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DYNAMIC STABILITY OF WEAK EQUATIONS OF RECTANGULAR PLATES

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The stability analysis method is developed for distributed dynamic problems with relaxed ssumptions imposed on solutions. The problem is motivated by structural vibrations with external time-dependent parametric excitations which are controlled using surfacemounted or embedded actuators and sensors. The strong form of equations involves irregularities which lead to computational difficulties for estimation and control problems. In order to avoid irregular terms resulting from differentiation of force and moment terms, dynamical equations are written in a weak form. The weak form of dynamical equations of linear mechanical structures is obtained using Hamilton's principle. The study of stability of a stochastic weak system is based on examining properties of the Liapunov functional along a weak solution. Solving the problem is not dependent on assumed boundary conditions.

Key words: weak formulation, dynamic stability, different boundary conditions, Liapunov method

1. Introduction

The strong form of plate equations involves irregularities which lead to computational difficulties for estimation and control problems. In order to avoid irregular terms resulting from differentiation of force and moment terms, dynamical equations are written in a weak form. The weak form of systems is useful for development of identification methods and general computational methods (Banks *et al.*, 1993). We consider dynamical systems with parametric excitations, e.g. plates with in-plane time dependent forces. The plate motion is described by partial differential equations that include time dependent coefficients implying parametric vibrations. The response of such systems can lead to a new increasing mode of oscillations and the structure dynamically buckles. The classical Liapunov technique for stability analysis of continuous elements is based on choosing or generating of a functional which is positive definite in the class of functions satisfying structure boundary conditions. The time-derivative of the Liapunov functional has to be negative in some defined sense. Almost sure stability of the beam equilibrium state in the strong formulation was examined by Kozin (1972). The technique of stability analysis was extended to plates and shells with in-plane or membrane time-dependent forces (Tylikowski, 1978). Uniform stochastic stability analysis of laminated beams and plates described by partial differential equations with time and space dependent variables was presented in Tylikowski and Hetnarski (1996). The weak form of a distributed controller in an active system consisting of electroded piezoelectric sensors/actuators with suitable polarization profiles (Tylikowski, 2005) is useful for the feedback theoretical developments.

For the purpose of active vibration and noise control, piezoelectric devices have shown great potential as elements of passive absorbers and active control systems as they are light-weight, inexpensive, small, and can be bonded to main structures. They can not be modeled as point force excitations, and partial differential equations should be used to describe the response of the structures driven by them. Active damping in composite structures with collocated sensors/actuators were studied by Tylikowski (2005), Tylikowski and Hetnarski (1996). Electrodes on sensors/actuators are spatially shaped to reduce the spillover between circumferential modes. In order to avoid irregular terms resulting from the modelling, the action of piezoelements as the Dirac delta function concentrated on their edges and dynamical equations are written in a weak form.

2. Weak formulation of plate dynamic equations

Consider an elastic rectangular plate of the length a, width b, thickness h, mass density ρ and bending stiffness D subjected to the in-plane time-dependent forces $F_X(t)$ and $F_Y(t)$ acting in the X and Y direction, respectively. Therefore, the onset of parametric vibrations is possible. In order to avoid developing modes of plate motion, viscous damping with the proportionality coefficient α is introduced. The strong form of plate dynamical equation in the transverse displacement w is given in the following form

$$\rho h w_{,TT} + \alpha w_{,T} + D\Delta^2 w + F_X(T)w_{,XX} + F_Y(T)w_{,YY} = 0 \tag{2.1}$$

where $(X,Y) \in \{0,a\} \times \{0,b\}$. Introducing dimensionless variables

$$x = \frac{X}{b} \qquad\qquad y = \frac{Y}{b} \qquad\qquad t = \frac{T}{k_t}$$

equation (2.1) becomes

$$w_{,tt} + 2\beta w_{,t} + \varDelta^2 w + [f_{ox} + f_x(t)]w_{,xx} + [f_{oy} + f_y(t)]w_{,yy} = 0 \tag{2.2}$$

where $(x,y) \in \{0,r\} \times \{0,1\}$, $r = a/b$ is the plate aspect ratio

$$k_t^2 = \frac{1}{D}\rho h b^4 \qquad\qquad \beta = \frac{\alpha k_t}{2\rho h}$$
$$f_{ox} + f_x(t) = \frac{1}{D}b^2 F_X(k_t t) \qquad\qquad f_{oy} + f_y(t) = \frac{1}{D}b^2 F_Y(k_t t)$$

and the solution should be fourth time partially differentiated with respect to x and y. If the plate is simply supported at the end, the transverse displacement and bending moment equal zero. For clamped edges, the transverse displacement and the slope equal zero.

