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RESPONSE OF BEAM ON VISCO-ELASTIC FOUNDATION TO MOVING DISTRIBUTED LOAD

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The paper is devoted to the study of several cases of stationary dynamical problems in which motion is driven by a distributed load acting on a beam on an elastic foundation at a moving position. The velocity of motion is assumed constant. In particular, cases of a load described by the Heaviside function (or its linear superposition) and a harmonic function are studied. Some problems examined by the authors in their previous investigations are reviewed.

Key words: dynamics, travelling load, wave propagation

1. Introduction

The development of various kinds of modern technology, like explosive bonding of layered materials or tracked high-speed transportation systems, becomes more and more important. This makes a strong need for simplified but reliable models of continuous or hybrid systems in order to study various dynamical effects which influence durability of structures, damage of the environment or comfort of transportation. The first study of beams on the Winkler foundation subjected to a simple concentrated force moving with a constant speed was initiated by Timoshenko (1926). The first stationary solution to a simple stationary case of the Bernoulli-Euler beam on an elastic foundation was properly obtained by Ludwig (1938). The case of a moving and oscillating force was formulated and partly solved by Mathews (1958). The first proper solution to the Mathews problem was given by Bogacz and Krzyżyński (1986). There are various extensions of this classic problem towards more complicated but also more realistic models of structures and loads. A great deal of new effects were recognized by Bogacz *et al.* (1998) who examined the problem of an oscillating load moving along a periodic (variable in space) structure. The dynamical effects for two or three-dimensional problems with moving loads have important practical engineering applications (Bogacz and Frischmuth, 2008). Some problems connected with a system of plates subjected to a traveling load can be found in Bogacz (2008), Bogacz and Frischmuth (2008). An application of the beam model to the railway track mechanics is connected with taking into account the axial force into the model (Kerr, 1972).

The aim of this paper is devoted to systematization and explanation of some new effects related to the moving distributed and oscillating load.

2. Beam on the Winkler foundation subjected to a uniformly distributed load acting on a segment

The problem of vibration of a flexibly supported beam with the stiffness EI, linear mass density mA, damping coefficient h and Winkler coefficient c, subjected to the distributed load moving with the velocity V_0, can be composed of the solution obtained for the limiting case of a load described by the following Heaviside function $F_0H(x-V_0t)$

$$EIw_{,xxxx} + Tw_{,xx} + mAw_{,tt} + hw_{,t} + cw = F_0H(x-V_0t) \tag{2.1}$$

A similar case of the beam (without the compression force, $T=0$) was studied by Bogacz and Rozenbajgier (1979). The beam on an elastic foundation was generalized there to the case of a beam on a visco-elastic semi-space. The boundary conditions, equivalent to the condition of radiation, in the visco-elastic case take the following form

$$\lim w(x) = \begin{cases} 0 & \text{for} \quad x \to \infty \\ \dfrac{c}{F_0} & \text{for} \quad x \to -\infty \end{cases} \tag{2.2}$$

and in the moving coordinate system

$$X = x - V_0t \tag{2.3}$$

where the displacement $w(X)$ as well as its derivatives $w_{,X}$, $w_{,XX}$, and $w_{,XXX}$ are continuous at $X=0$.

The equation of beam motion in the moving system of coordinates (2.3) takes the form

$$
\begin{aligned}
&EIw_{,XXXX} + Tw_{,XX} + mA(w_{,tt} - 2V_0 w_{,Xt} + V_0^2 w_{,XX}) + \\
&+h(w_{,t} - V_0 w_{,X}) + cw = F_0 H(X)
\end{aligned}
\tag{2.4}
$$

In the stationary case, a characteristic equation of Eq. (2.4) takes the following form

$$
R^4 + 4(Vq)^2 R^2 - 8Vbq^3 R + 4q^4 = 0 \tag{2.5}
$$

where

$$
\begin{aligned}
V &= \frac{V_0}{V_{cr}} & V_{cr} &= \sqrt{\sqrt{\frac{4cEI}{mA}} - \frac{T}{mA}} \\
q &= \sqrt{\frac{c}{4EI}} & b &= \frac{h}{2\sqrt{cmA}}
\end{aligned}
$$

