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NONLINEAR TRAVELING WAVES IN A THIN LAYER COMPOSED OF THE MOONEY-RIVLIN MATERIAL

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In this paper the problem of studies nonlinear traveling waves in the Mooney-Rivlin elastic layer is studied. By averaging the equations of motions over the width of the layer we obtain a system of partial differential equations in one dimensional space and time. A technique of phase planes is used to study the waves processes. Based on the phase trajectory method, we can make an interpretation of conditions of propagation of nonlinear traveling waves and can establish the existence conditions under which the phase plane contains physically acceptable solutions.

Key words: discontinuous surface, traveling waves, hyperelastic materials, phase plane

1. Introduction

The considered layer has two kinematically independent degrees of freedom which are represented by two independent functions describing motion in the layer. The effect of finite lateral dimensions and inertia of the elastic layer are considered by describing the layer as a one-dimensional elastic structure with one scalar variable representing transverse symmetric motion. In the simplest description, one scalar variable can be used to describe effects of finite transverse dimensions in the elastic layer that undergoes longitudinal and symmetrical transverse motion only.

Following this introduction, general equations describing motion of an incompressible, nonlinear elastic medium, symmetric lateral motion of the elastic layer and a procedure of averaging of equations of motion are presented in Section 2. The traveling waves are described in Section 3. We obtained a solution for the traveling wave propagating with the speed V in the direction of the coordinate X_1 depending on one parameter only.

In Section 4, we use phase plane methods to classify different solutions for traveling waves that are possible. Some of the solutions to the differential equations do not correspond to physically acceptable waves propagating in the layer, and so additional restrictions must be imposed from the physical problem. We explore such restrictions in Section 5. We are able to establish conditions for the existence of physically acceptable solutions as represented by individual paths in the phase plane. Finally, in Section 6 we present a numerical analysis for traveling waves in the layer composed of the Mooney-Rivlin material.

2. Symmetric motion of the layer

Motion of a continuum is represented by a set of functions (Truesdell and Toupin, 1960)

$$x_i = x_i(X_\alpha, t) \qquad i,\alpha = 1,2,3 \tag{2.1}$$

We assume that the traveling wave is propagating in the half-infinite elastic layer which occupies the material region $X_1 > 0$ (Fig. 1) in the direction of the axis X_1. At the frontal area of the layer $X_1 = 0$, the boundary conditions for deformations are given (Fu and Scott, 1989). We assume that motion described by equation (2.2) undergos without imposing additional contact forces at the lateral planes of the layer $X_2 = \pm h$ (Coleman and Newman, 1990; Wright, 1981).

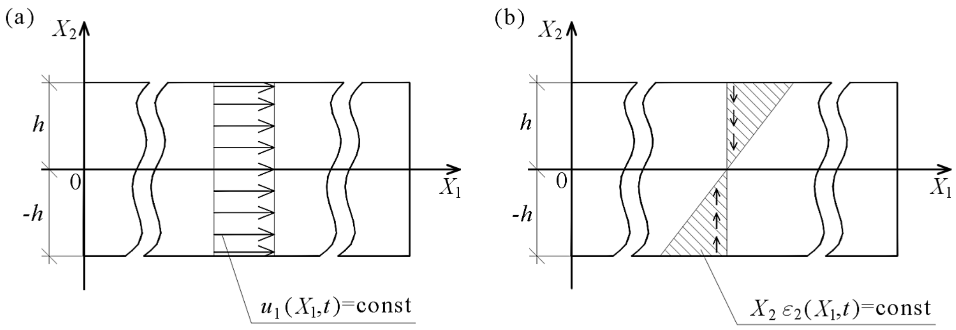

Fig. 1. Motion of the layer; (a) main motion in the longitudinal direction, (b) secondary motion in the transverse direction

Motion of the considered traveling wave is assumed as

$$\begin{aligned} x_1 &= X_1 + u_1(X_1,t) \qquad\qquad x_3 = X_3 \\ x_2 &= X_2 + f(X_2)\varepsilon_2(X_1,t) \end{aligned} \tag{2.2}$$

where $f(X_2)$ is a function of $f \in C^1\ (\langle -h; h\rangle \to R)$, which is odd for symmetric motion in the transverse direction, however $u_1, \varepsilon_2 \in C^3\ (\langle 0;\infty) \times \langle 0;\infty) \to R)$.

After multiplying by $\varepsilon_2(X_1,t)$, the function $f(X_2)$ describes motion in the transverse direction of the layer.

We assume the simplest form of the function $f(X_2) = X_2$, then for $(2.2)_2$ we have

$$x_2 = X_2 + X_2\varepsilon_2(X_1,t) \tag{2.3}$$

The strain ε_1, the gradient of the transversal strain κ and speeds of the particle of the medium ν_1 and ν_2 in both directions of the layer are equal, respectively

$$\varepsilon_1 = u_{1,1} \qquad\qquad \kappa = \varepsilon_{2,1} \tag{2.4}$$

$$\nu_1 = \dot{x}_1 = \dot{u}_1(X_1,t) \qquad\qquad \nu_2 = \dot{x}_2 = X_2\dot{\varepsilon}_2(X_1,t) \tag{2.5}$$

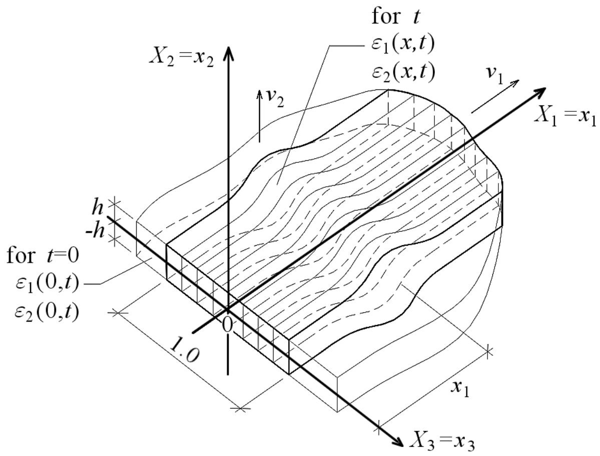

Fig. 2. Propagation of the traveling wave in the layer

For assumed motion (2.3), the deformation gradient and the left Cauchy-Green tensor have the form

$$\begin{gathered}
\mathbf{F} = [x_{i\alpha}] = \begin{bmatrix} 1+\varepsilon_1 & 0 & 0 \\ X_2\kappa & 1+\varepsilon_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{B} = \begin{bmatrix} (1+\varepsilon_1)^2 & (1+\varepsilon_1)X_2\kappa & 0 \\ (1+\varepsilon_1)X_2\kappa & (X_2\kappa)^2+(1+\varepsilon_2)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{gathered} \tag{2.6}$$

