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PLY THICKNESS TOLERANCES IN STACKING SEQUENCE OPTIMIZATION OF MULTILAYERED LAMINATE PLATES

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The paper deals with the impact of manufacturing tolerances of plies thicknesses on optimal design of multi-layered laminated plates in compression. It is assumed that the considered tolerances are represented by the maximum acceptable deviation of every individual ply thickness from its nominal design value. The robustness of the optimum is achieved diminishing the buckling load amplitude factor by the product of arbitrary assumed tolerances and appropriate sensitivities. The discussed optimization problem is solved numerically by the direct enumeration method. The proposed approach is illustrated with examples of the rectangular multi-layered laminated plate design under uni- and biaxial compression. The achieved results emphasise the robustness of the proposed method compared to the approaches with ignored tolerances.

Key words: laminate composite structures, optimization, manufacturing tolerances, robust design, structural stability

1. Introduction

Laminated composites are commonly used structural materials, primarily due to their excellent strength and stiffness to weight ratios. An additional advantage of these materials over conventional isotropic ones is a possibility of tailoring and optimizing their properties to match the specific requirements of a given application. In a general case, design variables available to the engineer include multiple material systems, plies orientations, thicknesses and their stacking sequence.

Numerous research papers deal with optimum design of composite plates, shells and similar structures. A vast majority of them are prepared on the assumption that design variables and thus the system performance are not subject to any deviations arising from manufacturing processes, simplifications in operating conditions, etc. Unfortunately, resulting from this approach optimal solutions might violate the constraints imposed on the system performance when real designs are considered. The proximate reason is a high sensitivity of the optimized structure to variations of design variables, and the fact that in optimal light-weight systems at least one constraint reaches its limit value. These observations seem to be particularly important for composite materials which experience larger uncertainties in their properties if compared to conventional materials. This is due to a number of parameters involved in their fabrication and manufacturing process.

Considering the above aspects, the expected uncertainties should be recognised and taken into account in the design code. The original approach to this matter is represented by the well-known concept of safety (or load) factors. With respect to composites, this has been discussed in numerous papers, e.g. by Chryssanthopoulos and Poggi (1995) to estimate limit load of cylinders made of a lay-up Kevlar composite. Although safety factors provide some latitude for variations between the actual structure and the specific instance considered in the computational analysis, this approach does not give any further detailed information about the design and on impact of uncertainty factors. Therefore, several more advanced approaches are being developed to face the discussed problem. All of them can be captured into a robust design philosophy, and most popular ones are stochastic analysis, interval programming and deterministic approach.

The concept of the stochastic based approach to laminate stability estimation is studied in several papers, e.g. by Kogiso *et al.* (1997). The authors derive a reliability analysis considering material properties, orientation angles and applied loads as independent random variables. To illustrate the method, numerical examples are given for a simply-supported 8-ply fiber laminated plate made of epoxy graphite, where the mean orientation angles are treated as design variables. Singh *et al.* (2001) present the application of the stochastic finite element approach to the eigenvalue problem of buckling arising from the dispersion of lamina material properties. Uncertain material data are modelled by random variables, while the other system parameters stay nominal. A mean-centred first order perturbation technique is used to find statistics of buckling loads of cylindrical panels with various boundary conditions.

The stochastic optimization is an accurate and exact approach to the discussed problem since it guarantees the assumed a priori level of system safety. Nevertheless, taking into account practicing engineers' expectations, this approach may have some drawbacks. The main reason could be the requirement of precise information on probability density functions and incomplete information about variables that are critical to the structure. Other issue is high complexity of the problem final formulation and – as a consequence – resulting tough calculations. Because of the above reasons, the alternative methods of robust design are under parallel development.

Interval programming requires only the bounds of the perturbed parameters to model the system uncertainty and no information about probability distributions is necessary. With respect to composite materials, this method is studied by e.g. Jiang *et al.* (2008). The authors formulated a problem to find the laminate plies orientations to maximise the stiffness of a plate if material properties are perturbed; next, the problem has been solved by a general nonlinear interval programming method with an arbitrary chosen possibility degree (accuracy). Three numerical examples are given. First two deal with the uncertain engineering constants of the laminate layer material. In the third example, the method is extended to incorporate perturbations in the thicknesses of the laminas as well.

The proposed interval number programming approach makes the uncertainty analysis convenient and computationally attractive. Comparing to the stochastic programming methods, the discussed approach can deal with problems of limited information about the uncertainty. However, examples given in the discussed paper show that the optimal design strongly depends on the arbitrary predetermined possibility degree level. Therefore, this value must be chosen appropriately, based on the class of the solved problem, the attitude of the engineer and his experience.