Similarly, we can anlyse other combinations of simple boundary conditions, i.e. a simply supported – clamped plate. We write the action integral of the plate without damping in the form

$$\begin{aligned} A[w] = \frac{1}{2}\int\limits_{t_1}^{t_2}\int\limits_0^r\int\limits_0^1 [w_{,t}^2 - (w_{,xx} + w_{,yy})^2 - 2(1-\nu)(w_{,xy}^2 - w_{,xx}w_{,yy}) + \\ + f_{ox}w_{,x}^2 + f_{oy}w_{,y}^2]\;dx\,dy\,dt \end{aligned} \tag{2.3}$$

where ν is Poisson's ratio, and applying Hamilton's principle

$$\frac{d}{d\varepsilon}A[w + \varepsilon\varPhi]\Big|_{\varepsilon=0} = 0 \tag{2.4}$$

where ε is a real number. Adding the viscous damping and the time-dependent components of the in-plane forces as external works, the dynamical equation can be written in the weak form as follows for all $\varPhi$

$$\begin{aligned} \int\limits_0^r\int\limits_0^1 \Big\{(w_{,tt} + 2\beta w_{,t})\varPhi + (w_{,xx} + \nu w_{,yy})\varPhi_{,xx} + (w_{,yy} + \nu w_{,xx})\varPhi_{,yy} + \\ + 2(1-\nu)w_{,xy}\varPhi_{,xy} - [f_{ox} + f_x(t)]w_{,x}\varPhi_{,x} - [f_{oy} + f_y(t)]w_{,y}\varPhi_{,y}\Big\}\;dx\,dy = 0 \end{aligned} \tag{2.5}$$

where Φ is a sufficiently smooth test function satisfying essential boundary conditions. There is no demand on the existence of higher derivatives than the second order. As detailed in Banks *et al.* (1993), usual integration by parts the terms containing derivatives of the test function Φ with respect to the in-plane variables x and y and the assumption of sufficient smoothness of the plate displacement leads to strong formulation (2.2).

3. Almost sure stability definition

Dynamical equations (2.2) and (2.5) contain terms explicitely dependent on time. The time dependency of the axial forces $f_x(t)$ and $f_y(t)$ parametrically excites the plate and an increasing form of vibrations can occur. In deterministic parametric vibrations, the stability properties are determined from the Mathieu equation together with the corresponding Ince-Strutt diagram. If the excitation is narrow-banded or has one latent periodicity, a series of wedges on the amplitude-frequency plane can be expected, analogously to the deterministic parametric resonance. The task is much more complex when the stochastic excitation is wide-band and continuous systems with an infinite number of natural fequencies are analysed. Due to the fact that the norms and metrics in infinite-dimensional spaces are not equivalent, the stability holds for just the norm or the metric used in the analysis. If the parametric excitation becomes random, the stability criteria depend on the statistical characteristics of the excitation and the systems parameters. The present paper examines dynamic stability due to an action of in-plane forces in the form of stochastic physically realizable time-dependent processes with given statistical properties. Equation (2.5) with zero initial conditions posseses the trivial solution

$$w = w_{,t} = 0 \tag{3.1}$$

The trivial solution to Eq. (2.2) or (2.5) is almost sure stable if with probability 1 if the measure of distance between the perturbed solution with nonzero initial conditions and the trivial one tends to zero as time tends to infinity (Kozin, 1972). Usually, the measure of distance is defined by a positive-definite functional. The trivial solution is called almost sure asymptotically stable if

$$P\{\lim_{t\to\infty} \|w(\cdot,t)\| = 0\} = 1 \tag{3.2}$$

where $\|w(\cdot,t)\|$ is a measure of the disturbed solution w with nontrivial initial conditions from the equilibrium state, and P is the probability measure. The almost sure stability is equivalent to the clasical Liapunov stability in linear systems.