For an incompressible material, there is identity

$$\det \mathbf{F} = 1 \tag{2.7}$$

then for the considered material

$$(1+\varepsilon_1)(1+\varepsilon_2) = 1 \tag{2.8}$$

We assume that the layer is made of the Mooney-Rivlin material characterized by the strain-energy function

$$W = \mu[C_1(I_1-3) + C_2(I_2-3)] \tag{2.9}$$

where C_1 and C_2 are constitutive constants. The invariants I_1 and I_2 of the deformation $\mathbf{B}$ are

$$I_1 = I_2 = (1+\varepsilon_1)^2 + (1+\varepsilon_2)^2 + (X_2\kappa)^2 + 1 \tag{2.10}$$

According to Wesołowski (1972a,b) or Dai (2001), the Cauchy tensor has the form

$$\mathbf{T} = -q\mathbf{I} + 2\mu(C_1\mathbf{B} - C_2\mathbf{B}^{-1}) \tag{2.11}$$

where q is an arbitrary hydrostatic pressure.

The nominal stress tensor (the Piola-Kirchhoff tensor $\mathbf{T}_R$) may be expressed by the Cauchy tensor $\mathbf{T}$

$$\mathbf{T}_R = \mathbf{T}\mathbf{F}^{-\top} \tag{2.12}$$

and its non-zero components are given by

$$\begin{aligned}
T_{R11} &= -(1+\varepsilon_2)[q + 2\mu C_2(X_2\kappa)^2] + 2\mu(1+\varepsilon_1)[C_1 - C_2(1+\varepsilon_2)^4]\\
T_{R12} &= 2\mu C_2 X_2\kappa[(1+\varepsilon_1)^2 + (1+\varepsilon_2)^2 + (X_2\kappa)^2] + qX_2\kappa\\
T_{R21} &= 2\mu X_2\kappa(C_1 + C_2)\\
T_{R22} &= -(1+\varepsilon_1)[q + 2\mu C_2(X_2\kappa)^2] + 2\mu(1+e_2)[C_1 - C_2(1+\varepsilon_1)^4]\\
T_{R33} &= -q + 2\mu(C_1 - C_2)
\end{aligned} \tag{2.13}$$

For deformation gradient $(2.6)_1$, the equations of motion

$$T_{Ri\alpha,\alpha} = \rho_R u_{i,tt} \tag{2.14}$$

are reduced to a system of equations for the plane strain deformation

$$\begin{aligned}
T_{R11,1} + T_{R12,2} &= \rho_R u_{1,tt}\\
T_{R21,1} + T_{R22,2} &= \rho_R X_2\varepsilon_{2,tt}\\
T_{R33,3} &= 0
\end{aligned} \tag{2.15}$$

The boundary conditions at the top and bottom surfaces of the layer have the form

$$T_{R12}(X_1, \pm h, X_3) = T_{R22}(X_1, \pm h, X_3) = 0 \tag{2.16}$$

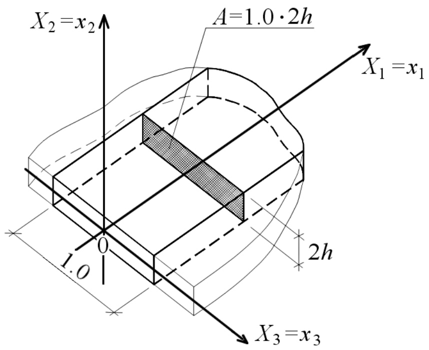

Fig. 3. A cross-section of the layer A (perpendicular to axis X_1)

We employ a procedure, which was described by Wright (1981), consisting in averaging equations of motion $(2.15)_{1,2}$ along the cross-section of the layer A (see Fig. 3).

We assume that in the averaging procedure, boundary conditions (2.16) at the lateral surface $X_2 = \pm h$ are satisfied. We multiply second equation of motion $(2.15)_2$ by X_2 and average both resulting equations and first equation $(2.15)_1$ over the width of the layer, thus obtaining

$$\begin{aligned}
&\frac{1}{2h}\int\limits_{-h}^{h} \frac{\partial T_{R11}}{\partial X_1}\,dX_2 + \frac{1}{2h}\int\limits_{-h}^{h} \frac{\partial T_{R12}}{\partial X_2}\,dX_2 = \frac{1}{2h}\int\limits_{-h}^{h} \rho_R \ddot{u}_1\,dX_2 \\
&\frac{1}{2h}\int\limits_{-h}^{h} \frac{\partial T_{R21}}{\partial X_1} X_2\,dX_2 + \frac{1}{2h}\int\limits_{-h}^{h} \frac{\partial T_{R22}}{\partial X_2} X_2\,dX_2 = \frac{1}{2h}\int\limits_{-h}^{h} \rho_R \ddot{\varepsilon}_2 X_2^2\,dX_2
\end{aligned} \tag{2.17}$$

In each of equations (2.17), the second integral admits explicit integration of the form

$$\begin{aligned}
&\int\limits_{-h}^{h} \frac{\partial T_{R12}}{\partial X_2}\,dX_2 = T_{R12}\Big|_{-h}^{h} = 0 \\
&\int\limits_{-h}^{h} \frac{\partial T_{R22}}{\partial X_2} X_2\,dX_2 = X_2 T_{R22}\Big|_{-h}^{h} - \int\limits_{-h}^{h} T_{R22}\,dX_2 = -\int\limits_{-h}^{h} T_{R22}\,dX_2
\end{aligned} \tag{2.18}$$