The third type of approach to the robust design is deterministic one which takes advantage of complicated and uncertain processes avoidance. The research effort is aimed at more in-depth analysis of a considered system but also at developing approximate simplified models as well. As an instance Fong *et al.* (2010) suggest taking into account second order effects into the design code of composite columns. Presented examples prove a more accurate reflection of the behaviour of the member and a considerable simplification in final calculations for robust design.

An alternative and simplified model of uncertainties representation in the optimization code is given by Lee and Park (2001). The authors proposed a new objective function defined as a linear combination of the mean and standard deviation of the original cost function. The introduction of a weighting factor into the cost function enables a change in the priority of the optimum – i.e. to have the low mean value of the objective function and simultaneously larger its standard deviation or to shift the solution to the region with a higher cost function mean value, but smaller deviations. Among numerous benefits, this method has an advantage in providing full information about the final solution. Nevertheless, the mean value and standard deviations of design variables still have to be known or assumed.

An interesting paper was published by Kristinsdottir *et al.* (1996). They propose their own technique for searching near-optimal safe designs. This is done by changing the right hand side of the inequality constraints and replacing the initial zero value in the nominal problem by a small positive number called a safety margin. Next the structure is re-optimized and checked, if new design values compared to the design variables in the original unperturbed problem have reached initially specified tolerances. If not, the appropriate safety margin has to be increased and the optimization procedure is run again. As an example, a hat-stiffened composite panel is optimized. The discussed method may ensure safe results for complicated problems, but since the assumed safety margin is not explicitly correlated to the uncertainties of design variables it leads to excessive near-optimal solutions only. Also, if re-optimization procedures are required, the choice of the appropriate safety margins to be increased faces some trouble.

Another method for optimum robust design of laminated plates was proposed by Walker and Hamilton (2005, 2006). The key idea of the method is the triple optimization of the structure. First the system is optimized without considering deviations in a design variable (angle ply for presented exemplary designs). Next, the optimization is re-run twice for the perturbed variable – i.e. for the upper and lower possible design variable value. Following these steps, a comparison of results of three objective functions is made to get the best solution. As an example, eight--layered plates with different side aspect ratios and loadings are optimized. Unfortunately, the given method is limited to designs, where the only one reinforcement orientation in all plies is allowed. Therefore, no solutions for multiple amgle-ply oriented laminates are possible.

The deterministic approach to the discussed robust design problem was also proposed by Gutkowski and Bauer (1999) and next by Gutkowski and Latalski (2003). The authors suggest to diminish the nominal limit values of state variables in design constraints by positive terms. These are the products of assumed arbitrary design tolerances and a vector of appropriate state variables sensitivities. In the paper by Gutkowski and Latalski (2003), the method is illustrated by composite plates optimum design with tolerances in fibers orientations.

In the current paper, the presented idea is further extended to cases where laminae thickness tolerances are considered. Updated formulas for buckling load factor in the composite optimization problem are derived. Recurrent sensitivity relations are also formulated. Based on these, the impact of thickness variations on the optimal laminate stacking sequence solution is examined in detail. The presented method is offered as a deterministic and computationally effective but an alternative approach to the exact original stochastic problem. Moreover, the proposed method overcomes certain limitations of other deterministic methods discussed above and directly correlates manufacturing tolerances to the structural response providing in-depth information about their actual impact. And finally, the proposed method might be extended to any type of other possible imperfections.

2. Problem statement

The impact of manufacturing imperfections defined by lamina tolerances on composite structures design is presented. In this study, the method is illustrated by optimization of a rectangular and simply supported sandwich panel subjected to compressive forces. However, this approach could be also applied to other structures, such as composite beams or shells and other sources of possible perturbations (material, geometric, etc.).

It is assumed that the value of an individual lamina thickness in the composite may be varied from its nominal dimension t_k. This variation corresponds to the accuracy of manufacture and is represented by a deterministic maximum allowable deviation Δt_k (lower or upper). This tolerance is assumed to be set up arbitrary as e.g. a ratio of the nominal lamina thickness t_k. Therefore, if upper and lower deviations stay the same, it is expected that the actual thickness of every k-th ply stays within a range from $(t_k - \Delta t_k)$ to $(t_k + \Delta t_k)$.

