big)^2\Big)dy}_{\text{conditions }(5.2)_{2,3}}\\
&+\underbrace{\int_0^{\varDelta t} dt\int_0^1\Big([w(x,0,t)]^2+[w(x,1,t)]^2+\Big(\frac{\partial^2 w(x,0,t)}{\partial y^2}\Big)^2+\Big(\frac{\partial^2 w(x,1,t)}{\partial y^2}\Big)^2\Big)dx}_{\text{conditions }(5.2)_{2,3}}
\end{aligned}
\tag{5.4}
$$

The necessary condition to minimize functional (5.4) has a form: $\partial I/\partial c_1=\ldots=\partial I/\partial c_N=0$.

Figure 1 shows the exact solution for the point $x=y=0.5$ and its approximation by polynomials from the order 0 to: (a) – 9, (b) – 11, (c) – 13 in the time interval $t\in(0,1/16)$.

In order to check the quality of the approximation, two kinds of errors can be calculated. The first, given by formula:

$$
\varepsilon=\sqrt{\frac{\int_0^{\varDelta t}\Big[w\Big(\frac{1}{2},\frac{1}{2},t\Big)-u\Big(\frac{1}{2},\frac{1}{2},t\Big)\Big]^2dt}{\int_0^{\varDelta t}\Big[u\Big(\frac{1}{2},\frac{1}{2},t\Big)\Big]^2dt}}\cdot 100\%
\tag{5.5}
$$

describes the error in the point $x = y = 0.5$. The second

$$\delta = \sqrt{\frac{\int\limits_0^{\Delta t} dt \int\limits_0^1 dy \int\limits_0^1 [w(x,y,t) - u(x,y,t)]^2 dx}{\int\limits_0^{\Delta t} dt \int\limits_0^1 dy \int\limits_0^1 [u(x,y,t)]^2 dx}} \cdot 100\% \tag{5.6}$$

describes the error in the entire time-space domain.

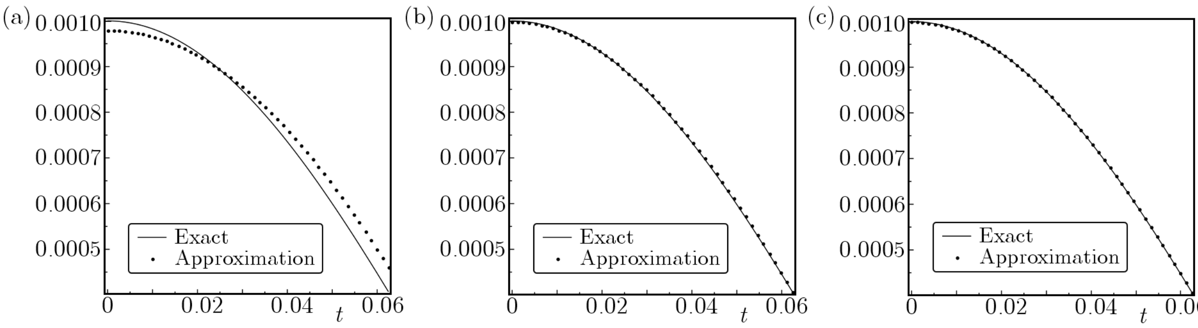

Fig. 1. Exact solution and their approximation by polynomials from order 0 to: (a) – 9, (b) – 11, (c) – 13

Table 2 shows values of errors (5.5) and (5.6) depending on the degree and the number of polynomials. For instance, if the approximation contains all polynomials from the order 0 to 9, we take 100 polynomials and so on. It is seen that the error decreases if the approximation contains more Trefftz functions. The error at the level of 0.0374% is remarkably low. Comparing errors (5.5) and (5.6), we can see that the error in the entire domain is bigger than in the particular point. However, this error remains at a low level.

Table 2. Error (5.5) and (5.6) of the approximation in dependance of the degree of the polynomials and the number of polynomials

Degree (No. of polynomials)	9 (100)	10 (121)	11 (144)	12 (169)	13 (196)
ε [%]	3.48	0.286	0.253	0.0683	0.0374
δ [%]	5.10	0.376	0.328	0.118	0.0667

The Trefftz functions can be used as base functions in the Finite Elements Method (see Maciąg, 2009, 2011a). When considering inverse problems, a particularly important question is how big the time-space element should be. Therefore, we examine the influence of the length of the time interval on the error. In the calculations presented above, the time interval $(0, 1/16)$ has been used. Now we take into consideration the time interval $(0, 1/4)$. Figure 2 shows the exact solution for the point $x = y = 0.5$ and its approximations by polynomials from the order 0 to: (a) – 9, (b) – 11, (c) – 13.