Taking into account boundary conditions (2.16), one obtains an averaged equation of motion

$$
\begin{aligned}
&\frac{\partial}{\partial X_1}\Big(\frac{1}{2h}\int\limits_{-h}^{h} T_{R11}\,dX_2\Big) = \rho_R \ddot{u}_1 \\
&\frac{\partial}{\partial X_1}\Big(\frac{1}{2h}\int\limits_{-h}^{h} X_2 T_{R21}\,dX_2\Big) - \frac{1}{2h}\int\limits_{-h}^{h} T_{R22}\,dX_2 = \rho_R \ddot{\varepsilon}_2 \frac{h^2}{3}
\end{aligned}
\tag{2.19}
$$

Equations (2.15) are the strict equations, however equations (2.19) are a consequence of the applied average procedure. For motion (2.15), the cross-section of the layer remains plane and the normal to the surface of cross-sections overlap the axis X_1 (Fig. 2). The analogical assumption was made in the paper by Braun and Kosiński (1999).

In further analysis, we take the advantage of averaged equation $(2.19)_1$ and equation $(2.15)_2$.

Substituting the components of Pioli-Kirchhoff stress tensor (2.13) into $(2.15)_2$ and integrating them with respect to X_2, we obtain the equation of motion in the direction of the axis X_2

$$
q = \frac{\mu X_2^2}{(1+\varepsilon_1)}\Big[\kappa_{,1}(C_1 + C_2) - 2\kappa^2 C_2(1+\varepsilon_1) - \frac{1}{2}\nu_o^{-2}\varepsilon_{2,tt}\Big] + \frac{q_1(X_1,t)}{1+\varepsilon_1} \tag{2.20}
$$

where $\nu_o = \sqrt{\mu/\rho_R}$ is the speed of infinitesimal shear waves and $q_1 = (X_1,t)$ is an arbitrary function.

Using boundary conditions (2.16) $T_{R22}\big|_{X_2=\pm h} = 0$, we determine the function $q_1 = (X_1,t)$. We substitute expression (2.20) into $(2.13)_4$. The obtained equation depends on $(X_2)^2$, then both boundary conditions are satisfied. For the Mooney-Rivlin material, we obtain

$$
\begin{aligned}
q_1 = {} & -2\mu C_2(1+\varepsilon_1)(h\kappa)^2 + 2\mu(1+\varepsilon_2)[C_1 - C_2(1+\varepsilon_1)^4] + \\
& -\mu h^2\Big[\kappa_{,1}(C_1 + C_2) - 2\kappa^2 C_2(1+\varepsilon_1) - \frac{1}{2}\nu_o^{-2}\varepsilon_{2,tt}\Big]
\end{aligned}
\tag{2.21}
$$

Finally (2.20) has the form

$$
\begin{aligned}
q = {} & \frac{\mu(X_2^2 - h^2)}{1+\varepsilon_1}\Big[-2\kappa^2 C_2(1+\varepsilon_1) + \kappa_{,1}(C_1 + C_2) - \frac{1}{2}\nu_o^{-2}\varepsilon_{2,tt}\Big] + \\
& + \frac{2\mu(1+\varepsilon_2)}{1+\varepsilon_1}\{C_1 - C_2(1+\varepsilon_1)^2[(1+\varepsilon_1)^2 + (h\kappa)^2]\}
\end{aligned}
\tag{2.22}
$$

Averaged equation of motion $(2.19)_1$ has the form

$$
\Big(\frac{1}{2h}\int\limits_{-h}^{h} T_{R11}\,dX_2\Big)_{,1} = \rho_R \ddot{u}_1 \tag{2.23}
$$

Including (2.22), the left-hand side of equation (2.23) for the Mooney-Rivlin material is

$$\begin{aligned}\Big(\frac{1}{2h}\int\limits_{-h}^{h} T_{R11}\, dX_2\Big)_{,1} &= 2\mu\Big\{C_1[(1+\varepsilon_1)-(1+\varepsilon_2)^3]+ \\ &+C_2(1+\varepsilon_2)\Big[(1+\varepsilon_1)^2-(1+\varepsilon_2)^2+\frac{2}{3}(h\kappa)^2\Big]+ \\ &-(1+\varepsilon_2)^2\frac{h^2}{3}\Big[2C_2\kappa^2(1+\varepsilon_1)+\frac{1}{2}\nu_o^{-2}\varepsilon_{2,tt}-\kappa_{,1}(C_1+C_2)\Big]\Big\}_{,1}\end{aligned} \tag{2.24}$$

After differentiation and transformation, we obtain from (2.8)

$$\begin{aligned}\kappa &= \varepsilon_{2,1} = -\varepsilon_{1,1}(1+\varepsilon_1)^{-2} \\ \kappa_{,1} &= 2\varepsilon_{1,1}^2(1+\varepsilon_1)^{-3}-\varepsilon_{1,11}(1+\varepsilon_1)^{-2}\end{aligned} \tag{2.25}$$

Including (2.8) and (2.25) in (2.24), we finally obtain an equation which contains the function $\varepsilon_1(X_1,t)$ only

$$\begin{aligned}&\Big\{C_1[(1+\varepsilon_1)-(1+\varepsilon_1)^{-3}]-\frac{h^2}{6}\nu_o^{-2}\varepsilon_{2,tt}(1+\varepsilon_1)^{-2}+ \\ &+C_2\Big[(1+\varepsilon_1)-(1+\varepsilon_1)^{-3}+\frac{2}{3}(h\varepsilon_{1,1})^2(1+\varepsilon_1)^{-5}\Big]+ \\ &-\frac{h^2}{3}[2C_2\varepsilon_{1,1}^2(1+\varepsilon_1)^{-5}-(C_1+C_2)(2\varepsilon_{1,1}^2(1+\varepsilon_1)^{-5}-\varepsilon_{1,11}(1+\varepsilon_1)^{-4})]\Big\}_{,1}= \\ &=\frac{1}{2}\nu_o^{-2}u_{1,tt}\end{aligned} \tag{2.26}$$

The above equation (2.26) is the governing equation describing nonlinear dynamics of the layers.