Roots of Eq. (2.5) are

$$
\begin{aligned}
R_1 &= S_1 + \mathrm{i}D_1 & R_2 &= S_1 - \mathrm{i}D_1 & R_3 &= S_2 + \mathrm{i}D_2 \\
R_4 &= S_2 - \mathrm{i}D_2 & S_1 &= -S_2
\end{aligned}
\tag{2.6}
$$

Using boundary conditions (2.2), the continuity conditions at $X = 0$, one can obtain the following kind of solution before and behind the front of the load:

— for $X < 0$

$$
\begin{aligned}
W_1(X) = \frac{F_0}{c} &+ \exp(nX)\Big\{A_1 \sin\Big[\Big(2V^2 + n^2 - \frac{2Vh}{n}\Big)X\Big] + \\
&+A_2 \cos\Big[\Big(2V^2 + n^2 - \frac{2Vh}{n}\Big)X\Big]\Big\}
\end{aligned}
\tag{2.7}
$$

— for $X > 0$

$$
\begin{aligned}
W_2(X) &= \exp(-nX)\Big\{A_3 \sin\Big[\Big(2V^2 + n^2 + \frac{2Vh}{n}\Big)X\Big] + \\
&+A_4 \cos\Big[\Big(2V^2 + n^2 + \frac{2Vh}{n}\Big)X\Big]\Big\}
\end{aligned}
\tag{2.8}
$$

where n is the positive root of the equation

$$
n^6 + 2V^2 n^4 + (V^4 - 1)n^2 - V^2 b^2 = 0 \tag{2.9}
$$

and

$$\begin{aligned}
A_1 &= -\frac{F_0}{2Kc}\,\frac{n}{2V^2+n^2-\frac{2Vh}{n}}\Big[2V^2(V^2+n^2)-3\Big(\frac{Vh}{n}\Big)^2-\frac{Vh}{n}(V^2+3n^2)\Big]\\
A_2 &= -\frac{F_0}{2Kc}\Big[2n^2(V^2+n^2)+\Big(\frac{Vh}{n}\Big)^2+\frac{Vh}{n}(V^2+3n^2)\Big]\\
A_3 &= -\frac{F_0}{2Kc}\,\frac{n}{2V^2+n^2+\frac{2Vh}{n}}\Big[2V^2(V^2+n^2)-3\Big(\frac{Vh}{n}\Big)^2+\frac{Vh}{n}(V^2+3n^2)\Big]\\
A_4 &= -\frac{F_0}{2Kc}\Big[-2n^2(V^2+n^2)-\Big(\frac{Vh}{n}\Big)^2+\frac{Vh}{n}(V^2+3n^2)\Big]\\
K &= 2n^2(V^2+n^2)-\Big(\frac{Vh}{n}\Big)^2
\end{aligned} \tag{2.10}$$

The solution for the purely elastic case can be obtained from Eqs. (2.7) and (2.8) for $h \to 0$. The solution has a different form for the sub-critical and super-critical case. The solution for the sub-critical case $(V < 1)$ behind the front of the load $W_1(X)$ and before the front $W_2(X)$ is described by following formulas

$$\begin{aligned}
W_1(X) &= \frac{F_0}{2c}\Big[2-\exp(\sqrt{1-V^2}X)\Big]\Big[\frac{V^2}{\sqrt{1-V^4}}\sin(\sqrt{1+V^2}X)+\cos(\sqrt{1+V^2}X)\Big]\\
W_2(X) &= -\frac{F_0}{2c}\exp(\sqrt{1-V^2}X)\Big[\frac{V^2}{\sqrt{1-V^4}}\sin(\sqrt{1+V^2}X)-\cos(\sqrt{1+V^2}X)\Big]
\end{aligned} \tag{2.11}$$

For the super-critical case $(V > 1)$ the displacements are as follows

$$\begin{aligned}
W_1(X) &= \frac{F_0}{2c}\Big[2-\Big(1+\frac{V^2}{\sqrt{V^4-1}}\Big)\cos[(\sqrt{1+V^2}-\sqrt{V^2-1})X]\Big]\\
W_2(X) &= -\frac{F_0}{2c}\Big[\Big(-1+\frac{V^2}{\sqrt{V^4-1}}\Big)\cos[(\sqrt{1+V^2}+\sqrt{V^2-1})X]\Big]
\end{aligned} \tag{2.12}$$

It is visible that in the stationary elastic case, for a super-critical velocity of load motion $V > V_{cr}$, the waves before and behind the front of the load do not decay for $|X| \to \infty$. Shorter waves with the phase velocity smaller than the group velocity propagate before the front of the load, and longer waves with the phase speed higher than the group velocity propagate behind the front of the load. The displacements in the sub-critical case are shown in Fig. 2.