The errors according to formulas (5.5) and (5.6) have been calculated to check the quality of the approximation in the time interval $(0, 1/4)$. The values of both errors are presented in Table 3.

Comparing Tables 2 and 3, it is seen that we get a better approximation in a shorter time interval. In a longer interval, the solution has more oscillations – this result has been expected. All tables show that more polynomials lead to better results. It means that if we want to obtain a better approximation in a longer time interval, more polynomials have to be taken. For example for 225 polynomials, the error (5.5) is at the level of 3.59% for time interval $(1, 1/4)$.

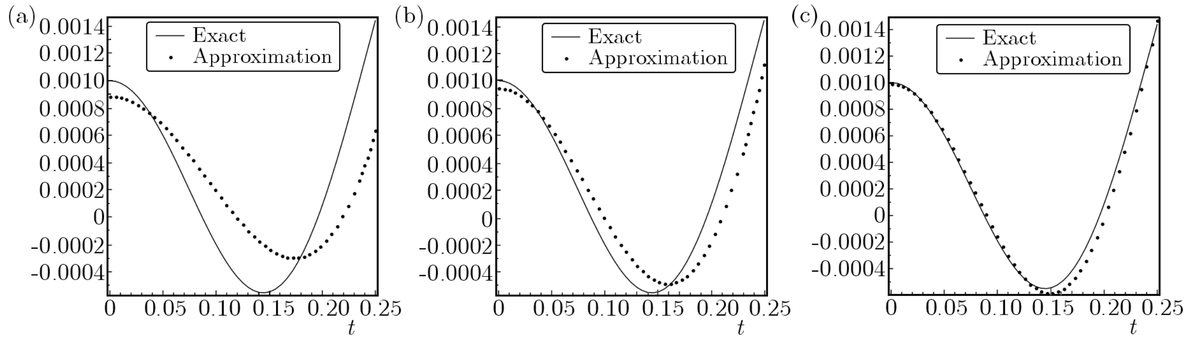

Fig. 2. The exact solution for the point $x = y = 0.5$ and its approximations by polynomials from order 0 to: (a) – 9, (b) – 11, (c) – 13

Table 3. Error (5.5) and (5.6) of approximation depending on the degree of polynomials and the number of polynomials

Degree (No. of polynomials)	9 (100)	10 (121)	11 (144)	12 (169)	13 (196)
ε [%]	64.3	58.3	30.7	21.4	7.67
δ [%]	65.8	56.7	28.4	19.5	7.09

In the second example let us consider the problem which is described by the non-homogeneous equation for $(x,y) \in (0,1) \times (0,1)$, $t > 0$

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^2 u}{\partial t^2} = \frac{2t^2\pi^4 \sin(\pi x)\sin(\pi y) + \sin(\pi x)\sin(\pi y)}{50} \tag{5.7}$$

and conditions (5.2). The exact solution has the form

$$u(x,y,t) = \frac{\sin(\pi x)\sin(\pi y)\cos(2\pi^2 t)}{1000} + \frac{t^2 \sin(\pi x)\sin(\pi y)}{100}$$

As an approximate solution we take a linear combination of polynomials completed by the particular solution

$$u(x,y,t) \approx w(x,y,t) = \sum_{n=1}^{N} c_n V_n + \frac{t^2 \sin(\pi x)\sin(\pi y)}{100} \tag{5.8}$$

The approximate solution will be calculated in the time interval $(0, 1/16)$. The coefficients c_n of inear combination (5.8) are calculated by minimizing functional (5.4). The errors of the approximation have been calculated according to formulas (5.5) and (5.6). The results are presented in Table 4.

