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# AXIALLY SYMMETRIC CONTACT INVOLVING FRICTION AND BOUNDARY ROUGHNESS

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In the paper, the problem of axially symmetric contact involving partial slip for a rigid flat-ended cylinder and elastic half-space is considered. Friction forces are taken into account. In addition, the surface of the half-space is assumed to be covered by some roughness. The model of the roughness involving the friction forces is developed. The problem is reduced to integral equations, which are solved numerically.

*Key words:* contact problem, boundary stick, slip

## 1. Introduction

The well-known Hertzian model of elastic contact (Hertz, 1981) provides that contacting surfaces are perfectly smooth and frictionless. The corresponding formulations involving friction forces in the contact area in the axially symmetric problem are presented in Mossakovskij (1954), Spence (1968, 1975). It was shown that for the limiting coefficients of friction, the contact area consists of the central region $(0 < r < c)$ with sticking conditions and an annular zone $(c < r < a)$, in which the contacting surfaces slip, see Fig. 1. The size of the stick region $c$ depends on the friction coefficient and Poisson's ratio. Even so friction is taken into account, the formulations presented in Mossakovskij (1954), Spence (1968, 1975) consider, from the geometrical point of view, ideally smooth contacting surfaces. However, real engineering surfaces have some geometrical microstructure, which consists of waviness and roughness. These geometrical imperfections play important role in the real contact.

The aim of this paper is to consider contact of a rigid flat punch and an elastic half-space involving simultaneously friction forces and surface roughness, Fig. 1.

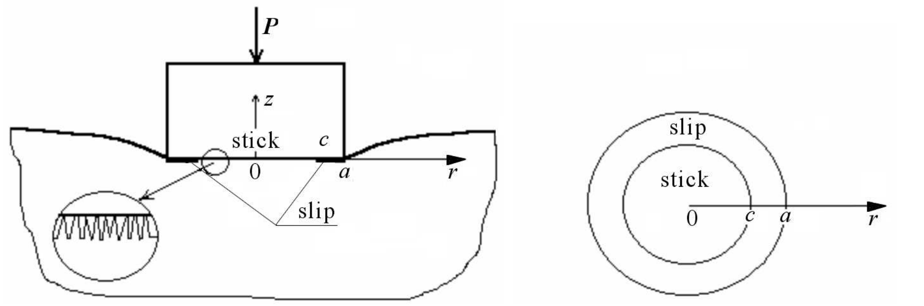


Fig. 1.

There are many approaches which enable one to take into account the boundary roughness in the contact area of elastic bodies. The well-known Greenwood-Williamson model (Greenwood and Williamson, 1966) treats the roughness as elastic asperities with a constant curvature and a Gaussian distribution of heights, which are very densely distributed over the contacting surface. The GW model has many modifications involving plastic deformation and other random distributions. Numerical solution to the normal spherical contact problam, involving the boundary roughness according to the GW model presented in Greenwood and Tripp (1967), shows a bigger contact radius and lower normal pressure in comparison to the Hertz solution. However, the GW model cannot be applied to the formulation of the contact problem involving friction forces, because it omits tangential forces and horizontal deformation on the rough surfaces. In addition, the GW model has been developed for parabolic bodies and cannot be used for flat-ended geometry.

In this paper, in order to take into account friction forces in the contact of a flat-ended rigid cylinder and an elastic rough half space, we propose another model of the boundary roughness. This model is phenomenological and states a modification of the Shtayerman model (Shtayerman, 1949) in the case of tangential forces and deformation. It permits one to reduce the considered contact problem to integral equations which can be solved numerically.

## 2. Problem formulation

Contact geometry of the considered problem is shown in Fig. 1 in the cylindrical co-ordinates $Or\varphi z$. We study an axially symmetric indentation of a rigid flat-ended cylinder (punch) into an elastic half-space. The punch is centrally pressed by the normal load $P$. As a result, the normal $p(r) = -\sigma_{zz}(r,0)$ and the radial $q(r) = \sigma_{rz}(r,0)$ tractions are aroused in the common contact area

$0 \leqslant r \leqslant a$. It is assumed that the contacting surfaces have some roughness. The proposed model of deformation of the rough half-space is inspired by the one given in Shtayerman (1949) for the normal frictionless contact.