In the linear case, superposition of the obtained solution for the Heviside function allows one to obtain various kinds of piece-wise constant loads distributed on a finite-length segment. For example, if we describe a load with a

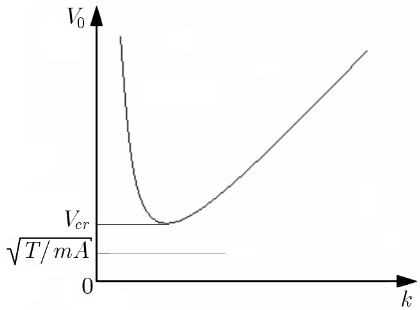


Fig. 1. Wave velocity V_0 versus wave number k for the Bernoulli-Euler beam on the Winkler foundation subjected to longitudinal force T

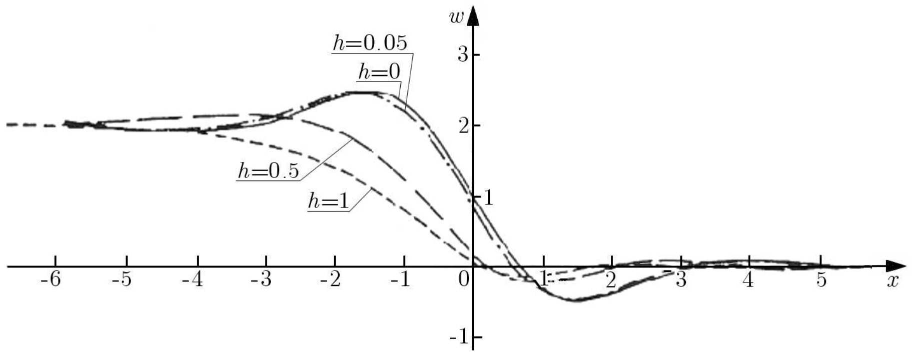


Fig. 2. Displacements of the Bernoulli-Euler beam on the Winkler foundation in the sub-critical case $(V = 0.8)$ for various damping coefficients

given value F_1 distributed between $x = 0$ and $x = L$ at $t = 0$, it is then possible to write the load as follows

$$F(x,t) = F_1[H(x - V_0t) - H(x - L - V_0t)] \tag{2.13}$$

In such a case, the solution must fulfill conditions (2.2) and, additionally, the continuity of displacements and derivatives $w_{,X}$, $w_{,XX}$, and $w_{,XXX}$ at $X = 0$ and $X = L$.

Let us now consider a more complicated model of the beam on an elastic foundation which takes into account shear deformation and rotary inertia of the cross-section – the Timoshenko beam. The case of the Timoshenko beam on an elastic foundation subjected to uniformly distributed moving loads has been studied by several authors (Fryba, 1972; Bogacz *et al.*, 1989). The equation of motion of the Timoshenko beam takes the following form

$$
\begin{aligned}
&EI\varphi_{,xx} + k'AG(w_{,,x} - \varphi) - mAI\varphi_{,tt} = 0 \\
&k'AG(w_{,xx} - \varphi_{,x}) - mAw_{,tt} - hw_{,t} - cw = -F_0 H(x - V_0 t)
\end{aligned}
\tag{2.14}
$$

where φ is the beam rotation due to pure shear, k' – shear coefficient, G – modulus of shear elasticity, A – cross-sectional area, and h – damping coefficient.

The first stationary solution obtained for the case of the Timoshenko beam on an elastic foundation was obtained by Achenbach and Sun (1965). The shape of displacement in this solution is shown in Fig. 3.