Table 4. Error (5.5) and (5.6) of the approximation depending on the degree of polynomials and number of polynomials

Degree (No. of polynomials)	9 (100)	10 (121)	11 (144)	12 (169)	13 (196)
ε [%]	3.52	0.289	0.256	0.0691	0.0379
δ [%]	5.16	0.381	0.332	0.120	0.0675

Similarly as before, it is seen that the error decreases if the approximation contains more Trefftz functions. The error remains at a low level and is bigger for the entire domain. Comparing Tables 2 and 4, it is seen that the error in both kinds of the non-homogeneity stays at the same level.

5.2. The identification of the load imposed to the plate – an inverse problem

Nowadays, there exist a lot of methods that can be applied to solving linear problems with partial differential equations. Unfortunately, most of them cannot be used for solving inverse problems. The Trefftz Functions Method seems to be a very convenient approach to solving such problems, which is its most important advantage. In general, different types of inverse problems exist, including a boundary inverse problem (identification of the boundary condition), geometric inverse problems (identification of the shape of the body) and many others. One of the kinds of inverse problems is also the source identification (identification of the load). In this paper, we propose to apply Trefftz functions for the identification of the load $Q(x,y,t)$ applied to the plate. Let us consider the non-homogeneous equation in dimensionless coordinates

$$L(u) = Q(x,y,t), \qquad (x,y) \in (0,1) \times (0,1) \qquad t > 0 \tag{5.9}$$

where function $Q(x,y,t)$ is an unknown load and the operator L has the form in (3.5). Equation (5.9) is completed by initial and boundary conditions (5.2). Additionally, we assume that the values of deflection of the plate in the internal points are known (internal responses)

$$u_{ijk} = u\Big(\frac{i}{10}, \frac{j}{10}, \frac{k\Delta t}{50}\Big) \qquad i,j = 1,\dots,9 \qquad k = 1,\dots,50 \tag{5.10}$$

For numerical simulation, we take the known function Q. Next, we calculate the internal responses and then identify the load. Because we know the function Q, we can calculate the error of the solution. We assume that the approximation of the solution $u(x,y,t)$ has the form

$$u(x,y,t) \approx w(x,y,t) = \sum_{n=1}^{N} c_n V_n + \sum_{k=1}^{K} \alpha_k v_k \tag{5.11}$$

where V_n are plate polynomials, v_k are values of the inverse operator L^{-1} for monomials

$$\begin{aligned}&\{1, x, y, t, x^2, t^2, y^2, xt, ty, xy, t^3, t^2y, t^2x, ty^2, xyt, tx^2, y^3, xy^2, x^2y, x^3, t^4, t^3y,\\ &t^3x, t^2y^2, t^2xy, t^2x^2, ty^3, txy^2, tyx^2, tx^3, y^4, xy^3, x^2y^2, yx^3, x^4, \dots\}\end{aligned} \tag{5.12}$$

In means that

$$Q(x,y,t) \approx L\Big(\sum_{k=1}^{K} \alpha_k v_k\Big) \tag{5.13}$$

The recurrent formulas for the inverse operator L^{-1} for monomials are presented in Maciąg (2011b). In order to determine the coefficients c_n and α_k, a functional similar to (5.4) should be build.

As the first example of an inverse problem, let us consider the problem described by equations (5.1) and (5.2). However, now we assume that the load

$$\begin{aligned}Q(x,y,t) &= \frac{24t^2(y^4 - 2y^3 + y) + 2t^2(12x^2 - 12x)(12y^2 - 12y)}{5}\\ &+ \frac{24t^2(x^4 - 2x^3 + x) + 2(x^4 - 2x^3 + x)(y^4 - 2y^3 + y)}{5}\end{aligned} \tag{5.14}$$

is unknown (we use the exact solution only for numerical simulation). Let us denote $\widetilde{Q}(x,y,t)$ as the approximation of the exact load $Q(x,y,t)$. Similarly as before, we can define two kinds of the error. The first describes the error in the point $x = y = 0.5$

$$\varepsilon = \sqrt{\frac{\int\limits_0^{\Delta t} \Big[\widetilde{Q}\Big(\frac{1}{2}, \frac{1}{2}, t\Big) - Q\Big(\frac{1}{2}, \frac{1}{2}, t\Big)\Big]^2 dt}{\int\limits_0^{\Delta t} \Big[Q\Big(\frac{1}{2}, \frac{1}{2}, t\Big)\Big]^2 dt}} \cdot 100\% \tag{5.15}$$

the second describes the error in the entire time space domain

$$\delta = \sqrt{\frac{\int\limits_0^{\Delta t} dt \int\limits_0^1 dy \int\limits_0^1 [\widetilde{Q}(x,y,t) - Q(x,y,t)]^2 dx}{\int\limits_0^{\Delta t} dt \int\limits_0^1 dy \int\limits_0^1 [Q(x,y,t)]^2 dx}} \cdot 100\% \tag{5.16}$$

The values of errors (5.15) and (5.16) for $\Delta t = 1/16$ are shown in Table 5.