4. Stability analysis in weak formulation

In order to examine the almost sure stability of the plate equilibrium (the trivial solution), the Liapunov functional is chosen in the form

$$
\begin{aligned}
V = \frac{1}{2}\int_0^r\int_0^1 \Big[& w_{,t}^2 + 2\beta w w_{,t} + 2\beta^2 w^2 + (w_{,xx}+w_{,yy})^2 + \\
& +2(1-\nu)(w_{,xy}^2 - w_{,xx}w_{,yy}) - f_{ox}w_{,x}^2 - f_{oy}w_{,y}^2\Big]\,dx\,dy
\end{aligned}
\tag{4.1}
$$

The functional is positive-definite if the constant components of the in-plane forces f_{ox} and f_{oy} fullfil the static buckling condition, i.e. are sufficiently small. Therefore, the measure of disturbed solutions is chosen as a square root of the functional V

$$
\|w(\cdot,t)\| = \sqrt{V}
\tag{4.2}
$$

As trajectories of the solution to equations (2.5) are physically realizable, classical calculus is applied to find the time-derivative of functional (4.1). Its time-derivative is given by

$$
\begin{aligned}
\frac{dV}{dt} = \int_0^r\int_0^1 \Big[& (w_{,t}+\beta w)w_{,tt} + \beta w_{,t}^2 + 2\beta^2 w w_{,t} + \\
& +w_{,xx}w_{,xxt} + w_{,yy}w_{,yyt} + 2(1-\nu)w_{,xy}w_{,xyt} + \\
& +\nu(w_{,xx}w_{,yyt} + w_{,yy}w_{,xxt}) - f_{ox}w_{,x}w_{,xt} - f_{oy}w_{,y}w_{,yt}\Big]\,dx\,dy
\end{aligned}
\tag{4.3}
$$

Substituting βw and $w_{,t}$ as the test functions in Eq.(2.5), we have two identities, respectively

$$
\begin{aligned}
& \int_0^r\int_0^1 \Big[(w_{,tt}+2\beta w_{,t})\beta w + (w_{,xx}+\nu w_{,yy})\beta w_{,xx} + (w_{,yy}+\nu w_{,xx})\beta w_{,yy} + \\
& +2(1-\nu)\beta w_{,xy}^2 - (f_{ox}+f_x(t))\beta w_{,x}^2 - (f_{oy}+f_y(t))\beta w_{,y}^2\Big]\,dx\,dy = 0 \\
& \int_0^r\int_0^1 \Big[(w_{,tt}+2\beta w_{,t})w_{,t} + (w_{,xx}+\nu w_{,yy})w_{,xxt} + (w_{,yy}+\nu w_{,xx})w_{,yyt} + \\
& +2(1-\nu)w_{,xy}w_{,xyt} - (f_{ox}+f_x(t))w_{,x}w_{,xt} - (f_{oy}+f_y(t))w_{,y}w_{,yt}\Big]\,dx\,dy = 0
\end{aligned}
\tag{4.4}
$$

Subtracting identities (4.4), from the time-derivative of functional (4.3), we obtain the following form

$$\frac{dV}{dt} = \int\limits_0^1 \Big\{ -\beta w_{,t}^2 - \beta[w_{,xx}^2 + w_{,yy}^2 + 2w_{,xx}w_{,yy} + 2(1-\nu)(w_{,xy}^2 - w_{,xx}w_{,yy})] + \tag{4.5}$$
$$+\beta f_{ox}w_{,x}^2 + \beta f_{oy}w_{,y}^2 + f_x(t)(w_{,xt} + \beta w_{,x})w_{,x} + f_y(t)(w_{,yt} + \beta w_{,y})w_{,y} \Big\}\, dx\, dy$$

After some algebra, we rewrite the time-derivative of functional as

$$\frac{dV}{dt} = -2\beta V + 2U \tag{4.6}$$

where U is an auxiliary functional given by

$$U = \frac{1}{2}\int\limits_0^1 \Big[2\beta^2 ww_{,t} + 2\beta^3 w^2 + \tag{4.7}$$
$$+f_x(t)(w_{,xt} + \beta w_{,x})w_{,x} + f_y(t)(w_{,yt} + \beta w_{,y})w_{,y} \Big]\, dx\, dy$$