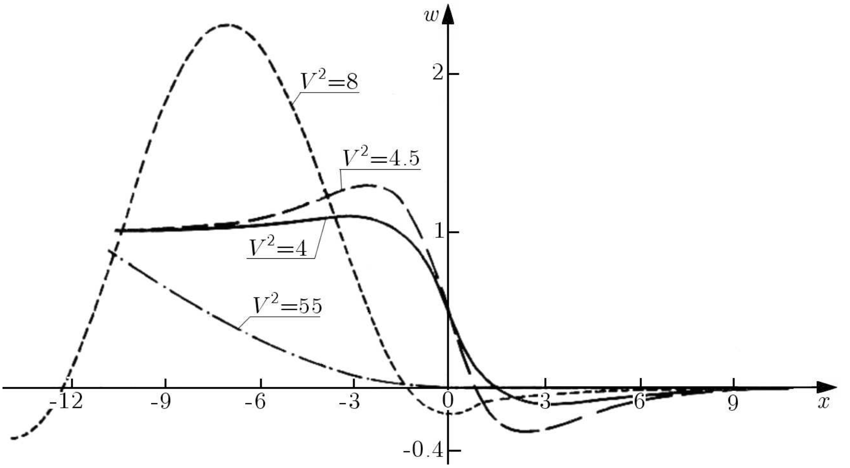


Fig. 3. Displacements of the Timoshenko beam on the Winkler foundation for various values of the load speed. The results are found for a parameter range similiar to that used by Achenbach and Sun (1965)

The solution obtained by Achenbach and Sun (1965) is qualitatively different from that shown in Fig. 4. Looking for the limiting stationary case, the set of equations (2.14) can be reduced to the following fourth-order equation with respect to displacement $W(X) = w(X)\sqrt{A/I}$, where I is the moment of inertia of the cross-section

$$
\begin{aligned}
&Q(V^2)W^{IV} + 2Vh(V^2 - V_E^2)W''' + [V^2(V_E^2 + 1) - V_G^2]W'' + \\
&+2VV_E^2 hW' + V_G^2 W = F[V_G^2 H(X) + (V^2 - V_E^2)H''(X)]
\end{aligned}
\tag{2.15}
$$

where

$$
F = F_0\Big(\frac{A}{Ic^2}\Big) \qquad Q(V^2) = (V^2 - V_G^2)(V^2 - V_E^2)
$$

$$
V_E^2 = \frac{EA^2}{Ic} \qquad V_G^2 = \frac{k'GA^2}{Ic}
$$

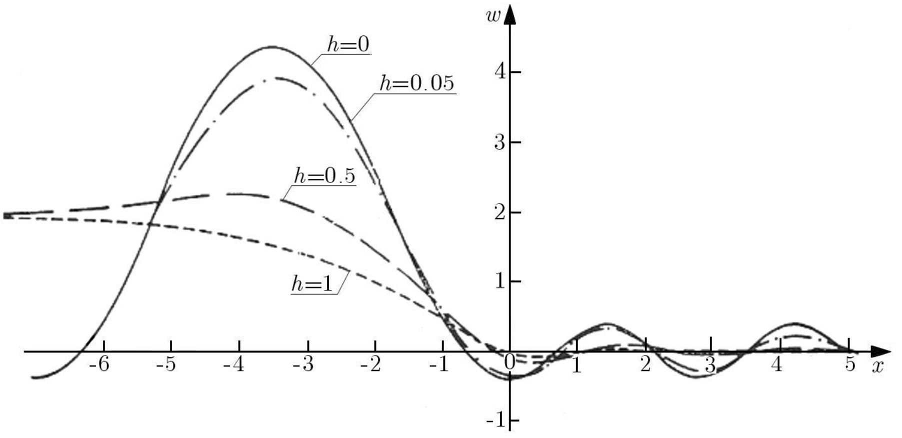


Fig. 4. Displacements of the Bernoulli-Euler beam on the Winkler foundation in the super-critical case $(V = 1.2)$ for various damping coefficients

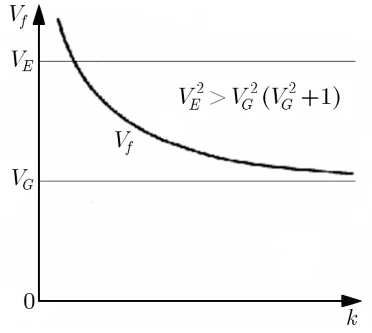
Making use of the above equation, we can determine the discontinuity values of the derivatives of $W(X)$ and rotation at the point $X = 0$. The solution to the problem consists in determination of displacements $W(X)$ and rotation that satisfy equations (2.15). We shall obtain them by applying the Fourier transformation to the equations of motion. To investigate the effect of the load speed on the qualitative character of the solutions for the elastic system, i.e. $h \to 0$, let us consider two sets of parameters.