According to this model, the vertical $u_z$ and radial $u_r$ displacements of the rough surface of the half-space under the punch consist of two parts

$$
\begin{aligned}
u_z(r,0) &= u_z^{(a)}(r) + u_z^{(b)}(r,0) \qquad & 0 \leqslant r \leqslant a \\
u_r(r,0) &= u_r^{(a)}(r) + u_r^{(b)}(r,0) \qquad & 0 \leqslant r \leqslant a
\end{aligned}
\tag{2.1}
$$

where $u_z^{(a)}(r)$ and $u_r^{(a)}(r)$ describe displacements due to deformation of asperities, and $u_z^{(b)}(r,0)$ and $u_r^{(b)}(r,0)$ are bulk displacements. For the first ones, a phenomenological model is used, according to which

$$
u_z^{(a)}(r) = \alpha p(r) \qquad u_r^{(a)}(r) = \beta q(r) \qquad 0 \leqslant r \leqslant a \tag{2.2}
$$

where $\alpha$, $\beta$ are called roughness parameters.

The bulk displacements can be obtained as elastic solutions for the half-space subjected to action of the traction $p(r)$ and $q(r)$ (Johnson, 1985)

$$
\begin{aligned}
u_z^{(b)}(r,0) &= \frac{4(1-\nu^2)}{\pi E}\int\limits_0^a p(\xi)K_1(\xi,r)\,d\xi - \frac{(1-2\nu)(1+\nu)}{2E}\int\limits_r^a q(\xi)\,d\xi \\
u_r^{(b)}(r,0) &= \frac{4(1-\nu^2)}{\pi E}\int\limits_0^a q(\xi)K_2(\xi,r)\,d\xi - \frac{(1-2\nu)(1+\nu)}{rE}\int\limits_0^r \xi p(\xi)\,d\xi
\end{aligned}
\tag{2.3}
$$

where $E$, $\nu$ are Young's modulus and Poisson's ratio of the half-space material. The kernels $K_1$, $K_2$ are known

$$
\begin{aligned}
K_1(\xi,r) &= \frac{\xi}{\xi+r}K(k) \qquad k = \frac{2\sqrt{\xi r}}{\xi+r} \\
K_2(\xi,r) &= \frac{\xi}{\xi+r}\Big[\Big(\frac{2}{k^2}-1\Big)K(k) - \frac{2}{k^2}E(k)\Big]
\end{aligned}
\tag{2.4}
$$

Here, $K(x)$ and $E(x)$ are the complete elliptic integrals of the first and second kind, respectively.

As we see, the model proposed here states extension of Shtayerman's one on upon action of friction forces and involves radial deformation of the roughness due to these forces. It depends on two parameters $\alpha$, $\beta$ which should be determined experimentally. Notice that the first one describes stiffness of the asperities in the normal direction and the second one – in the radial direction. Another way to determine these parameters is presented in Pauk and Zastrau

(2004). Comparing the solution to the normal contact problem obtained in the framework of the GW model and Shtayerman's one, it is possible to find relations between the roughness parameter $\alpha$ and the parameters from the GW model. Relations (2.2) describe also the well-known model of Winkler's foundation. This means that the developed model treats the roughness as a set of springs or rods distributed over the surface.

## 3. Problem of normal contact

For further analysis, we make the following assumption: the frictional traction $q(r)$ has no effect on the normal displacements. It is known (Johnson, 1985) that this effect is generally small, which means that the second term in formula $(2.3)_1$ can be omitted. This assumption permits one to consider separately the problem of determination of the normal pressure $p(r)$.

Satisfying with the help of formulae $(2.1)_1$, $(2.2)_1$, $(2.3)_1$ and with the above assumption the boundary condition of the normal problem

$$u_z(r,0) = \delta \qquad 0 \leqslant r \leqslant a \tag{3.1}$$

where $\delta$ is the rigid approach of the punch, we arrive at the integral equation of the Fredholm type of the second kind

$$\alpha p(r) + \frac{4(1-\nu^2)}{\pi E} \int\limits_0^a p(\xi) K_1(\xi, r)\, d\xi = \delta \qquad 0 \leqslant r \leqslant a \tag{3.2}$$

This equation with the equilibrium condition

$$2\pi \int\limits_0^a p(r) r\, dr = P \tag{3.3}$$

states a closed system for the unknown function $p(r)$.