3. Traveling waves

The phase ξ is defined by

$$\xi = X_1 - Vt \tag{3.1}$$

where V is the speed of propagation of the traveling wave with a constant profile displaced along the axis X_1. For the traveling wave with any profile, we express motion as a function of one parameter ξ only

$$u_1(X_1,t) = u_1(\xi) \qquad\qquad \varepsilon_2(X_1,t) = \varepsilon_2(\xi) \tag{3.2}$$

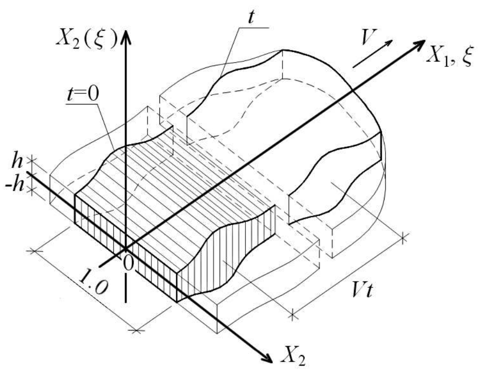

Fig. 4. Propagation of the traveling wave with speed V

According to equations (3.1) and (3.2), the derivatives with respect to t and X_1 are equal

$$
\begin{aligned}
&u_{1,t} = -Vu_{1,\xi} \qquad && u_{1,tt} = V^2 u_{1,\xi\xi} \qquad && u_{1,1} = \frac{\partial u_1}{\partial \xi} = u_{1,\xi} \\
&u_{1,11} = \frac{\partial^2 u_1}{\partial \xi^2} = u_{1,\xi\xi} \qquad && \varepsilon_{2,tt} = V^2 \varepsilon_{2,\xi\xi} = V^2\Big[-\frac{\varepsilon_1}{1+\varepsilon_1}\Big]_{,\xi\xi}
\end{aligned}
\tag{3.3}
$$

Including (3.3) and integrating with respect to ξ, equation (2.26) for the traveling wave has the form

$$
\begin{aligned}
(C_1 + C_2)[(1+\varepsilon_1) - (1+\varepsilon_1)^{-3}] + \frac{h^2}{6}\nu(1+\varepsilon_1)^{-2}\Big[\frac{\varepsilon_1}{1+\varepsilon_1}\Big]_{,\xi\xi} + \\
+\frac{h^2}{3}(C_1 + C_2)[2\varepsilon_{1\xi}^2(1+\varepsilon_1)^{-5} - \varepsilon_{1\xi\xi}(1+\varepsilon_1)^{-4}] = \frac{1}{2}\nu\varepsilon_1 + d_1
\end{aligned}
\tag{3.4}
$$

where $\nu = V^2/\nu_o^2$ and d_1 is integration constant.

Multiplying (3.4) by $\varepsilon_{1,\xi}$, we integrate once more to obtain

$$
\begin{aligned}
\frac{1}{2}(C_1 + C_2)[(1+\varepsilon_1)^2 + (1+\varepsilon_1)^{-2}] + \frac{h^2}{12}\nu\varepsilon_{1,\xi}^2(1+\varepsilon_1)^{-4} + \\
-\frac{h^2}{6}(C_1 + C_2)\varepsilon_{1,\xi}^2(1+\varepsilon_1)^{-4} = \frac{1}{4}\nu\varepsilon_1^2 + d_1\varepsilon_1 + d_2
\end{aligned}
\tag{3.5}
$$

where d_2 is another integration constant.

This equation gives a solution for the traveling wave propagating with the speed V in the X_1 direction and depends on one parameter ξ (3.1) only.

If the constant $\nu = \rho V^2/\mu$ and the constants of integration d_1 and d_2 are known, we could find the solution.

4. Phase plane analysis of propagation of traveling waves in the layer

By constructing phase portraits of the solution in the $(\varepsilon_1, \varepsilon_{1,\xi})$ plane, we can made an interpretation of the conditions of propagation of a nonlinear traveling wave (Dai, 2001; Major and Major, 2006).

First, we introduce dimensionless variables

$$\begin{aligned} b_1 &= \frac{2C_1}{\nu - 2C_1} & \qquad b_2 &= \frac{2C_2}{\nu - 2C_1} \\ D_1 &= \frac{2d_1}{\nu - 2C_1} & \qquad D_2 &= \frac{2d_2}{\nu - 2C_1} \end{aligned} \tag{4.1}$$

Multiplying the equations of motion in form (3.5) by $4/(\nu - 2C_1)$ and including (4.1), we obtain an approximate form

$$\begin{aligned} &(b_1 + b_2)[(1+\varepsilon_1)^2 + (1+\varepsilon_1)^{-2}] + \frac{h^2}{3}\varepsilon_{1,\xi}^2(1+\varepsilon_1)^{-4}(1-b_2) = \\ &= \varepsilon_1^2(1+b_1) + 2D_1\varepsilon_1 + 2D_2 \end{aligned} \tag{4.2}$$

Now we introduce the following transformation

$$\zeta = \frac{\sqrt{3}}{h}\xi \tag{4.3}$$

Apart from a scaling factor, ζ is just the current configuration coordinate X_1 in terms of the phase ξ, and (4.2) takes the form

$$\varepsilon_{1,\zeta}^2 = F(\varepsilon_1, D_2) \tag{4.4}$$

where

$$\begin{aligned} F(\varepsilon_1, D_2) &= \frac{1}{(1-b_2)(1+\varepsilon_1)^2} \cdot \\ &\cdot\{(1+\varepsilon_1)^6[2D_1\varepsilon_1 + 2D_2 + \varepsilon_1^2(1+b_1)] - (b_1+b_2)[(1+\varepsilon_1)^8 + (1+\varepsilon_1)^4]\} \end{aligned} \tag{4.5}$$

We have written D_2 explicitly as an argument of F because different curves in the phase plane correspond to different values of D_2. More precisely, the parameters b_1, b_2 and D_1 uniquely determine a portrait, and then D_2 determines the curves in that portrait.