Case I

$$V_E^2 > V_G^2(V_G^2 + 1) \tag{2.16}$$

In this case, there exist three main ranges of the load speed in which there are three corresponding different solutions like those obtained by Achenbach and Sun (1965) and shown in Fig. 3.

Within range No. 1, for $|V| < V_1$, the solutions tend to the asymptotes $W = 0$ and $W = 1$ in monotonous ways. Within range No. 2 for $V_1 < V < V_2$, the solution vanishes monotonously before the load front and oscillates periodically around the value $W = 1$ behind the load front. Within range No. 3 for $V > V_2$, the displacement and rotation before the load front are equal to zero, and behind the load the solution consists of superposition of two particular periodic solutions. This solution, unobservable in Fig. 3, can be seen in Fig. 6.



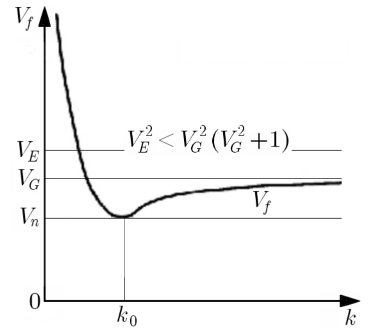


Fig. 5. Phase velocity V_f versus wave number k for two qualitatively different cases of the solution

Case II

$$V_E^2 < V_G^2(V_G^2 + 1) \tag{2.17}$$

In this case, which is illustrated in Fig. 6, we have four ranges of velocities with qualitatively different solutions. At the critical speed $V^2 = V_0^2$, the displacement and rotation increase infinitely. To the speed range $V^2 < V_0^2$ there corresponds a solution with properties being characteristic for range No. 1 in case I. This case is represented by the curve $V^2 = 3$ in Fig. 6. In the range of speed $V_0 < V < V_1$, the solution substantially changes in its quantitative feature. Namely, the solution consists then of two visible waves; the wave with a small amplitude and wavelength before the load front and a much greater amplitude and wavelength behind the front of the load. The solution in this case is represented by the curve $V^2 = 4.5$ (Fig. 6). Within the range $V_1 < V < V_2$, the periodic character of the wave behind the load front remains periodic but before it the displacement vanishes monotonously with distance from the load front. This case is represented by the solution for $V^2 = 6$. The solution for $V^2 > V_1$ has a similar feature as in case I. The solution shown in Fig. 6 for $V^2 = 15$ illustrates qualitative behaviour of the beam in this region.

The above case shows that there exists a set of parameters for the Timoshenko beam which can be taken qualitatively as the limiting case, i.e. transition to the Bernoulli- Euler beam. The change between case I and case II is connected with the change from the hyperbolic to parabolic type of the equation. This is the reason why the solution obtained by Achenbach and Sun (1965) is valid in the whole range of velocity, but only for the set of parameters fulfilling inequality (2.16).

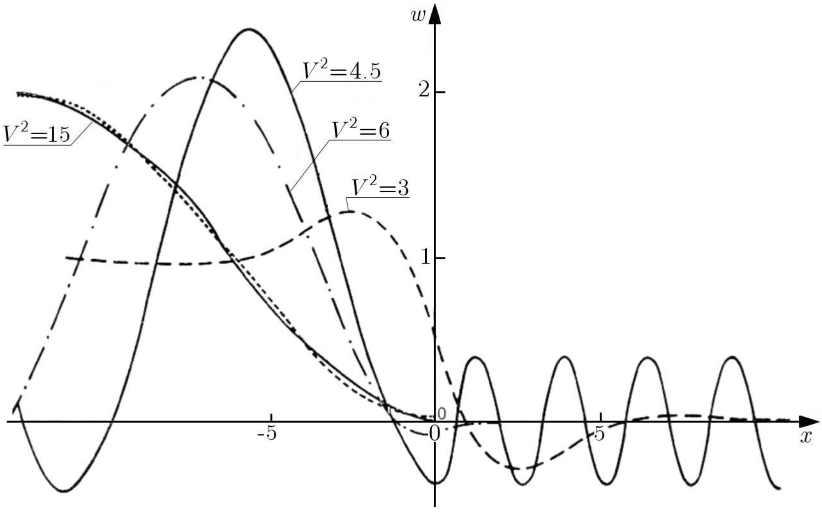


Fig. 6. Displacements of the Timoshenko beam on the Winkler foundation for various values of the load speed, in the case of parameters $V_E^2 < V_G^2(V_G^2 + 1)$ (which was not considered by Achenbach and Sun, 1965)