Now we attempt to construct a bound

$$\lambda V \geqslant U \tag{4.8}$$

where the stochastic function λ is to be determined. In an explicite notation, the function λ has to satisfy the following equation for arbitrary functions w and $w_{,t}$ satisfying suitable boundary conditions

$$\int\limits_0^r \int\limits_0^1 \Big\{ \lambda[w_{,t}^2 + 2\beta ww_{,t} + 2\beta^2 w^2 + (w_{,xx} + w_{,yy})^2 +$$
$$+2(1-\nu)(w_{,xy}^2 - w_{,xx}w_{,yy}) - f_{ox}w_{,x}^2 - f_{oy}w_{,y}^2] + \tag{4.9}$$
$$-[2\beta^3 w^2 + 2\beta^2 ww_{,t} + f_x(t)(w_{,xt} + \beta w_{,x})w_{,x} + f_y(t)(w_{,yt} + \beta w_{,y})w_{,y}] \Big\}\, dx \geqslant 0$$

It should be noticed that the way to obtain estimation (4.8) is purely algebraic contrary to systems described by strong equations, where the derivation of stability conditions is based on integrations by parts and manipulations with higher order partial derivatives. Usually, the Liapunov stability analysis of plates is performed for all four simply supported edges (Tylikowski, 1978).

In order to extend the field of possible applications, let us assume the following combinations of boundary conditions: a) c-c-c-c, b) c-c-c-s, c) c-c-s-c, d) c-c-s-s, e) s-c-s-s, f) s-s-s-s, shown in Fig. 1, where "s" denotes a simply supported edge, and "c" denotes a clamped edge. Contrary to the Levy method of determination of displacements of a rectangular plate with two simply supported opposite edges, the proposed technique of determining the stability domains can be applied to plates with arbitrary combinations of simply supported and clamped edges. In the first combinations of boundary conditions for plates with all four edges simply supported, we have

$$w(x,y,t) = \sum_{m,n=1}^{\infty} w_{mn}(t) \sin \frac{m\pi x}{r} \sin(n\pi y) \tag{4.10}$$

If the plate with mixed simply supported-clamped edges is considered, the plate displacement is written in the following form

$$w(x,y,t) = \sum_{m,n=1}^{\infty} w_{mn}(t) X_m\Big(\frac{x}{r}\Big) Y_n(y) \tag{4.11}$$

where X_m and Y_n are beam functions depending on boundary conditions for $x = 0$, r and $y = 0$, 1, respectively. For example, if all four edges of plate are clamped, the beam functions in Eq. (4.11) have the following form

$$\begin{aligned}
X_m &= \Big(\sin \frac{\beta_n x}{r} - \sinh \frac{\beta_n x}{r}\Big)(\cos\beta_n - \cosh\beta_n) + \\
&\quad -(\sin\beta_n - \sinh\beta_n)\Big(\cos \frac{\beta_n x}{r} - \cosh \frac{\beta_n x}{r}\Big) \\
Y_n &= (\sin\gamma_n y - \sinh\gamma_n y)(\cos\gamma_n - \cosh\gamma_n) + \\
&\quad -(\sin\gamma_n - \sinh\gamma_n)(\cos\gamma_n y - \cosh\gamma_n y)
\end{aligned} \tag{4.12}$$

Integrating, we have the following equality

$$\int_0^r X_{m,xx}^2 \, dx = \kappa_m \beta_m^2 \int_0^r X_{m,x}^2 \, dx \qquad \int_0^1 Y_{n,yy}^2 \, dy = \chi_n \gamma_n^2 \int_0^1 Y_{n,y}^2 \, dy \tag{4.13}$$

