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FATIGUE LIFE ESTIMATION OF NOTCHED SPECIMENS UNDER BENDING AND TORSION WITH STRAIN ENERGY DENSITY PARAMETER

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This paper presents an energy damage parameter based on the critical plane under a complex loading state and stress concentration. Two criteria including combinations of the normal and shear strain energy density parameters have been proposed. The plane in which the normal or shear strain energy density parameter reaches its maximum is assumed to be the critical plane. The predictions of the proposed model have been compared with constant amplitude fatigue tests under combined proportional bending and torsion carried out on ring-notched specimens made of 42CrMo4V steel.

Key words: multiaxial fatigue, notch, energy approach

Notations

C	–	coefficient determining stress in circumferential direction depending on its concentration
E	–	Young's modulus
$\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$	–	unit vectors in Cartesian coordinate system
K_t	–	theoretical stress concentration factor
$\widehat{l}_\eta,\widehat{m}_\eta,\widehat{n}_\eta$	–	direction cosines of unit vector $\boldsymbol{\eta}$
$\widehat{l}_s,\widehat{m}_s,\widehat{n}_s$	–	direction cosines of unit vector $\boldsymbol{s}$
N_f	–	number of cycles to failure
$R_{p0.2}$	–	yield stress
R_m	–	ultimate tensile strength
$\mathrm{sgn}\,[i,j]$	–	two-argument logical function of signs of i, j variables
t	–	time
W	–	strain energy density parameter

x, y, z	–	spatial coordinates of specimen
β, κ	–	coefficients determining criterion form obtained from uniaxial fatigue test
ε	–	strain
ν	–	Poisson's ratio
γ	–	engineering shear strain
σ	–	normal stress
τ	–	shear stress

Subscripts

a	–	amplitude	s	–	shear
b	–	bending	t	–	torsion
eq	–	equivalent	η	–	normal

1. Introduction

Problems connected with notches are among the most important and widely tested ones in the material fatigue. In such cases, stresses can be much greater than calculated nominal ones and they cause decrease of the fatigue strength.

Problems connected with notch occur in the case of section discontinuities. They cause local stress concentrations which can be much greater than calculated ones. Their magnitudes depend on notch geometry, material properties, loading path, etc. The notch occurrence always leads to fatigue strength. Thus, definition of an efficient method for fatigue life determination under stress concentration belongs to the most important and widely tested problems of fatigue of materials.

The first stage of fatigue life determination under stress concentration should include the definition of local stresses and elastic-plastic strains in the notch root. Neuber's method (Neuber, 1961) or the strain energy density method (ESED) (Molski and Glinka, 1981) are the most widely applied for their determination under cyclic loading. Both methods were generalized to a multiaxial loading state (Moftakar *et al.*, 1995). The two methods differ only in the way of determination of the plastic strain energy density. Let us note that Neuber's method gives overestimated values of strains and stresses in the notch root, whereas the values obtained with ESED are lowered. The results obtained according to the model proposed in Łagoda and Macha (1998) are included between the results obtained according to the models proposed by Molski and Glinka (1981), Neuber (1961). In the literature, we can also find other energy models formulated by Inoue *et al.* (1996) and Ye *et al.* (2004)

as well as the empirical model proposed by Grubisic and Sonsino (1982). Determination of local stresses and strains in the notch root requires application of Hencky's constitutive equations and complex numerical calculations. Under a non-proportional loading, the calculations become even more complex because the influence of the loading path must be taken into account and an incremental notation of the chosen method should be used. Moreover, in order to determine relations between stresses and plastic strains, it is necessary to apply a model of plasticity (for example, the model formulated by Mróz or Chu). Thus, pseudoelastic strains and stresses are often applied in fatigue calculations. In Susmel (2004), a criterion based on the assumption of the elastic stress state is proposed; the elastic stress state occurs in the notch neighbourhood and it is determined with the finite element method. The fatigue life has been determined from normal and shear stresses in the plane of maximum normal stresses. Elastic stresses are also used in the criterion proposed by Grubisic and Simburger (1976), where the maximum shear stress was assumed as the equivalent stress amplitude.