We introduce a denotation

$$y = \varepsilon_{1,\zeta} = \frac{d\varepsilon_1}{d\zeta} = \sqrt{F(\varepsilon_1, D_2)} \tag{4.6}$$

whose first derivative with respect to ζ is

$$y_{,\zeta} = \varepsilon_{1,\zeta\zeta} = \frac{dy}{d\zeta} = \frac{F'(\varepsilon_1, D_2)}{2\sqrt{F(\varepsilon_1, D_2)}}\varepsilon_{1,\zeta} = \frac{1}{2}F'(\varepsilon_1, D_2) \tag{4.7}$$

then

$$\tan\beta = \frac{dy}{d\varepsilon_1} = \frac{d}{d\varepsilon_1}\sqrt{F(\varepsilon_1, D_2)} = \frac{\frac{1}{2}F'(\varepsilon_1, D_2)}{\sqrt{F(\varepsilon_1, D_2)}} = \frac{F'}{2y} \tag{4.8}$$

where derivatives of $F(\varepsilon_1, D_2)$ with respect to ε_1 are denoted by prime. This system shows immediately that equilibria in the phase plane satisfy $y = 0$, $F'(\varepsilon_1, D_2) = 0$.

This indicates a specific character of a nonlinear system, which have one or several equilibrium positions, and depends on the function $F(\varepsilon_1, D_2)$ (Dai, 2001).

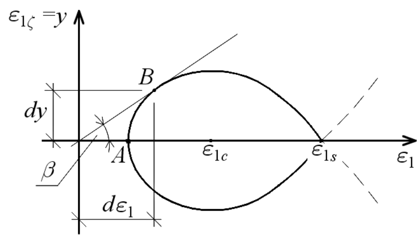

Fig. 5. The slope of a straight line tangent to the phase trajectory at the phase plane $(\varepsilon_{1,\zeta}, \varepsilon_1)$ at the point B, (ε_{1c} – denotes the center point, ε_{1s} – saddle point)

Equation (4.8) describes a straight line tangent to the trajectory in function of the phase coordinates (ε_1, y). The phase points are called ordinary or regular points if the tangent is determinated, however if the tangent is indeterminate, i.e.

$$\frac{dy}{d\varepsilon_1} = \frac{y_\zeta}{\varepsilon_{1,\zeta}} \to 0 \tag{4.9}$$

the points are called singular points or equilibrium points.

Equilibria are solutions to the simultaneous system

$$\begin{aligned} y = \varepsilon_{1,\zeta} = 0 \quad &\Rightarrow \quad F(\varepsilon_1, D_2) = 0 \\ y_\zeta = 0 \quad &\Rightarrow \quad F'(\varepsilon_1, D_2) = 0 \end{aligned} \tag{4.10}$$

Discharging necessary equilibrium condition (4.10) after substituting (4.5), we have

$$\begin{aligned} &(1+\varepsilon_1)^6[2D_1\varepsilon_1 + 2D_2 + \varepsilon_1^2(1+b_1)] - (b_1+b_2)[(1+\varepsilon_1)^8 + (1+\varepsilon_1)^4] = 0 \\ &6(1+\varepsilon_1)^5[2D_1\varepsilon_1 + 2D_2 + \varepsilon_1^2(1+b_1)] + 2(1+\varepsilon_1)^6[D_1 + \varepsilon_1(1+b_1)] + \\ &\qquad -4(b_1+b_2)[2(1+\varepsilon_1)^7 + (1+\varepsilon_1)^3] = 0 \end{aligned} \tag{4.11}$$

Eliminating D_2 and simplifying, we obtain a polynomial equation

$$\Big\{(1+\varepsilon_1)^5[D_1+\varepsilon_1(1+b_1)]+(b_1+b_2)[(1+\varepsilon_1)^2-(1+\varepsilon_1)^6]\Big\}(1+\varepsilon_1)^2=0 \tag{4.12}$$

The character of each equilibrium can be found by linearization of (4.6) and (4.7). If $\varepsilon_1=\varepsilon_{1e}$, $y=0$ is the solution to (4.12), and according with (4.10) we have

$$F(\varepsilon_{1e},D_2)=F'(\varepsilon_{1e},D_2)=0 \tag{4.13}$$

then close to the equilibrium point

$$y=Y \qquad\qquad \varepsilon=\varepsilon_{1e}+\Lambda \tag{4.14}$$

where Y and Λ are small perturbations.

Substituting (4.14) into (4.6), we have

$$Y=(\varepsilon_{1e}+\Lambda)_\zeta \tag{4.15}$$

which entails that

$$\Lambda_\zeta=Y \tag{4.16}$$

Similarly, substituting (4.13) into (4.7), we obtain

$$\begin{aligned}Y_\zeta&=\frac{1}{2}F'[(\varepsilon_{1e}+\Lambda),D_2]=\\&=\frac{1}{2}F'(\varepsilon_{1e},D_2)+\frac{1}{2}F''(\varepsilon_{1e},D_2)(\varepsilon_{1e}+\Lambda-\varepsilon_{1e})\end{aligned} \tag{4.17}$$

According to (4.13) $\frac{1}{2}F'(\varepsilon_{1e},D_2)=0$, then

$$Y_\zeta=\frac{1}{2}F''(\varepsilon_{1e},D_2)\Lambda \tag{4.18}$$

where D_2 is a parameter value representing the equilibrium point.

It follows from the analysis described by Dai (2001) and Osiński (1980) that if $F''(\varepsilon_{1e},D_2)<0$ the singular point (equilibrium point) at the phase plane is a center. Such a point sets a stable state of equilibrium. However, if $F''(\varepsilon_{1e},D_2)>0$, the singular point is a saddle and the state of equilibrium is unstable. In the degenerate case in which $F''(\varepsilon_{1e},D_2)=0$, we obtain a cusp point (see Fig. 6).