Since for the considered tolerances only nominal values of design parameters and their maximum acceptable deviations are known, the exact magnitude of the system performance is unknown. Therefore, it is estimated by calculating the state variable λ for nominal values of design parameters and next diminishing this by a penalty term, which is a product of thicknesses admissible imperfections Δt and appropriate sensitivities $d\lambda/dt$. Since the actual deviations are of unknown sign (upper or lower) and also the sensitivities might be of positive or negative sign, the moduli of successive $\Delta t_k d\lambda/dt_k$ products are considered in resultant summation

$$\Delta\lambda = \sum_{k=1}^{N} \Big| \frac{d\lambda}{dt_k} \Delta t_k \Big| \tag{2.1}$$

Finally, the obtained difference of the nominal system performance and the mentioned penalty term is considered to be the approximation of the imperfect system performance. This approach ensures the design to stay on the safe side. As it has been shown in previous research papers by Gutkowski and Bauer (1999) and Gutkowski and Latalski (2003) the use of the penalty term as proposed above results in more effective solutions if compared to simple worst case designs, where the system safety is ensured by considering the structure with all nominal design variables t_k arbitrarily increased (or decreased) by Δt_k.

The imperfect composite material as described above is used for a rectangular and simply supported on all four edges sandwich panel given in Fig. 1. The plate consists of N plies in total, each of the equal nominal thickness $t = t_1 = \ldots = t_N$ with fibers orientation denoted by angle θ_k $(k = 1, \ldots, N)$. The uniform longitudinal stress resultants λN_x and λN_y are applied at the edges of the panel, where λ is the amplitude parameter; no shear forces are considered.

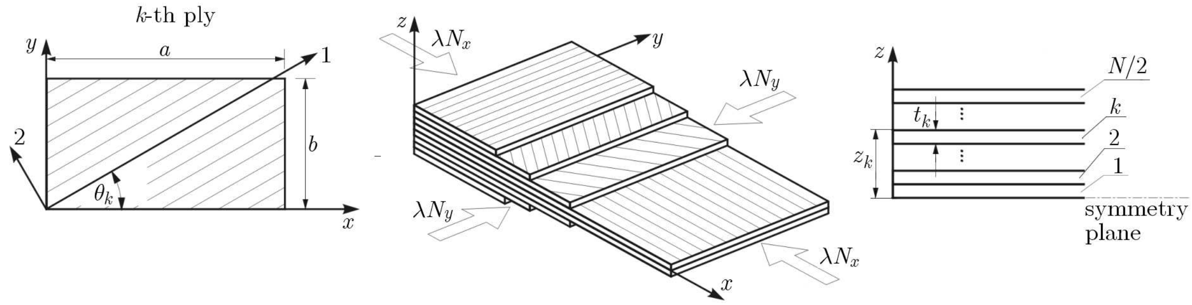

Fig. 1. Laminate sandwich plate

In the present research, the lamina stacking sequence to maximise the lowest (over all possible modes) value of estimated buckling load λ is looked for. The admissible direction of fibers in every lamina is limited to one of the four angles: $0°$, $90°$ and $\pm 45°$ and they are not subject to any deviations. It is also assumed that the sandwich plate is symmetric and balanced, i.e. the number of $(+45°)$ plies is equal to the number of $(-45°)$ ones. Moreover, all the assumptions of classical buckling theory for plates are in force.

3. Optimization problem formulation

Following the above considerations, the discussed optimum design problem with lamina thickness tolerances is formulated as follows:

1) find the vector $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_{N/2}]^{\mathrm{T}}$,
2) such, that

$$\max_{\boldsymbol{\theta}} \Big(\lambda_{cr}(m,n) - \sum_{k=1}^{N/2} \Big| \varDelta t_k \frac{d\lambda_{cr}}{dt_k} \Big| \Big) \tag{3.1}$$

3) with constraints

$$\begin{aligned} & \theta_k \in (0^\circ, 90^\circ, 45^\circ, -45^\circ) \qquad \text{where} \quad k = 1, \ldots, N/2 \\ & \sum_{k=1}^{N/2} (\theta_k = 45^\circ) = \sum_{k=1}^{N/2} (\theta_k = -45^\circ) \end{aligned} \tag{3.2}$$

Set of constraints $(3.2)_1$ limits the fiber orientation to admissible values only, while relation $(3.2)_2$ represents the condition of the plate to be balanced. Plate symmetry requirement is guaranteed by the fact that the summation index k stays within the range $(1, \ldots, N/2)$ in relations (3.1) and (3.2). Appearing in (3.1) (m,n) parameters are mode shape indices to ensure all possible buckling modes are examined.