In previous papers by Kardas *et al.* (2004), Łagoda and Macha (2000), Sonsino and Łagoda (2004), the strain energy density parameter in the critical plane was applied for fatigue life estimation of smooth specimens. In this paper, the same parameter is used for notched components.

The aim of this paper is to formulate an energy based criterion formulated according to stress and strain histories of multiaxial fatigue including stress concentration. The predictions of the proposed model have been compared with constant amplitude fatigue tests under combined proportional bending and torsion carried out on ring-notched specimens made of 42CrMo4V steel (Pötter, 2000).

2. Energy damage parameter under complex loading state

2.1. The strain energy density parameter in the uniaxial loading

In the uniaxial loading, the strain energy density parameter SEDP (Łagoda, 2001) is defined as

$$W_{axx}(t) = \frac{1}{2}\sigma_{axx}(t)\varepsilon_{axx}(t)\,\mathrm{sgn}\,[\sigma_{axx}(t), \varepsilon_{axx}(t)] \tag{2.1}$$

where

$$\mathrm{sgn}\,[i,j] = \frac{\mathrm{sgn}\,[i] + \mathrm{sgn}\,[j]}{2} \tag{2.2}$$

For a constant amplitude loading, we have the amplitude of SEDP

$$W_{axx} = \frac{1}{2}\sigma_{axx}\varepsilon_{axx} \tag{2.3}$$

and for an elastic material

$$W_{axx} = \frac{\sigma_{axx}^2}{2E}(1 - \nu C) \tag{2.4}$$

where $0 \leqslant C \leqslant \nu$.

An interpretation of this coefficient, C, is shown in Fig. 1. Its approximate value can be determined from

$$C = \frac{1.84\nu}{K_t}(K_t - 1)^{1-\nu} \tag{2.5}$$

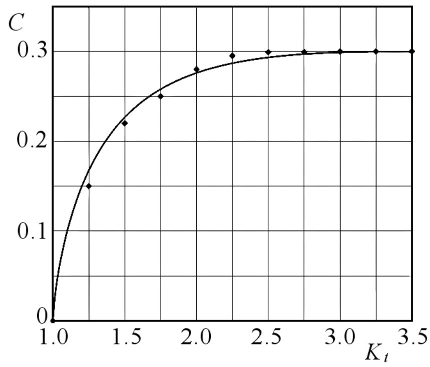

Fig. 1. The coefficient C depending on the stress concentration factor

The function given by Eq. (2.5) was based on calculations with the finite element method for different notch geometries, $K_t = K_{ta}$ for the axial loading and $K_t = K_{tb}$ for bending.

2.2. The strain energy density parameter in a multiaxial loading

For the proposed damage parameter under stress concentration, the assumptions are the same as those for smooth specimens (Kardas *et al.*, 2004; Łagoda and Macha, 2000; łagoda *et al.*, 1999; Łagoda and Ogonowski, 2005; Sonsino and Łagoda, 2004, i.e.

- Fatigue cracking is caused by the part of strain energy density corresponding to the work of normal stress $\sigma_\eta(t)$ on the normal strain $\varepsilon_\eta(t)$, i.e. $W_\eta(t)$ and the work of shear stress on shear strain $\varepsilon_{\eta s}(t)$ in the direction $\boldsymbol{s}$ in the plane with the normal $\boldsymbol{\eta}$, i.e. $W_{\eta s}(t)$;
- The direction $\boldsymbol{s}$ on the critical plane is the mean direction, in which the shear strain energy density reaches its maximum, $W_{\eta s_{max}}(t)$;