Introducing dimensionless variables and functions

$$\begin{aligned} s &= \frac{r}{a} & \eta &= \frac{\xi}{a} & p^*(s) &= \frac{a^2 p(r)}{P} \\ \alpha^* &= \frac{\alpha E}{4(1-\nu^2)a} & \delta^* &= \frac{a\delta E}{4(1-\nu^2)P} & k^* &= \frac{2\sqrt{\eta s}}{\eta + s} \\ K_1^*(\eta, s) &= \frac{\eta}{\eta + s} K(k^*) \end{aligned} \tag{3.4}$$

the system of integral equations (3.2), (3.3) can be rewritten in the form

$$\begin{aligned}
&\alpha^* p^*(s) + \frac{1}{\pi}\int\limits_0^1 p^*(\eta) K_1^*(\eta - s)\, d\eta = \delta^* \qquad\qquad 0 \leqslant s \leqslant 1 \\
&2\pi \int\limits_0^1 p^*(s) s\, ds = 1
\end{aligned} \tag{3.5}$$

Equations (3.5) were solved numerically for different values of the dimensionless roughness parameter $\alpha^*$. The effects of this parameter on the normal pressure distribution are presented in Fig. 2. For $\alpha^* = 0$, the well-known solution for the rigid flat-ended cylinder resting on the smooth half-space (Johnson, 1985)

$$p^*(s) = \frac{1}{2\pi\sqrt{1 - s^2}} \qquad\qquad 0 \leqslant s < 1 \tag{3.6}$$

is obtained. This classical solution is unbounded for $s \to \pm 1$. But in the problem under consideration the contact pressure is bounded at the punch edge. This result is due to the boundary roughness and was first discovered by Shtayerman (1949).

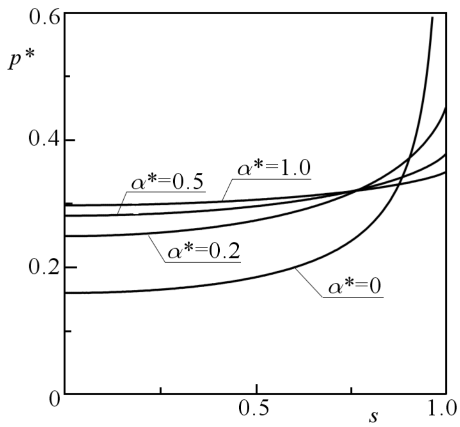


Fig. 2.

## 4. Punch perfectly connected with half-space

Let us initially consider a situation in which friction forces are sufficiently big to prevent slip in the whole contact area. In this case, the boundary conditions of the tangential problem are as follows $(0 \leqslant r \leqslant a)$

$$u_r(r,0) = 0 \qquad\qquad q(r) < fp(r) \tag{4.1}$$

where $f$ is the Coulomb friction coefficient.

Satisfying condition (4.1), with the help of presentation $(2.1)_2$, $(2.2)_2$, $(2.3)_2$ we obtain the following integral equation for the unknown function $q(r)$ $(0 \leqslant r \leqslant a)$

$$\beta q(r) + \frac{4(1-\nu^2)}{\pi E} \int\limits_0^a q(\xi)K_2(\xi,r)\, d\xi = \frac{(1-2\nu)(1+\nu)}{rE} \int\limits_0^r \xi p(\xi)\, d\xi \tag{4.2}$$

Here, the contact pressure $p(r)$ is assumed to be known from the previous Section.

The total friction force $Q$ is unknown and can be found as

$$Q = 2\pi \int\limits_0^a q(r)r\, dr \tag{4.3}$$

Equation (4.2) was solved numerically similarly to equation $(3.5)_1$ in normalized variables (3.4) and

$$q^*(s) = \frac{a^2 q(r)}{P} \qquad\qquad \beta^* = \frac{\beta E}{4(1-\nu^2)a} \tag{4.4}$$

Figure 3 presents the distribution of frictional traction for some values of the parameters $\beta^*$ and for $\alpha^* = 0.5$, $\nu = 0.3$. In contrast to the case of the perfectly smooth surface $\beta^* = 0$ (Johnson, 1985), the friction forces in the problem under consideration are bounded at the punch edge. Upon the increasing parameter $\beta^*$, these forces decrease.