Using the orthogonality condition, we can also write

$$\int_0^r X_{m,x}^2 \, dx = \frac{\beta_m^2}{\kappa_m} \int_0^r X_m^2 \, dx \qquad \int_0^1 Y_{n,y}^2 \, dy = \frac{\gamma_n^2}{\chi_m} \int_0^1 Y_n^2 \, dy \tag{4.14}$$

where the first few constants κ_m and χ_n used in equalities (4.13) and (4.14) are given in Table 1 for the most commonly used boundary conditions: a clamped-clamped beam (c-c), and a clamped-simply supported beam (c-s). The constants tend to 1 as $n \to \infty$. In order to unify the notations in the case of the simply supported beam, we assume $\beta_n = n\pi$ and $\kappa_n = 1$. Using properties of the functions X_m and Y_n, we have

$$\int_0^r \int_0^1 w_{,xt}(x,y,t) w_{,x}(x,y,t)\, dx = \sum_{m,n=1}^{\infty} w_{mn,t}(t) w_{mn}(t) \int_0^r X_{m,x}^2\, dx \int_0^1 Y_n^2\, dy \tag{4.15}$$

Substituting Eq. $(4.14)_2$, yields

$$\begin{aligned}\sum_{m,n=1}^{\infty} w_{mn,t}(t) w_{mn}(t) \int_0^r X_{m,x}^2\, dx \int_0^1 Y_n^2\, dy = \\ = \sum_{m,n=1}^{\infty} \frac{\beta_m^2}{\kappa_m} w_{mn,t}(t) w_{mn}(t) \int_0^r X_m^2\, dx \int_0^1 Y_n^2\, dy\end{aligned} \tag{4.16}$$

Similarly, we have

$$\begin{aligned}\int_0^r \int_0^1 w_{,xx}(x,y,t) w_{,yy}(x,y,t)\, dx\, dy = \\ = \sum_{m,n=1}^{\infty} \sum_{i,j=1}^{\infty} w_{mn}(t) w_{ij}(t) \int_0^r X_{m,xx} X_i\, dx \int_0^1 Y_n Y_{j,yy}\, dy\end{aligned} \tag{4.17}$$

Integrating by parts the terms in the right-hand-side of Eq. (4.17) for simply supported or clamped edges, we have

$$\int_0^r X_{m,xx} X_i\, dx = X_{m,x} X_i \Big|_0^r - \int_0^r X_{m,x} X_{i,x}\, dx = -\delta_{mi} \int_0^r X_{m,x}^2\, dx \tag{4.18}$$

where δ_{mi} denotes the Kronecker delta function

$$\int_0^1 Y_n Y_{j,yy}\, dy = Y_{n,y} Y_j \Big|_0^1 - \int_0^1 Y_{n,y} Y_{j,y}\, dx = -\delta_{nj} \int_0^1 Y_{n,y}^2\, dy \tag{4.19}$$

Substituting Eqs. (4.18) and (4.19) into Eq. (4.17), yields

$$\int_0^r\int_0^1 w_{,xx}(x,y,t)w_{,yy}(x,y,t)\,dx\,dy = \int_0^r\int_0^1 w_{,xy}^2\,dx\,dy \tag{4.20}$$

Table 1. Numbers κ_n

$m,\ n$	1	2	3	4	5	6	7
c-s	1.3396	1.1649	1.1086	1.0809	1.0645	0.90205	1.0537
c-c	1.8185	1.3392	1.2224	1.1648	1.1309	1.10857	1.09276

Combining Eqs. (4.9) and (4.20) and substituting expansion (4.11), we have

$$\begin{aligned}&\sum_{m,n=1}^{\infty}\Big\{[\lambda w_{mn,t}^2 + 2\beta(\lambda-\beta)w_{mn,t}w_{mn} + 2\beta^2(\lambda-\beta)w_{mn}^2]\int_0^r X_m^2\,dx\int_0^1 Y_n^2\,dy +\\
&+\lambda w_{mn}^2\Big(\int_0^r X_{m,xx}^2dx\int_0^1 Y_n^2dy + 2\int_0^r X_{m,x}^2dx\int_0^1 Y_{n,y}^2dy + \int_0^r X_m^2dx\int_0^1 Y_{n,yy}^2dy\Big) +\\
&-[(\lambda f_{ox}+\beta f_x(t))w_{mn}^2 + f_x(t)w_{mn,t}w_{mn}]\int_0^r X_{m,x}^2\,dx\int_0^1 Y_n^2\,dy +\\
&-[(\lambda f_{oy}+\beta f_y(t)]w_{mn}^2 + f_y(t)w_{mn,t}w_{mn}]\int_0^r X_m^2\,dx\int_0^1 Y_{n,y}^2\,dy\Big\} \geqslant 0\end{aligned} \tag{4.21}$$