- In the limit state, the material effort is determined by the maximum linear combination of the strain energy density parameters $W_\eta(t)$ and $W_{\eta s}(t)$, which satisfies the following equation under a random multiaxial loading
$$W_{eq}(t) = \beta W_{\eta s}(t) + \kappa W_\eta(t) \tag{2.6}$$
where
$$\begin{aligned} W_\eta(t) &= \frac{1}{2}\sigma_\eta(t)\varepsilon_\eta(t)\,\mathrm{sgn}\,[\sigma_\eta(t), \varepsilon_\eta(t)] \\ W_{\eta s}(t) &= \frac{1}{2}\tau_{\eta s}(t)\varepsilon_{\eta s}(t)\,\mathrm{sgn}\,[\tau_{\eta s}(t), \varepsilon_{\eta s}(t)] \\ \varepsilon_{\eta s}(t) &= \frac{1}{2}\gamma_{\eta s}(t) \end{aligned} \tag{2.7}$$
The constants β and κ and the chosen criterion defining the critical plane position lead to selection of one from the two special forms of criterion (2.6).

2.2.1. The criterion of the strain energy density parameter on the plane determined by normal strain energy density

The equivalent strain energy density parameter is a linear combination of normal and shear strain energy densities. Participation of particular energies in the damage process depends on the coefficients β and κ. The critical plane is determined by the normal strain energy density parameter. It is assumed that the critical plane with the normal $\boldsymbol{\eta}$ is determined by the normal loading. The position of the vector $\boldsymbol{s}$ is defined by one of the directions determined by the given shear loading, where the scalar product

$$\boldsymbol{\eta} \cdot \boldsymbol{s} = 0 \tag{2.8}$$

as it is presented in Fig. 2

$$\boldsymbol{\eta} = \widehat{l}_\eta \boldsymbol{i} + \widehat{m}_\eta \boldsymbol{j} + \widehat{n}_\eta \boldsymbol{k} \qquad\qquad \boldsymbol{s} = \widehat{l}_s \boldsymbol{i} + \widehat{m}_s \boldsymbol{j} + \widehat{n}_s \boldsymbol{k} \tag{2.9}$$

The following equation for the equivalent strain energy density parameter results from (2.6)
$$W_{eq}(t) = \beta W_{\eta s}(t) + W_\eta(t) \tag{2.10}$$

Considering stress and strain states for pure torsion and tension-compression or bending under a constant-amplitude loading, it is possible to determine relationships between the coefficients β and κ.

For bending on the plane of maximum tension, we obtain the same values as in the case of torsion, so it is not possible to determine the coefficient β in an

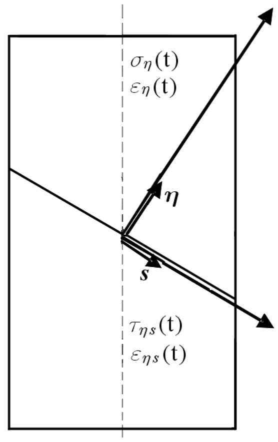

Fig. 2. Interpretation of the normal stress $\sigma_\eta(t)$, normal strain $\varepsilon_\eta(t)$, shear stress $\tau_{\eta s}(t)$ and shear strain $\varepsilon_{\eta s}(t)$ acting in the $\boldsymbol{s}$ direction, on the plane with the normal $\boldsymbol{\eta}$

analytical way. This coefficient can be selected for a given material from non-proportional tests. It could be done in amplitude-constant amplitude fatigue tests with a phase shift equal to $\pi/2$ between bending and torsion.

Criterion (2.10) is proposed for cast irons and welded steel joints (Łagoda and Ogonowski, 2005).

2.2.2. The criterion of the strain energy density parameter on the plane determined by shear strain energy density

The equivalent strain energy density is a linear combination of normal and shear strain energy densities. Participation of particular energies in the damage process is dependent on the coefficients β and κ as shown in Section 2.1. In such a case, the critical plane is determined by the shear strain energy density parameter. It is assumed that $Q = W_{af}$ and that the critical plane with the normal $\boldsymbol{\eta}$ and tangent $\boldsymbol{s}$ is determined as the mean position of one of the two planes, where the maximum shear strain energy density occurs. As previously, the equivalent strain energy density parameter is determined from Eq. (2.6).