We obtain the curve in the phase plane directly by taking square roots of $F(\varepsilon_1,D_2)$.

The foregoing discussion indicates connection between the location and the nature of equilibria as well as the form of graphs of $F(\varepsilon_1,D_2)$. The real curves in the phase plane are described by equation (4.5) $y=\pm\sqrt{F(\varepsilon_1,D_2)}$.

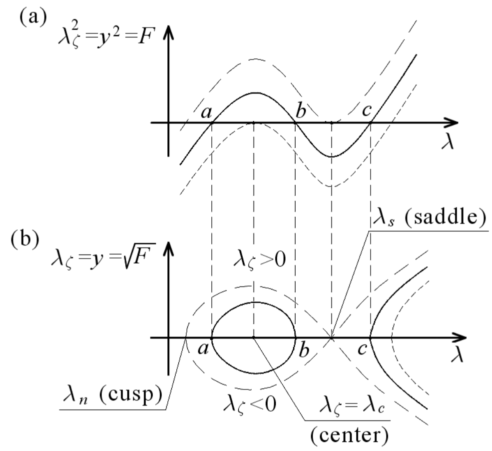

Fig. 6. Graphs of functions F (a) and phase trajectory y in the phase plane (b)

5. Discussion about physically acceptable solutions

The phase portrait method allows one to find solutions to differential system (4.4). However, not all curves in the phase plane are interesting for the physical problem at hand. Our main task consists in characterizing such portraits, whose values of b_1, b_2 and D_1, represent physically meaningful behaviour.

With some approximation we can assume that in the case of compression or tension of a thin rubber layer the physically acceptable value ε_1 is in the interval from -0.5 to 0.5.

According to Theorem 1 from the paper by Dai (2001, p. 104), in order that there be a physically acceptable solution we must obtain for the function $F(\varepsilon_1, D_2)$ a center point in the region of physically acceptable value ε_1. Supposing that this point exists for $\varepsilon_1 = \varepsilon_{1c}$ (then $F(\varepsilon_{1c}, D_2) = 0$ and $F'(\varepsilon_{1c}, D_2) = 0$), we can find D_1 and D_2 as functions of ε_{1c}, which determines this center

$$\begin{aligned} D_1 &= \frac{(b_1+b_2)[(1+\varepsilon_{1c})^6 - (1+\varepsilon_{1c})^2]}{(1+\varepsilon_{1c})^5} - \varepsilon_{1c}(1+b_1) \\ D_2 &= \frac{\varepsilon_{1c}(b_1+b_2)[(1+\varepsilon_{1c})^2 - (1+\varepsilon_{1c})^6]}{(1+\varepsilon_{1c})^5} + \\ &\quad + \frac{(b_1+b_2)[(1+\varepsilon_{1c})^6 + (1+\varepsilon_{1c})^2]}{2(1+\varepsilon_{1c})^4} + \frac{1}{2}\varepsilon_{1c}^2(1+b_1) \end{aligned} \tag{5.1}$$

After substituting (5.1) into (4.12), we obtain

$$
\begin{aligned}
&(\varepsilon_1-\varepsilon_{1c})(1+\varepsilon_1)^2\Big\{(1+\varepsilon_1)^5(1+b_1)-\frac{(b_1+b_2)(1+\varepsilon_1)^2}{(1+\varepsilon_{1c})^3}\cdot\\
&\quad\cdot[(1+\varepsilon_1)^3(1+\varepsilon_{1c})^3+3(1+\varepsilon_1+\varepsilon_{1c})+\varepsilon_1^2+\varepsilon_1\varepsilon_{1c}+\varepsilon_{1c}^2]\Big\}=0
\end{aligned}
\tag{5.2}
$$

Other equilibrium points are then given by the roots of

$$
\begin{aligned}
&(1+\varepsilon_1)^5(1+b_1)-\frac{(b_1+b_2)(1+\varepsilon_1)^2}{(1+\varepsilon_{1c})^3}\cdot\\
&\quad\cdot[(1+\varepsilon_1)^3(1+\varepsilon_{1c})^3+3(1+\varepsilon_1+\varepsilon_{1c})+\varepsilon_1^2+\varepsilon_1\varepsilon_{1c}+\varepsilon_{1c}^2]=0
\end{aligned}
\tag{5.3}
$$

Finally, by computing $F''(\varepsilon_{1c},D_2)$

$$
\begin{aligned}
F''(\varepsilon_{1c},D_2)=&\frac{1}{(1-b_2)(1+\varepsilon_{1c})^2}\Big\{12(1+\varepsilon_{1c})^4[5D_1\varepsilon_{1c}+2D_1(1+\varepsilon_{1c})+5D_2]+\\
&+2(1+b_1)(1+\varepsilon_{1c})^4[15\varepsilon_{1c}^2+12\varepsilon_{1c}(1+\varepsilon_{1c})+(1+\varepsilon_{1c})^2]+\\
&-4(b_1+b_2)[14(1+\varepsilon_{1c})^6+3(1+\varepsilon_{1c})^2]\Big\}
\end{aligned}
\tag{5.4}
$$

and substituting (5.1) into (5.4), we find that ε_{1c} will be a center if

$$
\frac{1}{2}F''(\varepsilon_{1c},D_2)=\frac{(1+b_1)(1+\varepsilon_{1c})^6-(b_1+b_2)[3(1+\varepsilon_{1c})^2+(1+\varepsilon_{1c})^6]}{(1-b_2)(1+\varepsilon_{1c})^2}<0
\tag{5.5}
$$

In order to obtain physically acceptable solutions, we must have $\nu>2C_1$. If $\nu<2C_1$, equation (5.5) is not satisfied.

It results from the paper by Dai (2001), that there is a second point except for the point of stable state of equilibrium. It is a point of unstable state of equilibrium – the saddle point.