Table 5. Error (5.15) and (5.16) depending on the degree of polynomials and the number of polynomials ($\Delta t = 1/16$)

Degree (No. of polynomials)	10 (121)	11 (144)	12 (169)	13 (196)
ε [%]	2.97	2.18	0.635	0.447
δ [%]	6.86	3.00	2.47	2.66

Table 5 shows that more polynomials in the approximation decreases the error. Obviously, the error is bigger in the entire time-space domain than in a particular point. However, the error at the level of 2.66% is very small, considering that the problem is inverse. The values of errors (5.15) and (5.16) for the bigger time interval ($\Delta t = 1/4$) are shown in Table 6.

Table 6. Error (5.15) depending on the degree of polynomials and the number of polynomials ($\Delta t = 1/4$)

Degree (No. of polynomials)	10 (121)	11 (144)	12 (169)	13 (196)
ε [%]	20.9	14.5	3.87	5.65
δ [%]	43.6	23.8	11.4	12.6

Obviously, the approximation in a bigger time interval is worse. Moreover, we observe a slight increase of the error in the last column. This may be caused by the Runge effect (waving of the polynomials of the high degree). It means that if we intend to solve an inverse problem, the time interval cannot be too long.

The second example of an inverse problem is described by equations (5.7) and conditions (5.2). Similarly as before we assume that the exact load is unknown (the exact solution is used to generate internal responses in numerical simulation).

As before, in this case the exact load is known. Therefore, we can calculate the errors (5.15) and (5.16). The results for $\Delta t = 1/16$ and for $\Delta t = 1/4$ are presented in Tables 7 and 8 correspondingly.

Table 7. The error (5.15) and (5.16) depending on the degree of polynomials and the number of polynomials ($\Delta t = 1/16$)

Degree (No. of polynomials)	10 (121)	11 (144)	12 (169)	13 (196)
ε [%]	5.66	4.33	1.45	0.999
δ [%]	14.4	6.79	4.81	5.48

Also in this case, more polynomials in the approximation decrease the error, and the error is bigger in the entire time-space domain than in a particular point. The approximation in a bigger time interval is worse also for the second kind of the load. It confirms the conclusion that if we solve an inverse problem, the time interval cannot be too long.

Table 8. Error (5.15) and (5.16) depending on the degree of polynomials and the number of polynomials ($\Delta t = 1/4$)

Degree (No. of polynomials)	10 (121)	11 (144)	12 (169)	13 (196)
ε [%]	36.1	26.3	7.73	10.4
δ [%]	88,7	48,3	25,4	28,1

5.2.1. Noisy data

The solution of the inverse problem can be very sensitive to disturbances of the input data. Therefore, it is very important to examine each new method according to sensitivity to noisy data. To this end, the internal responses have been randomly disturbed according to the formula

$$u_{ijk}^{dis} = u_{ijk}(1 + \delta_{ijk}) \qquad i,j = 1,\dots,9 \qquad k = 1,\dots,50 \tag{5.17}$$

where δ_{ijk} has a normal distribution with the mean equal to 0 and standard deviation 0.02. The disturbed internal responses have been used to calculate the approximation of the load $Q(x,y,t)$. Table 9 shows error (5.15) of the approximation of the load in polynomial form (5.14).

Table 9. Error (5.15) of the identification of the load in polynomial form for disturbed data ($\Delta t = 1/4$)

Degree (No. of polynomials)	10 (121)	11 (144)	12 (169)	13 (196)
ε [%]	20.8	14.2	3.94	5.18

We expect that the error for the noisy data should be bigger. However, comparing the results presented in Tables 6 and 9, we can observe that the noisy data does not cause any considerable increase in the error. It means that the presented approach is resistant to disturbance of the internal responses. This is a key quality of the method in terms of solving inverse problems.