Considering the stress and strain states for pure torsion and tension-compression or bending under a constant-amplitude loading, we can determine equations relating the coefficients β with κ.

Thus, the β parameter is

$$\beta = k^2 \frac{1 - \upsilon C}{1 + \upsilon} \tag{2.11}$$

and κ parametr is

$$\kappa = \frac{[4 - k^2(1 - C^2)](1 - \upsilon C)}{(1 - \upsilon)(1 + C)^2} \tag{2.12}$$

where

$$k(N_f) = \frac{\sigma_{axx}(N_f)}{\tau_{axy}(N_f)} \tag{2.13}$$

Having found the coefficients β and κ, Eq. $(2.7)_1$ takes the following form

$$W_{eq}(t) = k^2\frac{1 - \upsilon C}{1 + \upsilon}W_{\eta s}(t) + \frac{[4 - k^2(1 - C^2)](1 - \upsilon C)}{(1 - \upsilon)(1 + C)^2}W_{\eta}(t) \tag{2.14}$$

Let us note that for $C = 0$ equation (2.14) is equivalent to the parameter used in the proposed criterion for smooth specimens, see Kardas *et al.* (2004), Łagoda and Macha (2000), Łagoda and Ogonowski (2005), Sonsino and Łagoda (2004).

Criterion (2.14) is proposed for steel and aluminium alloy as well as welded alloys (Łagoda and Küppers, 2006; Łagoda and Ogonowski, 2005).

3. Verification of proposed parameter

Specimens made of 42CrMo4V steel were tested by Pötter (2000). Notched specimens were subjected to tests as shown in Fig. 3. Calculations were performed for bending, torsion and combined bending with torsion. Static properties of the steel tested are given in Table 1, and its chemical composition – in Table 2. The calculated stress concentration factors are $K_{tb} = 2.0$ for bending and $K_{tt} = 1.5$ for torsion.

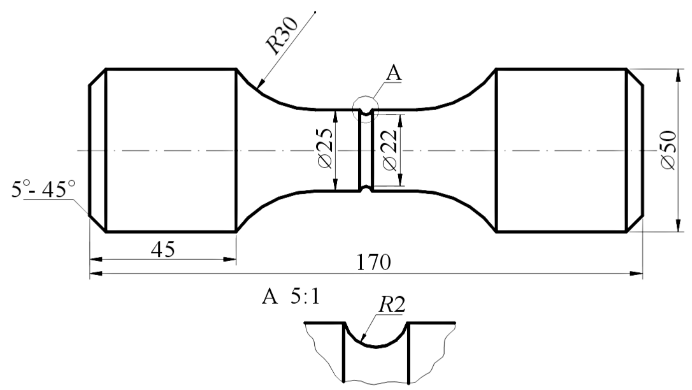

Fig. 3. Geometry of tested specimens

Table 1. Static properties of 42CrMo4V steel (Neuber, 1961)

$R_{p0.2}$ [MPa]	R_m [MPa]	RA	E [GPa]	ν
743	920	69	210	0.3

Table 2. Chemical composition of 42CrMo4V steel in % (Neuber, 1961)

C	Mn	Si	P	S	Cr	Mo	Fe
0.38-0.45	0.5-0.8	< 0.4	< 0.035	< 0.03	0.9-1.2	0.15-0.30	remaining

The criterion of maximum parameter on the plane of maximum shear strain energy density was used for verification. In the tests, $C = \nu$ for $K_{tb} \geqslant 2.0$ was assumed, so Eq. (2.14) was reduced to the following form

$$W_{eq}(t) = k^2(1-\upsilon)W_{\eta s}(t) + \frac{4-k^2(1-\upsilon)^2}{1+\upsilon}W_{\eta}(t) \tag{3.1}$$