3. Beam on the Winkler foundation subjected to a harmonically distributed moving load

In the case when the load is described by a continuous and oscillating (harmonic) function, and is moving with a given velocity V_0, the beam equation is described as follows

$$EIw_{,xxxx} + Tw_{,xx} + m_0 w_{,tt} + hw_{,t} + cw = F_0 \sin[k(x - V_0 t)] \tag{3.1}$$

or in the moving system of coordinates

$$\begin{aligned} &EIw_{,XXXX} + Tw_{,XX} + m_0(w_{,tt} - 2V_0 w_{,Xt} + V_0^2 w_{,XX}) + \\ &+h(w_{,t} - V_0 w_{,X}) + cw = F_0 \sin(kX) \end{aligned} \tag{3.2}$$

In the case of Eq. (3.2) the solution has the following form

$$w(x,t) = W_s \sin[k(x - V_0 t)] + W_c \cos[k(x - V_0 t)] \tag{3.3}$$

while in the case of Eq. (3.2), we can write it as follows

$$w(X,t) = W_s \sin(kX) \tag{3.4}$$

In the elastic case $(h = 0)$, $W_c = 0$ and the dependence between W_s and F_0 takes the following form

$$f(V_0) = \frac{W_s}{F_0} = \frac{1}{EIk^4 - Tk^2 - m_0 V_0^2 k^2 + c} = \frac{1}{m_0 k^2 (R_0^2 - V_0^2)} \tag{3.5}$$

It is visible that $W_s \to \infty$ for given F_0, k and $V_0 = \sqrt{(EIk^2 - T + c/k^2)/m_0}$ greater than V_{cr}, which is described by the formula

$$V_{cr} = \sqrt{\frac{4cEI}{m_0} - \frac{T}{m_0}} \qquad\qquad R_0^2 = \frac{1}{m_0}\Big(EIk^2 - T + \frac{c}{k^2}\Big) \tag{3.6}$$

An example of the dependence $W_0/F_0 = f(V_0)$ is presented in Fig. 7a. We can see that in the case of Bernoulli-Euler beam for $|V_0| = R_0$, similarly as in the case of vibration resonance, the solution changes from being "in phase" to "out of phase". This solution is important for applications in railway engineering, when the track dynamics can be studied as the Bernoulli-Euler beam on an elastic or visco-elastic foundation subjected to a longitudinal force T. This force can have a destabilizing character in the case of increasing temperature that produces a compressing force in the rails.

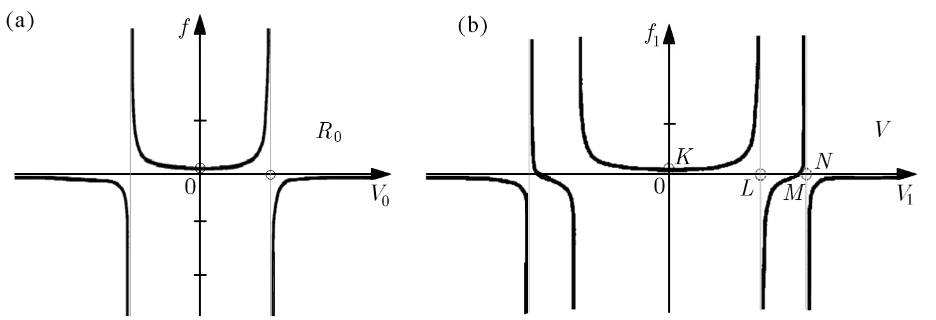


Fig. 7. (a) Function $W_0/F_0 = f(V_0)$ for the Bernoulli-Euler beam, and (b) for the Timoshenko beam