Using Eqs. (4.14)-(4.17), yields

$$\begin{aligned}&\sum_{n=1}^{\infty}\Big\{\lambda w_{mn,t}^2 + \Big(2\beta(\lambda-\beta) - f_x(t)\frac{\beta_m^2}{\kappa_m} - f_y(t)\frac{\gamma_n^2}{\chi_n}\Big)w_{mn,t}w_{mn} +\\
&+\Big[2\beta^2(\lambda-\beta) + \lambda\Big(\beta_m^4 + 2\frac{\beta_m^2}{\kappa_m}\frac{\gamma_n^2}{\chi_n} + \gamma_n^4\Big) - (\lambda f_{ox}+\beta f_x(t))\frac{\beta_m^2}{\kappa_m} +\\
&-(\lambda f_{oy}+\beta f_y(t))\frac{\gamma_n^2}{\chi_n}\Big]w_{mn}^2\Big\}\int_0^r X_m^2\,dx\int_0^1 Y_n^2\,dy \geqslant 0\end{aligned} \tag{4.22}$$

Therefore, the variational inequality is reduced to an infinite system of quadratic inequalities

$$\lambda w_{mn,t}^2 + 2A_{mn}(\lambda,t)w_{mn,t}w_{mn} + B_{mn}(\lambda,t)w_{mn}^2 \geqslant 0 \tag{4.23}$$

where

$$A_{mn}(\lambda,t) = \beta(\lambda-\beta) - f_x(t)\frac{\beta_m^2}{2\kappa_m} - f_y(t)\frac{\gamma_n^2}{2\chi_n}$$

$$\begin{aligned}B_{mn}(\lambda,t) &= 2\beta^2(\lambda-\beta) + \lambda g\Big(\beta_m^4 + 2\frac{\beta_m^2}{\kappa_m}\frac{\gamma_n^2}{\chi_n} + \gamma_n^4 g\Big) + \\ &\quad -[\lambda f_{ox} + \beta f_x(t)]\frac{\beta_m^2}{\kappa_m} - [\lambda f_{oy} + \beta f_y(t)]\frac{\gamma_n^2}{\chi_n}\end{aligned}$$

The unknown function λ is determined from the zero determinant condition for all m and n of the form

$$\begin{vmatrix} \lambda & A_{mn}(\lambda,t) \\ A_{mn}(\lambda,t) & B_{mn}(\lambda,t) \end{vmatrix} = 0 \tag{4.24}$$

After some algebraic manipulations, (4.24) yields

$$\lambda = \max_{m,n=1,2,\dots} \lambda_{mn} \tag{4.25}$$

where

$$\lambda_{mn} = \frac{\Big|\beta^2 + f_x(t)\frac{\beta_m^2}{2\kappa_m} + f_y(t)\frac{\gamma_n^2}{2\chi_n}\Big|}{\sqrt{\beta_m^4 + 2\frac{\beta_m^2}{\kappa_m}\frac{\gamma_n^2}{\chi_n} + \gamma_n^4 - f_{ox}\frac{\beta_m^2}{\kappa_m} - f_{oy}\frac{\gamma_n^2}{\chi_n} + \beta^2}} \tag{4.26}$$

Combining inequality (4.6) and (4.8), yields an upper estimation of the time-derivative of the functional

$$\frac{dV}{dt} = -2\beta V + 2U \leqslant -2(\beta-\lambda)V \tag{4.27}$$

Integrating with respect to time, yields the following upper estimation of V

$$V(t) \leqslant V(0)\exp\Big[-2\Big(\beta - \frac{1}{t}\int\limits_0^t \lambda(\tau)\,d\tau\Big)t\Big] \tag{4.28}$$

Therefore

$$\lim_{t\to\infty} \|w(\cdot,t)\| = 0 \tag{4.29}$$

if the exponent in (4.28) is negative

$$\beta \geqslant \lim_{t\to\infty}\frac{1}{t}\int\limits_0^t \lambda(\tau)\,d\tau \tag{4.30}$$