For a constant amplitude loading, criterion (3.1) reduces to

$$W_{aeq} = k^2(1-\upsilon)W_{a\eta s} + \frac{4-k^2(1-\upsilon)^2}{1+\upsilon}W_{a\eta} \tag{3.2}$$

In the proposed parameter, the criterion form depends on the coefficient $k(N_f)$. In Kardas *et al.* (2004), the coefficient k was determined at the level of fatigue limit for bending. Pötter (2000) used the stress criterion where the ratio of shear to normal stresses was determined for 10^5 cycles. However, the considered material is characterized by a large difference of slopes on the Wöhler curve for bending $m_\sigma = 5.7$ and for torsion $m_\tau = 11.9$ (Fig. 4), so assuming the stress ratio for the determined number of cycles does not give good results (Fig. 5 and Fig. 6). In Fig. 5, the coefficient k is: $k(N_f) = k(10^5$ cycles) and in Fig. 6 – $k(N_f) = k(\sigma_{af}/\tau_{af}) = k(2.5 \cdot 10^5$ cycles). Much better results could be obtained if the coefficient k is a number of cycles dependent (Fig. 7) according to relationship (2.13). Thus, k and the criterion form take different values for successive numbers of cycles. In Table 3, values of constant k as well as β and κ for three numbers of cycles are presented.

Table 3. Values of the constant k and coefficients β and κ depending on the number of cycles

N_f [cycles]	σ_a [MPa]	τ_a [MPa]	k	β	κ
10^4	687	355	6.67	4.67	0.56
10^5	406	292	3.43	2.40	1.78
$2.5 \cdot 10^5$	330	271	2.65	1.85	2.08

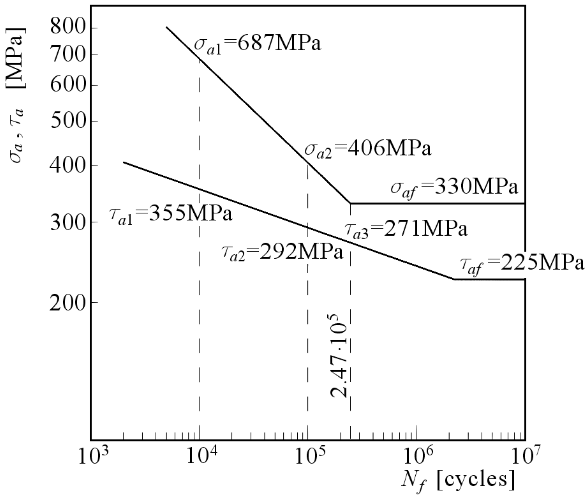

Fig. 4. Wöhler's curves for bending and torsion in the stress notation

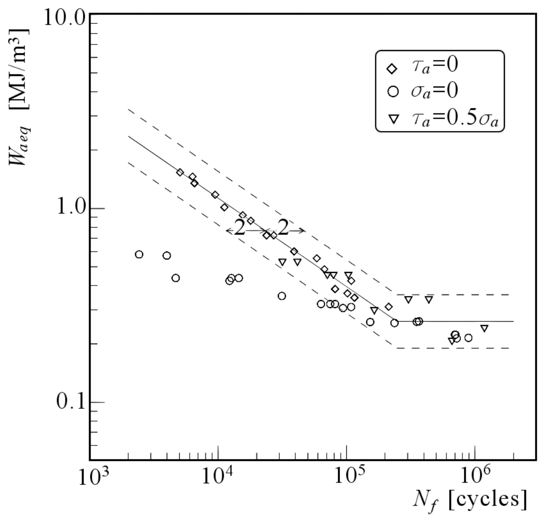

Fig. 5. Wöhler's curves in the energy approach based on calculations against the graph for pure bending (k for the fatigue limit in bending)

It appears from Fig. 5 and Fig. 6 that most results for torsion are located outside the scatter band with a factor of 2, characteristic for pure bending. In the case of k for 10^5 cycles, also under the combined bending with torsion, some points are located outside the scatter band. If the coefficient k is determined by an iterative method versus the number of cycles, all of the results are included in the scatter band with a factor of 2, as in the case of pure bending.