Let us consider a more complicated case – the Timoshenko beam on an elastic foundation subjected to a continuous harmonic load $F_0 \sin(kX)$ moving with a given speed V_0, $X = x - V_0 t$. The equation of motion is described now by a set of equations similar to (2.15)

$$\begin{aligned} &Q(V^2)W^{IV} + 2Vh(V^2 - V_E^2)W''' + [V^2(V_G^2 + 1) - V_E^2]W'' + \\ &+2VV_G^2 hW' + V_G^2 W = F[V_G^2 - (V^2 - V_E^2)k^2]\sin(kX) \end{aligned} \tag{3.7}$$

where

$$F = F_0\Big(\frac{A}{Ic^2}\Big) \qquad\qquad Q(V^2) = (V^2 - V_G^2)(V^2 - V_E^2)$$

Now the ratio $W/F = f_1(V)$ is given by the following equation

$$f_1(V) = \frac{V_G^2 - (V^2 - V_E^2)k^2}{k^4 Q(V^2) - k^2[V^2(V_G^2 + 1) - V_E^2]} + V_G^2 \tag{3.8}$$

where the velocities illustrated in Fig. 7b are as follows

$$\begin{aligned} V_L &= \overline{OL} = \sqrt{\frac{1}{2}\Big(\beta - \sqrt{\beta^2 - 4\gamma}\Big)} & \beta &= V_G^2 + V_E^2 + \frac{1}{k^2}(V_G^2 + 1) \\ \gamma &= V_G^2 V_E^2 + \frac{1}{k^2} V_E^2 V_G^2 + \frac{1}{k^4} V_G^2 & V_M &= \overline{OM} = \frac{1}{k}\sqrt{V_G^2 + k^2 V_E^2} \end{aligned} \tag{3.9}$$

As can be seen in Fig. 7b, there exist two values of the speed V_L and V_M for which some kind of resonance occurs. The critical speed $V_L = V_c$ and the related wave number k_0 as function of V_E are shown in Fig. 8.

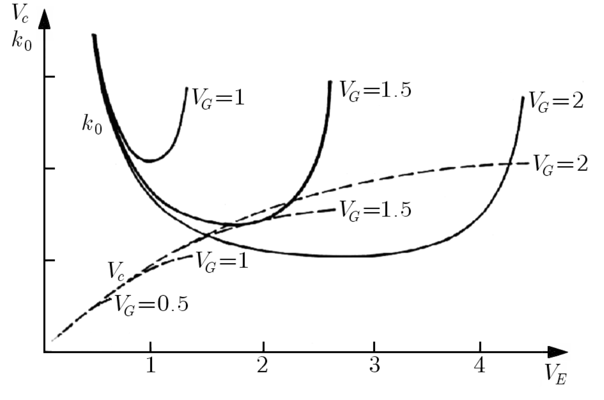


Fig. 8. Critical speed V_c and related wave number k_0 versus the value of longitudinal wave V_E for selected values of V_G

For these relationships, the denominator of (3.8) is equal to zero, which corresponds to the case when beam displacements tend to infinity. It is visible that for a given value of V_G, there exists a limiting value of V_E determined by inequality (2.17), where k_0 is bounded. For a value greater than V_E, the phase velocity tends to V_G for the wave number approaching infinity.

From the point of view of applications in railway engineering, it is particularly important to study the problem of response of a periodic structure to a distributed load. Generalization of the problem investigated by Jezequel (1981) and Mead (1986) will be studied in a separate paper.

4. Summary

Several cases of stationary dynamics of a continuous system are considered. The investigated problem seems to be important for applications in railway engineering. The considered one-dimensional continuous system is subjected to a distributed moving load. The load is described by the Heaviside function (or its linear superposition) and by a moving load harmonically distributed in space. The velocity of load motion is assumed to be constant. The results obtained in this paper will be a basis for generalization of the problem of the response of periodic structures to periodically distributed loads.

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Drgania belki na lepko-sprężystym podłożu pod wpływem ruchomego, rozłożonego obciążenia

Streszczenie

Niniejszy artykuł jest poświęcony badaniu wybranych przypadków stacjonarnych zagadnień dynamicznych, w których belka na sprężystym podłożu poddana jest ruchomemu obciążeniu. Rozważono zagadnienie stałej prędkości ruchu obciążenia opisanego w przestrzeni funkcją Heaviside'a (lub liniowej kombinacji tych funkcji) oraz obciążenia harmonicznie zmiennego. Niektóre z zagadnień badanych wcześniej zostały krytycznie omówione i uzupełnione.

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