Additionally, assuming the ergodicity of in-plane forces, we substitute the time-averaging in (4.30) by the average over a probabilistic space

$$\lim_{t\to\infty} \frac{1}{t} \int\limits_0^t \lambda(\tau)\, d\tau = E\{\lambda\} \tag{4.31}$$

Finally, the trivial solution to Eq. (2.5) is almost surely stable with respect to the measure of distance defined as the square root of functional (4.1), if the transcendental inequality is fulfilled

$$\beta \geqslant E\{\lambda\} \tag{4.32}$$

Therefore, stability domains for the clamped-clamped plate and the clamped-simply supported plate are defined as follows

$$\beta \geqslant E\Bigg\{ \max_{m,n=1,2,\dots} \Bigg[\frac{\Big|\beta^2 + f_x(t)\frac{\beta_m^2}{2\kappa_m} + f_y(t)\frac{\gamma_n^2}{2\chi_n}\Big|}{\sqrt{\beta_m^4 + 2\frac{\beta_m^2\gamma_n^2}{\kappa_m\chi_n} + \gamma_n^4 - f_{ox}\frac{\beta_m^2}{\kappa_m} - f_{oy}\frac{\gamma_n^2}{\chi_n} + \beta^2}} \Bigg] \Bigg\} \tag{4.33}$$

We mention that Eq. (4.33) is a generalization of the similar form as the analytical formula defining the stability region obtained by Kozin (1972) for the simply supported beam in the strong formulation. It should be emphasized that formulae (4.32) and (4.33) are obtained without dealing with the third and fourth order spatial derivatives. If the density functions for the time-dependent componenets of in-plane forces are known, numerical integration can be used to evaluate Eqs. (4.32) and (4.33) for different values of parameters such as for example the variance σ_x^2 and σ_y^2. As the damping coefficient β is involved in the right hand side of the equations they have to be solved in an iterative way.

The structure of stability conditions for all boundary conditions is similar. They contain different values of constants admitting Eqs. (4.13)-(4.14)$_1$. The stability domains are examined for quadratic plates uniaxially loaded $f_{oy} = f_y(t) = 0$ with each combination of simple and clamped supports, cf. Fig. 1. The dependence of stability regions on different boundary conditions is shown in Fig. 2 for $f_{ox} = 0$ on the plane β, σ^2 for the Gaussian in-plane force. The influence of boundary conditions on the shape and size of stability domains is rather weak. The effect is more severe (Fig. 3) for $f_{ox} = 39$ corresponding to the critical loading of the quadratic plate with all four edges simply supported, Fig. 1f. The influence of the constant component $f_{ox} = 64$ corresponding to the critical loading (Fig. 1d) and $f_{ox} = 83$ corresponding to the critical loading (Fig. 1b) on stability domains is shown in Fig. 4 and Fig. 5, respectively.

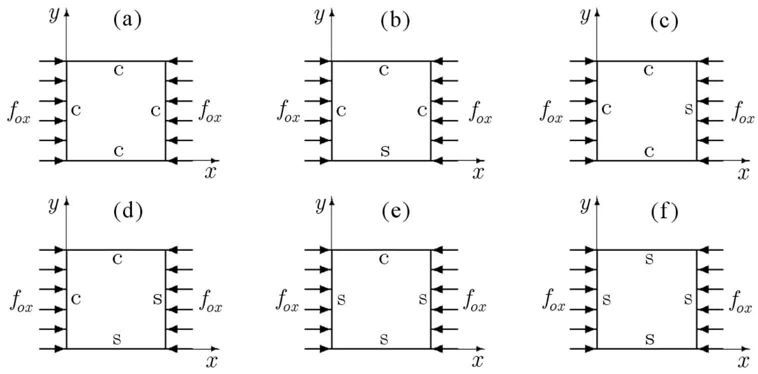

Fig. 1. Rectangular plate under uniaxial in-plane loading and boundary conditions