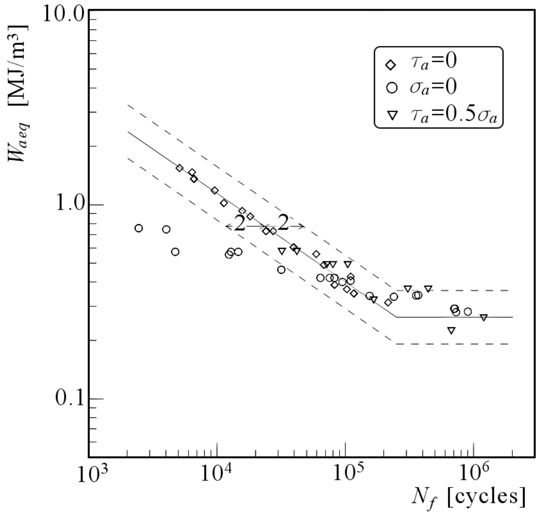

Fig. 6. Wöhler's curve in the energy approach based on calculations against the graph for pure bending (k for 10^5 cycles)

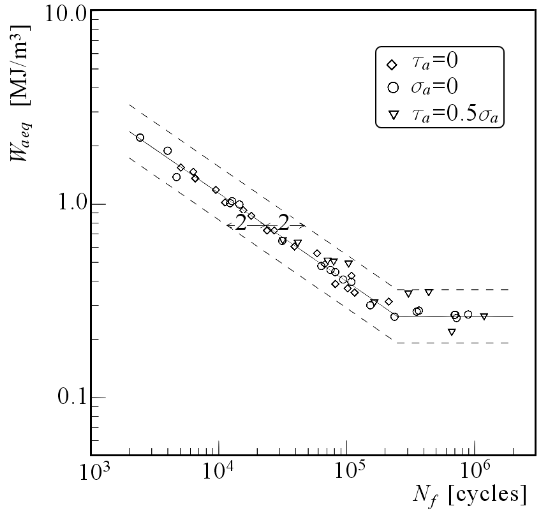

Fig. 7. Wöhler's curve in the energy approach based on calculations against the graph for pure bending (k determined versus the number of cycles)

4. Conclusions

- In this paper, an energy damage parameter is proposed for a combined loading by bending with torsion when stress concentration occurs.
- Two forms of the parameter are proposed, which assume that the plane of maximum normal or shear strain energy density is the critical plane.

- The proposed parameter was investigated for 42CrMo4V steel, and the best results were obtained if the coefficient k depended on the number of cycles for the criterion of the strain energy density parameter in the critical plane. The critical plane is one of the planes where the shear strain energy density parameter reaches its maximum (maximum damage).
- Further verification of the proposed model should be done for other materials, other stress concentration factors and other loadings.

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Ocena trwałości zmęczeniowej próbek z karbem poddanych zginaniu ze skręceniem z wykorzystaniem parametru gęstości energii odkształceń

Streszczenie

W pracy przedstawiono energetyczny parametr uszkodzenia oparty na płaszczyźnie krytycznej w warunkach złożonego stanu obciążenia i spiętrzenia naprężeń. Zaproponowano dwa kryteria uwzględniające kombinacje parametrów gęstości energii

odkształcenia normalnego i postaciowego. Za płaszczyznę krytyczną przyjęto płaszczyznę, w której parametr gęstości energii odkształcenia normalnego lub postaciowego osiąga wartość maksymalną. Wyniki obliczeń uzyskane za pomocą zaproponowanego modelu porównano z wynikami stałoamplitudowych badań zmęczeniowych próbek ze stali 42CrMo4V z karbem obrączkowym wykonanych w warunkach kombinacji proporcjonalnego zginania ze skręcaniem.

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