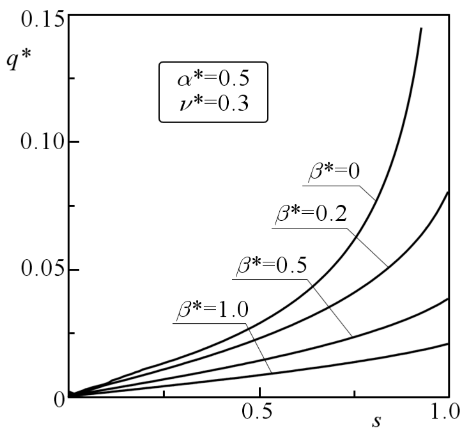


Fig. 3.

For the investigation of possible slip between the contacting surfaces, the ratio of tangential to normal tractions should be investigated. In the classical case (Mossakovskij, 1954; Spence, 1968, 1975), it is a function of Poisson's ratioo $\nu$ and the coordinate $r$, and it is unbounded at the punch edge $r = a$. This means that condition $(4.1)_2$ can not be satisfied in this point for physically accepted values of the friction coefficient $f < \infty$. As a consequence, slip is expected on the annulus of the contact area edge. The corresponding contact problem involving partial slip was considered in Spence (1975).

Quite different situation is observed in the contact problem under consideration. The ratio $q(r)/p(r)$ is now a function of $\nu$, $r$ as well as of the roughness parameters $\alpha^*$ and $\beta^*$. This ratio at the punch edge is shown in Fig. 4 versus Poisson's ratio for some sets of $\alpha^*$ and $\beta^*$. We can see that it is bounded, and there are values of parameters $\nu$, $\alpha^*$, $\beta^*$ for which stick condition $(4.1)_2$ is satisfied for the given friction coefficient $f$. For example, if $f = 0.1$ and the roughness parameters $\alpha^* = 0.2$, $\beta^* = 0.5$, there is a range of Poisson's ratio $0.255 < \nu < 0.5$ (see Fig. 4), for which condition $(4.1)_2$ is satisfied and the contacting surfaces remain stuck within the whole contact region. If $0 < \nu < 0.255$, this condition is disturbed at $r = a$, and slip must occur near the punch edge.

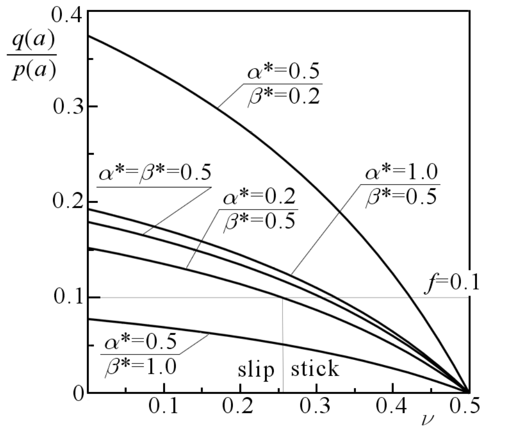


Fig. 4.

Such formulation of the contact will be considered in the next Section. The possibility of expected slip for other sets of parameters $f$, $\alpha^*$, $\beta^*$ can be read from Fig. 4. Generally, the possibility of slip occurence is bigger for smaller $\beta^*$ when $\alpha^*$ is fixed, compare the curves $\alpha^* = 0.5$, $\beta^* = 0.2$; $\alpha^* = 0.5$, $\beta^* = 0.5$; $\alpha^* = 0.5$, $\beta^* = 1.0$.

## 5. Partial slip contact problem

Let us now pass to the problem in which the contacting surfaces are stuck in the central part of the contact area, while on the annulus of the punch edge slip takes place. The boundary conditions of the considered problem are as follows:

— for $0 < r < c$

$$u_r(r,0) = 0 \qquad\qquad q(r) < fp(r) \tag{5.1}$$

— for $c < r < a$

$$q(r) = fp(r) \qquad\qquad u_r(r,0) > 0 \tag{5.2}$$

Notice that the stick area size $c$ is unknown.