Since $\nu>2C_1$, we see from (4.1) that $b_1>0$ and $b_2>0$. The difference in the signs of marks of terms in expression (5.3) suggests that there is a positive root, which we assume to be equal $\varepsilon_1=\varepsilon_{1s}$.

Equation (5.2) take the form

$$
\begin{aligned}
&\frac{(\varepsilon_1-\varepsilon_{1c})(\varepsilon_1-\varepsilon_{1s})(1+\varepsilon_1)^2}{(1+\varepsilon_{1c})^2}\frac{(1-b_2)(1+\varepsilon_1)^2(1+\varepsilon_{1c})^2}{3(1+A_1)+\varepsilon_{1s}^2+\varepsilon_{1s}\varepsilon_{1c}+\varepsilon_{1c}^2}\cdot\\
&\quad\cdot\Big[8E+3E^2+3(\varepsilon_1^2A_1+\varepsilon_{1c}^2A_2+\varepsilon_{1s}^2A)+6(1+F)+AF+\\
&\quad+\varepsilon_{1s}^2B+\varepsilon_1^2\varepsilon_{1c}^2+3(\varepsilon_1\varepsilon_{1c}+\varepsilon_1\varepsilon_{1s}+\varepsilon_{1c}\varepsilon_{1s})\Big]=0
\end{aligned}
\tag{5.6}
$$

where we have eliminated b_1 using the fact that $\varepsilon_1 = \varepsilon_{1s}$ is a root

$$b_1 = \frac{(1-b_2)(1+\varepsilon_{1s})^3(1+\varepsilon_{1c})^3}{3(1+\varepsilon_{1s}+\varepsilon_{1c})+\varepsilon_{1s}^2+\varepsilon_{1s}\varepsilon_{1c}+\varepsilon_{1c}^2} - b_2 \tag{5.7}$$

and we used the following variables

$$\begin{aligned} A &= \varepsilon_1 + \varepsilon_{1c} & A_1 &= \varepsilon_{1s} + \varepsilon_{1c} \\ A_2 &= \varepsilon_{1s} + \varepsilon_1 & B &= \varepsilon_1^2 + \varepsilon_1\varepsilon_{1c} + \varepsilon_{1c}^2 \\ E &= \varepsilon_1 + \varepsilon_{1s} + \varepsilon_{1c} & F &= \varepsilon_1\varepsilon_{1s}\varepsilon_{1c} \end{aligned} \tag{5.8}$$

Substituting b_1 (see (5.7)) into (5.5), we obtain the following expressions for ε_{1c} and ε_{1s}, respectively

$$\begin{aligned} \frac{1}{2}F''(\varepsilon_{1c}, D_2) &= \frac{\varepsilon_{1c}-\varepsilon_{1s}}{(1-b_2)(1+\varepsilon_{1c})^2} \cdot \\ &\cdot \frac{(1-b_2)(1+\varepsilon_{1c})^5(6+4\varepsilon_{1c}+\varepsilon_{1c}^2+2\varepsilon_{1c}\varepsilon_{1s}+8\varepsilon_{1s}+3\varepsilon_{1s}^2)}{3(1+\varepsilon_{1s}+\varepsilon_{1c})+\varepsilon_{1s}^2+\varepsilon_{1s}\varepsilon_{1c}+\varepsilon_{1c}^2} \\ \frac{1}{2}F''(\varepsilon_{1s}, D_2) &= \frac{\varepsilon_{1s}-\varepsilon_{1c}}{(1-b_2)(1+\varepsilon_{1s})^2} \cdot \\ &\cdot \frac{(1-b_2)(1+\varepsilon_{1s})^5(6+4\varepsilon_{1s}+\varepsilon_{1s}^2+2\varepsilon_{1s}\varepsilon_{1c}+8\varepsilon_{1c}+3\varepsilon_{1c}^2)}{3(1+\varepsilon_{1c}+\varepsilon_{1s})+\varepsilon_{1c}^2+\varepsilon_{1c}\varepsilon_{1s}+\varepsilon_{1s}^2} \end{aligned} \tag{5.9}$$

According to conclusions featured at condition (4.18), we can see that if

$$\frac{1}{2}F''(\varepsilon_{1c}, D_2) < 0 \qquad \text{or} \qquad \frac{1}{2}F''(\varepsilon_{1s}, D_2) > 0 \tag{5.10}$$

we obtain a center point or saddle point in the phase plane, respectively.

6. Numerical analysis

The numerical analysis is carried out for the function $F(\varepsilon_1, D_2)$ based on equation (4.4) obtained for the Mooney-Rivlin material

$$\begin{aligned} &F(\varepsilon_1, D_2) = \\ &= \frac{(1+\varepsilon_1)^6[2D_1\varepsilon_1+2D_2+\varepsilon_1^2(1+b_1)] - (b_1+b_2)[(1+\varepsilon_1)^8+(1+\varepsilon_1)^4]}{(1-b_2)(1+\varepsilon_1)^2} \end{aligned} \tag{6.1}$$

The constant D_1 depends on ε_1, and according to (4.12) we have

$$D_1(\varepsilon_1) = \frac{(b_1+b_2)[(1+\varepsilon_1)^6 - (1+\varepsilon_1)^2]}{(1+\varepsilon_1)^5} - \varepsilon_1(1+b_1) \tag{6.2}$$

analogously, the constant D_2 (which depends on ε_1 too), according to $(4.10)_1$ is equal

$$\begin{aligned} D_2(\varepsilon_1) = {}& \frac{\varepsilon_1(b_1+b_2)[(1+\varepsilon_1)^2 - (1+\varepsilon_1)^6]}{(1+\varepsilon_1)^5} + \\ & + \frac{(b_1+b_2)[(1+\varepsilon_1)^6 + (1+\varepsilon_1)^2]}{2(1+\varepsilon_1)^4} + \frac{1}{2}\varepsilon_1^2(1+b_1) \end{aligned} \tag{6.3}$$

Then, for a chosen value of ε_1 (in this paper $\varepsilon_1 = 0.5$) we can determine the constants D_1 and D_2 from (6.1) and (6.2), respectively.