6. Concluding remarks

In the paper, a method of solving problems described by an inhomogeneous equation of plate vibration is presented. The approach is relatively simple and suitable for solving both direct and inverse problems. The theorem of linear independence of the plate polynomials has been proved. The greatest advantage of the method is its usability for solving the ill-posed inverse problems. In the paper, the way of identification of the load (non-homogeneity) has been presented. The results described in the paper show a remarkable efficiency of the method for solving inverse problems. Moreover, the approach proposed here is relatively invulnerable to disturbance of the input data.

References

1. Al-Khatib M.J., Grysa K., Maciąg A., 2008, The method of solving polynomials in the beam vibration problems, *Journal of Theoretical and Applied Mechanics*, **46**, 2, 347-366

2. Blanc L., Blanze C., Rouch P., 2007, A multiscale "Trefftz" computational method for medium-frequency vibrations of assemblies of heterogeneous plates with uncertainties, *Computers and Structures*, **85**, 595-605

3. Ciałkowski M.J., Frąckowiak A., 2000, *Heat Functions and Their Application for Solving Heat Transfer and Mechanical Problems*, Poznań University of Technology Publishers [in Polish]

4. Grysa K., 2010, *Trefftz Functions and their Applications in Solving the Inverse Problems*, Kielce University of Technology Publishers [in Polish]

5. Grysa K., Maciąg A., 2011, Solving direct and inverse thermoelasticity problems by means of Trefftz base functions for finite element method, *Journal of Thermal Stresses*, **34**, 4, 378-393

6. Kołodziej J.A., Zieliński A.P., 2009, *Boundary Collocation Techniques and their Application in Engineering*, WIT Press, Southampton, Boston

7. Li Z.-C., Qiu Lu T.-T., Hu H.-Y., Cheng H.-D., 2008, *The Trefftz and Collocation Methods*, WIT Press, Southampton, Boston

8. Maciąg A., 2004, Solution of the three-dimensional wave equation by using wave polynomials, *PAMM – Proceedings in Applied Mathematics and Mechanics*, **4**, 706-707

9. Maciąg A., 2005, Solution of the three-dimensional wave polynomials, *Mathematical Problems in Engineering*, **5**, 583-598

10. Maciąg A., 2007, Wave polynomials in elasticity problems, *Engineering Transactions*, **55**, 2, 129-153

11. Maciąg A., 2009, *Trefftz Functions for Some Direct and Invers Problems of Mechanics*, Kielce University of Technology Publishers [in Polish]

12. Maciąg A., 2011a, The usage of wave polynomials in solving direct and inverse problems for two-dimensional wave equation. *International Journal for Numerical Methods in Biomedical Engineering*, **27**, 1107-1125

13. Maciąg A., 2011b, Trefftz function for a plate vibration problem, *Journal of Theoretical and Applied Mechanics*, **49**, 1, 97-116

14. Maciąg A., Wauer J., 2005a, Solution of the two-dimensional wave equation by using wave poly-nomials, *Journal of Engineering Mathematics*, **51**, 4, 339-350

15. Maciąg A., Wauer J., 2005b, Wave polynomials for solving different types of two-dimensional wave equations, *Computer Assisted Mechanics and Engineering Sciences*, **12**, 87-102

16. Qin Q.H., 2000, *The Trefftz Finite and Boundary Element Method*, WIT Press, Southampton, Boston

17. Reutskiy S.Yu., 2007, The method of fundamental solutions for problems of free vibrations of plates, *Engineering Analysis with Boundary Elements*, **31**, 10-21

18. Rosenbloom P.C., Widder D.V., 1956, Expansion in terms of heat polynomials and associated functions, *Transactions of the American Mathematical Society*, **92**, 220-266

19. Trefftz E., 1926, Ein Gegenstuek zum Ritz'schen Verfahren, [In:] *Proceedings 2nd International Congres of Applied Mechanics*, Zurich, 131-137

20. Vanmaele C., Vandepitte D.. Desmet W., 2007, An efficient wave based prediction technique for plate bending vibrations, *Computer Methods in Applied Mechanics and Engineering*, **196**, 3178-3189

21. Wu C.S., Young D.L., Fan C.M., 2011, Frequency response analyses in vibroacoustics using the method of fundamental solutions, *Computational Mechanics*, **47**, 519-533

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