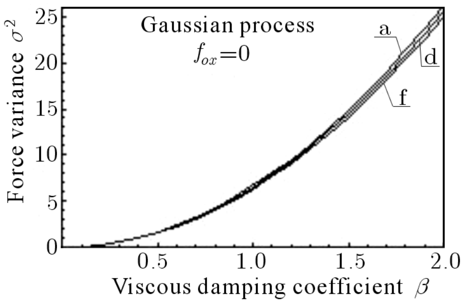

Fig. 2. Stability domains of a plate subjected to the uniaxial in-plane Gaussian parametric excitation for different boundary conditions, $f_{ox} = 0$

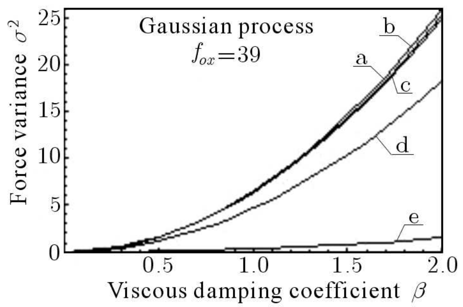

Fig. 3. Stability domains of a plate subjected to the uniaxial in-plane harmonic parametric excitation for different boundary conditions, $f_{ox} = 39$

Although, the stability regions are similar qualitatively for different boundary conditions, the diferences of critical values of σ^2 (the force variance) are significant for compressive forces close to critical loadings. Therefore, a more

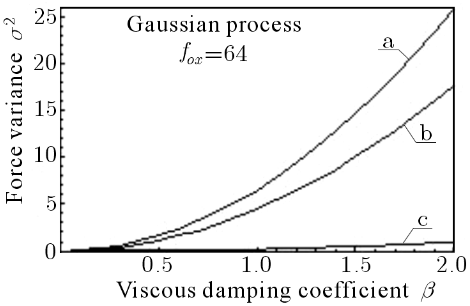

Fig. 4. Stability domains of a plate subjected to the uniaxial in-plane harmonic parametric excitation for different boundary conditions, $f_{ox} = 64$

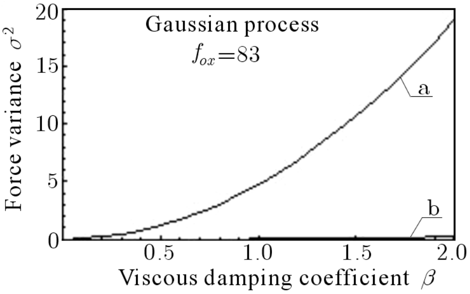

Fig. 5. Stability domains of a plate subjected to the uniaxial in-plane harmonic parametric excitation for different boundary conditions, $f_{ox} = 83$

carefull analysis of boundary conditions is needed in the analysis of dynamic stability of continuous systems.

5. Conclusions

The stability analysis method is developed for distributed dynamic problems with relaxed assumptions imposed on solutions. The Lyapunov method can be used to stability analysis of equations in weak formulation. The results are obtained in the frame of the distributed parameter approach without earlier discretization or truncation. Without any viscous damping, the plate motion is unstable due to the parametric excitation. The stability domains are presented for plates with each combination of simple and clamped supports.

Stability domains of plates compressed by forces close to the critical loading substantially depend on the assumed boundary conditions. Stability results obtained for a plate with simply supported boundary conditions in strong formulations and stability conditions obtained for simply supported plates described by strong equations are also valid under weak formulation.

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Dynamiczna stateczność słabych równań płyt prostokątnych

Streszczenie

W pracy rozszerzono możliwości analizy stabilności układów ciągłych na układy z osłabionymi warunkami nakładanymi na rozwiązania. Układy aktywnego tłumienia drgań cienkościennych elementów płytowych mogą zawierać elementy piezoelektryczne oddziaływujące na konstrukcję. W uproszczonym modelu oddziaływanie to sprowadza się do działania momentów gnących lub sił rozłożonych na krawędziach elementu piezoelektrycznego. Wprowadzenie dystrybucji δ-Diraca i jej pochodnej prowadzi do analitycznego zapisu obciążenia i wprowadza nieregularności do rozwiązania zadania drgań wymuszonych układu ciągłego. Słabą postać równań płyty otrzymano za pomocą zasady Hamiltona. Badanie staeczności stochastycznych układów w formie słabej jest oparte na analizie funkcjonału Lapunowa wzdłuż słabego rozwiązania. Rozwiązanie zadania jest niezależne od przyjętych warunków brzegowych.

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