The plate load factor λ is evaluated according to the classical buckling theory for simply supported equivalent orthotropic plate subject to in-plane loading. The system stability is assured for the load amplitude $\lambda = \lambda_{cr}(m,n)$ sustaining the relation as given by Adali and Duffy (1990), Haftka and Walsh (1992), Nemeth (1986)

$$1 \leqslant \lambda_{cr}(m,n) = \frac{D_{11}m^4 + 2(D_{12} + 2D_{66})m^2n^2(a/b)^2 + D_{22}n^4(a/b)^4}{m^2N_x + n^2(a/b)^2N_y} \tag{3.3}$$

where m and n are natural numbers corresponding to the number of buckling half-waves (mode shape) in the x and y directions respectively.

Variables D_{11}, D_{12}, D_{22} and D_{66} are flexural stiffnesses and can be expressed in terms of material invariants U_i $(i = 1, \ldots, 5)$ and three integrals V_0, V_1, V_3 as follows

$$\begin{aligned} D_{11} &= U_1V_0 + U_2V_1 + U_3V_3 \qquad & D_{12} &= U_4V_0 - U_3V_3 \\ D_{22} &= U_1V_0 - U_2V_1 + U_3V_3 \qquad & D_{66} &= U_5V_0 - U_3V_3 \end{aligned} \tag{3.4}$$

Invariants U_i $(i = 1, \ldots, 5)$ are related to composite material properties

$$\begin{aligned} U_1 &= \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}) \qquad & U_2 &= \frac{1}{2}(Q_{11} - Q_{22}) \\ U_3 &= \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}) \qquad & U_4 &= \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}) \\ U_5 &= \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66}) \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} Q_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}} \qquad & Q_{12} &= \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \\ Q_{22} &= \frac{E_2}{1 - \nu_{12}\nu_{21}} \qquad & Q_{66} &= G_{12} \end{aligned} \tag{3.6}$$

In the above relations, E_1, E_2, ν_{12}, ν_{21} and G_{12} denote Young's moduli in the fibers direction 1 and lateral direction 2 (see Fig. 1), Poisson's ratios and shear modulus, respectively.

The variables V_0, V_1 and V_3 comprise information about fibers orientations, about layers thicknesses and their stacking sequence

$$
\begin{aligned}
V_0 &= \int\limits_{-h/2}^{h/2} z^2\,dz = \frac{1}{3}\sum_{k=1}^{N}(z_k^3 - z_{k-1}^3) = \frac{2}{3}\sum_{k=1}^{N/2}(z_k^3 - z_{k-1}^3)\\
V_1 &= \int\limits_{-h/2}^{h/2} z^2\cos 2\theta\,dz = \frac{2}{3}\sum_{k=1}^{N/2}(z_k^3 - z_{k-1}^3)\cos 2\theta_k\\
V_3 &= \int\limits_{-h/2}^{h/2} z^2\cos 4\theta\,dz = \frac{2}{3}\sum_{k=1}^{N/2}(z_k^3 - z_{k-1}^3)\cos 4\theta_k
\end{aligned}
\tag{3.7}
$$

where $h = Nt$ is the total thickness of the laminate, z is the distance from symmetry plane as given in Fig. 1.

4. Sensitivity of the buckling load

Following the proposed in this research approach to robust design – as given in "Problem statement" section – the system performance penalty term $\Delta\lambda$ (2.1) is to be calculated. This requires the information about the sensitivity of the buckling load factor $\lambda_{cr}(m,n)$ — see Eq. (3.3) with respect to each lamina thickness

$$
\frac{d\lambda_{cr}}{d\mathbf{t}} = \Big[\frac{d\lambda_{cr}}{dt_1},\ldots,\frac{d\lambda_{cr}}{dt_N}\Big]^{\mathrm{T}} = \pi^2\frac{\frac{dD_{11}}{d\mathbf{t}}m^4 + 2\Big(\frac{dD_{12}}{d\mathbf{t}} + 2\frac{dD_{66}}{d\mathbf{t}}\Big)m^2n^2\Big(\frac{a}{b}\Big)^2 + \frac{dD_{22}}{d\mathbf{t}}n^4\Big(\frac{a}{b}\Big)^4}{m^2N_x + n^2\Big(\frac{a}{b}\Big)^2N_y}
\tag{4.1}
$$

According to (3.3) the appropriate derivatives of flexural stiffnesses are to be calculated

$$
\begin{aligned}
\frac{dD_{11}}{d\mathbf{t}} &= U_1\frac{dV_0}{d\mathbf{t}} + U_2\frac{dV_1}{d\mathbf{t}} + U_3\frac{dV_3}{d\mathbf{t}} &\qquad \frac{dD_{12}}{d\mathbf{t}} &= U_4\frac{dV_0}{d\mathbf{t}} - U_3\frac{dV_3}{d\mathbf{t}}\\
\frac{dD_{22}}{d\mathbf{t}} &= U_1\frac{dV_0}{d\mathbf{t}} - U_2\frac{dV_1}{d\mathbf{t}} + U_3\frac{dV_3}{d\mathbf{t}} &\qquad \frac{dD_{66}}{d\mathbf{t}} &= U_5\frac{dV_0}{d\mathbf{t}} - U_3\frac{dV_3}{d\mathbf{t}}
\end{aligned}
\tag{4.2}
$$

To derive the above relations a simple case is analysed first.