Satisfying boundary condition $(5.1)_1$ with the help of formulae $(2.1)_2$, $(2.2)_2$, $(2.3)_2$, we arrive at an integral equation for the frictional traction $(0 < r < c)$

$$\beta q(r) + \frac{4(1-\nu^2)}{\pi E}\int\limits_0^a q(\xi)K_2(\xi,r)\,d\xi = \frac{(1-2\nu)(1+\nu)}{rE}\int\limits_0^r \xi p(\xi)\,d\xi \tag{5.3}$$

where the contact pressure $p(s)$ as assumed to be known from the solution to the normal problem.

Integral equation (5.3) can not be solved in the present form because it is written within the stick area, while the unknown function $q(s)$ is defined on the whole contact region. To avoid this difficulty, the distribution of friction forces is expressed as follows

$$q(r) = \begin{cases} fp(r) + q_0(r) & \text{for} \quad 0 < r < c \\ fp(r) & \text{for} \quad c < r < a \end{cases} \tag{5.4}$$

where $q_0(r)$ is a new unknown called the corrective traction. Notice that function (5.4) satisfies boundary conditions $(5.1)_2$, $(5.2)_1$ if $q_0(r) < 0$.

Substituting (5.4) into (5.3), the integral equation for the corrective traction can be derived $(0 < r < c)$

$$\begin{aligned} &\beta q_0(r) + \frac{4(1-\nu^2)}{\pi E}\int\limits_0^c q_0(\xi)K_2(\xi,r)\,d\xi = \\ &\qquad = \frac{(1-2\nu)(1+\nu)}{rE}\int\limits_0^r \xi p(\xi)\,d\xi - f\beta p(r) - \frac{4(1-\nu^2)}{\pi E}f\int\limits_0^a p(\xi)K_2(\xi,r)\,d\xi \end{aligned} \tag{5.5}$$

The unknown total friction force $Q$ can be now calculated from (4.3) and (5.4) as

$$Q = fP + 2\pi \int\limits_0^c q_0(r)r\, dr \tag{5.6}$$

In the normalized variables and functions (3.4), (4.4) and

$$q_0^*(s) = \frac{a^2 q_0(r)}{P} \qquad c^* = \frac{c}{a} \qquad Q^* = \frac{Q}{P} \tag{5.7}$$

equations (5.5), (5.6) read

$$\begin{aligned} &\beta^* q_0^*(s) + \frac{1}{\pi}\int\limits_0^{c^*} q_0^*(\eta)K_2^*(\eta,s)\, d\eta = \frac{1-2\nu}{4(1-\nu)}\int\limits_0^s \frac{\eta}{s}p^*(\eta)\, d\eta + \\ &\qquad -f\beta^* p^*(s) - \frac{f}{\pi}\int\limits_0^1 p^*(\eta)K_2^*(\eta,s)\, d\eta \qquad 0 < s < c^* \\ &Q^* = f + 2\pi\int\limits_0^{c^*} q_0^*(s)s\, ds \end{aligned} \tag{5.8}$$

Integral equation $(5.8)_1$ was solved numerically. The unknown normalized size of the stick zone $c^*$ was calculated iteratively satisfying the physical condition

$$q_0^*(c^*) = 0 \tag{5.9}$$

which, according to formula (5.4), provides the continuity of frictional traction within the whole contact area.

After calculation of the corrective traction, the distribution of friction forces can be found from function (5.4).

The input data for calculations are: Poisson's ratio $\nu$, roughness parameters $\alpha^*$ and $\beta^*$, and friction coefficient $f$.

Figures 5a and 5b show distributions of tangential traction for some values of the friction coefficient and for two values of Poisson's ratio. We observe this traction decreasing upon a decrease in the friction coefficient. Simultaneously, the stick zone shrinks. These phenomena are more evident for lower Poisson's ratios. If the friction coefficient increases, the distributions of the shear traction tend to those obtained in the previous Section and are shown by dotted curves.