In the analysis, we assumed the rubber density $\rho = 1190\,\mathrm{kg/m^3}$ and the shear modulus $\mu = 1.432 \cdot 10^5\,\mathrm{N/m^2}$. The constants C_1 and C_2 are characteristic for a kind of rubber described by Zahorski (1962) and take the values

$$C_1 = 4.299 \cdot 10^4 \frac{\mathrm{N}}{\mathrm{m^2}} \qquad\qquad C_2 = 0.604 \cdot 10^4 \frac{\mathrm{N}}{\mathrm{m^2}} \tag{6.4}$$

the constants b_1 and b_2 are calculated according to $(4.1)_{1,2}$.

In Fig. 7, there are four graphs of the functions $F_i(\varepsilon_1) \equiv F_i(\varepsilon_1, D_2)$, $i = 1,2,3,4$ for constant D_2 calculated according to (6.3) and for $\varepsilon_1 = 0.5$. The functions $y(\varepsilon_1)$ denote respectively

$$y_i(\varepsilon_1) \equiv \sqrt{F_i(\varepsilon_1)} \qquad\qquad y_{ia}(\varepsilon_1) \equiv -\sqrt{F_i(\varepsilon_1)} \qquad \text{for} \quad i = 1,2,3,4$$

Figure 7b shows phase trajectories in the coordinate system $(\varepsilon_{1,\zeta} = y, \varepsilon_1)$ for the functions $F_i(\varepsilon_1)$, $i = 1,2,3,4$, found from Fig. 7a for the Mooney-Rivlin material.

The constants D_1 and D_2 for $\varepsilon_1 = 0.5$ are calculated according to (6.2) and (6.3). In Fig. 7, the constant D_1 is -0.207 and the constant D_2 calculated from (6.3) is 0.452. The constants $D_2 = 0.46$, $D_2 = 0.44$ and $D_2 = 0.37$ have been established arbitrarily, but here it is fixed at $D_2 = 0.452$.

The center point is obtained for $\varepsilon_1 \cong 0.062$, and the graph contains physically acceptable solutions in the interval $\varepsilon_1 = \langle -0.435; 0.5\rangle$ (see Section 5).

We find that propagation of the traveling wave in a thin layer is possible for compression and tension. The solution has a periodic character for closed curves in the area limited by the solid line shown in Fig. 7b, and can be a solitary wave for solutions represented by a homoclinic orbit (see the solid line in Fig. 7b).

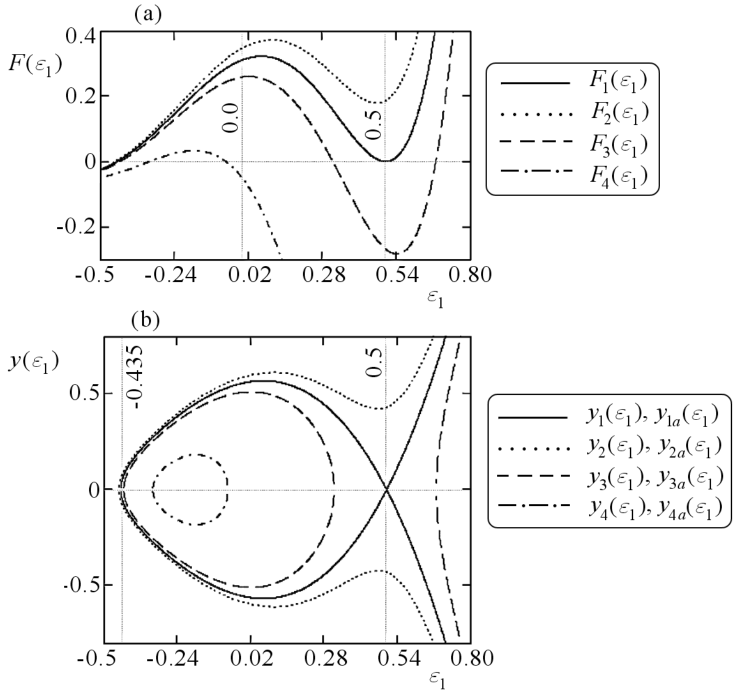

Fig. 7. Graphs for the rubber OKA-1 made of the Mooney-Rivlin material ($\mu = 1.46\,\mathrm{kG/cm^2}$, $\rho = 1190\,\mathrm{kg/m^3}$) for the speed $V = 20.5\,\mathrm{m/s}$ and constants $b_1 = 0.335$, $b_2 = 0.047$ and $D_1 = -0.207$ (according to (6.2) for $\varepsilon_1 = 0.5$); (a) distribution of functions: for $F_1(\varepsilon_1)$ the constant D_2 is 0.452 (according to (6.3) for $\varepsilon_1 = 0.5$), for $F_2(\varepsilon_1) - D_2 = 0.46$, for $F_3(\varepsilon_1) - D_2 = 0.44$ and for $F_4(\varepsilon_1) - D_2 = 0.37$, respectively, (b) phase trajectory

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Nieliniowe fale biegnące w cienkiej warstwie wykonanej z materiału Mooneya-Rivlina

Streszczenie

Referat dotyczy propagacji nieliniowej fali biegnącej w cienkiej sprężystej warstwie wykonanej z materiału Mooneya-Rivlina. Dla przybliżonego rozwiązania zagadnienia propagacji fali biegnącej w warstwie hipersprężystej zastosowano metodę polegającą na uśrednieniu równań ruchu w przekroju poprzecznym warstwy przy założeniu, że uśrednione wielkości spełniają równania ruchu i warunki brzegowe. Otrzymane w ten sposób równania zastosowano do opisu procesów falowych dla rozpatrywanych w pracy fal biegnących. Do analizy procesów falowych użyta została technika płaszczyzny fazowej. W oparciu o metodę trajektorii fazowej zinterpretowano warunki propagacji nieliniowej fali oraz ustalono warunki istnienia fizycznie akceptowalnych rozwiązań.

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