The dimensionless stick region size $c^*$ and total shear force $Q^*$ are respectively presented in Fig. 6a and Fig. 6b, as functions of the friction coefficient $f$ for some values of Poisson's ratio. An increase of both these parameters with

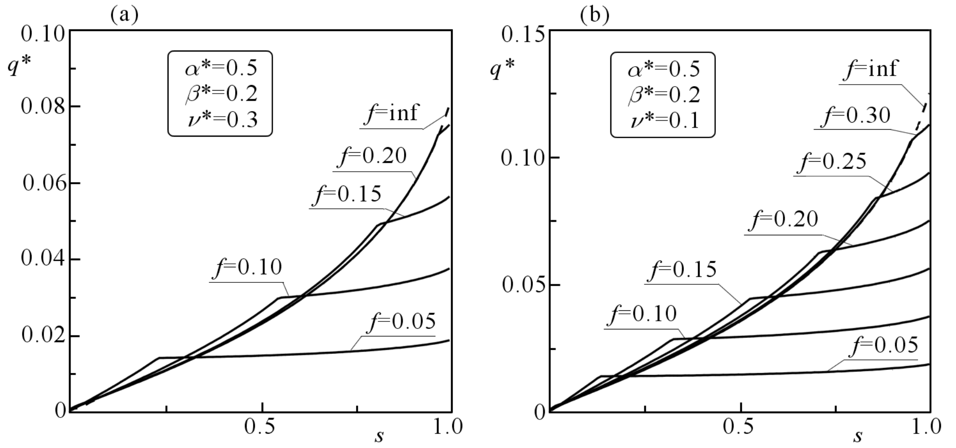


Fig. 5.

the friction coefficient is observed. At some values of the friction coefficient, we obtain $c^* = 1$ which means that the slip zone vanishes and the contacting surfaces are in the stick conditions.

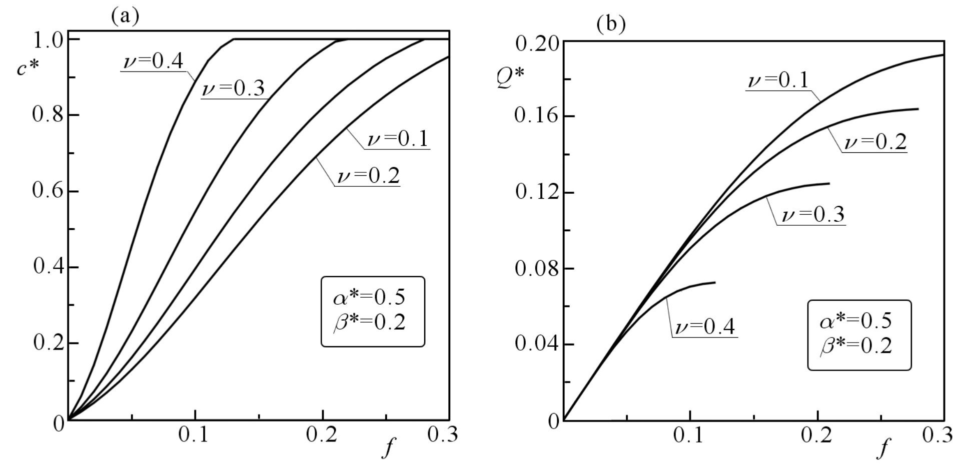


Fig. 6.

## 6. Conclusions

The axially symmetric contact of a rigid flat-ended punch and an elastic half-space is studied. Simultaneous effects due to friction under the punch and roughness of the half-space surface are investigated. A new model of the surface roughness is developed. This model enables one to solve the considered contact problems by the boundary integral equation method. The derived integral equations are solved numerically.

New phenomena brought about to the surface roughness are observed. In contrast to the corresponding classical contact, in the problem under consideration the contact with perfectly connected surfaces is physically possible for the limiting coefficients of friction.

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## Osiowo-symetryczne zagadnienie kontaktu z uwzględnieniem tarcia i chropowatości brzegu


Streszczenie

W pracy zajęto się zagadnieniem osiowo-symetrycznego kontaktu płasko zakończonego walca z podatną półprzestrzenią przy uwzględnieniu częściowego poślizgu

oraz sił tarcia. Dodatkowo założono, że powierzchnia rozważanej półprzestrzeni jest chropowata. Zaproponowano odpowiedni model chropowatości z siłami tarcia. Całe zagadnienie sprowadzono do równań całkowych, które rozwiązano numerycznie.

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