4.1. Simple example

Let us consider a 6-layered composite plate fulfilling all the assumptions given in "Problem statement" section. The only exception is that this time all the layers are temporarily assumed to have different thicknesses. Thus the vector of nominal thicknesses is $\mathbf{t} = [t_1, t_2, t_3]^{\mathrm{T}}$ since the symmetry condition is still preserved.

According to (3.7), the integrals V_i for the discussed simple composite are

$$
V_0 = \int\limits_{-h/2}^{h/2} z^2\,dz = \frac{2}{3}\sum_{k=1}^{3}(z_k^3 - z_{k-1}^3) = \frac{2}{3}(z_1^3 + z_2^3 - z_1^3 + z_3^3 - z_2^3) = \frac{2}{3}z_3^3
$$

$$
\begin{aligned}
V_1 &= \int\limits_{-h/2}^{h/2} z^2 \cos 2\theta \; dz = \frac{2}{3}\sum_{k=1}^{3}(z_k^3 - z_{k-1}^3)\cos 2\theta_k \\
&= \frac{2}{3}[z_1^3 \cos 2\theta_1 + (z_2^3 - z_1^3)\cos 2\theta_2 + (z_3^3 - z_2^3)\cos 2\theta_3] \\
V_3 &= \int\limits_{-h/2}^{h/2} z^2 \cos 4\theta \; dz = \frac{2}{3}\sum_{k=1}^{3}(z_k^3 - z_{k-1}^3)\cos 4\theta_k \\
&= \frac{2}{3}[z_1^3 \cos 4\theta_1 + (z_2^3 - z_1^3)\cos 4\theta_2 + (z_3^3 - z_2^3)\cos 4\theta_3]
\end{aligned}
\tag{4.3}
$$

Therefore, bearing in mind that $h/2 = z_3 = t_1+t_2+t_3$, the derivatives $\frac{dV_0}{d\mathbf{t}} = \Big[\frac{dV_0}{dt_1}, \frac{dV_0}{dt_2}, \frac{dV_0}{dt_3}\Big]^{\rm T}$ requested by (4.2) are given by the subsequent relations

$$
\begin{aligned}
\frac{dV_0}{dt_1} &= \frac{d}{dt_1}\Big(\frac{2}{3}z_3^3\Big) = \frac{2}{3}\frac{d}{dt_1}(t_1+t_2+t_3\Big)^3 = 2(t_1+t_2+t_3)^2 = \frac{h^2}{2} \\
\frac{dV_0}{dt_2} &= \frac{d}{dt_2}\Big(\frac{2}{3}z_3^3\Big) = \frac{2}{3}\frac{d}{dt_2}(t_1+t_2+t_3)^3 = 2(t_1+t_2+t_3)^2 = \frac{h^2}{2} \\
\frac{dV_0}{dt_3} &= \frac{d}{dt_3}\Big(\frac{2}{3}z_3^3\Big) = \frac{h^2}{2}
\end{aligned}
\tag{4.4}
$$

Sensitivity of V_1 variable to t_1 may be calculated keeping in mind that $z_1 = t_1$, $z_2 = t_1 + t_2$ and $z_3 = t_1 + t_2 + t_3$; thus

$$
\begin{aligned}
\frac{dV_1}{dt_1} &= \frac{2}{3}\frac{d}{dt_1}\Big\{t_1^3 \cos 2\theta_1 + [(t_1+t_2)^3 - t_1^3]\cos 2\theta_2 + [(t_1+t_2+t_3)^3 - (t_1+t_2)^3]\cos 2\theta_3\Big\} \\
&= 2\Big\{t_1^2 \cos 2\theta_1 + [(t_1+t_2)^2 - t_1^2]\cos 2\theta_2 + [(t_1+t_2+t_3)^2 - (t_1+t_2)^2]\cos 2\theta_3\Big\}
\end{aligned}
\tag{4.5}
$$

The same may be done for t_2 and t_3 parameters

$$
\begin{aligned}
\frac{dV_1}{dt_2} &= \frac{2}{3}\frac{d}{dt_2}\Big\{t_1^3 \cos 2\theta_1 + [(t_1+t_2)^3 - t_1^3]\cos 2\theta_2 + [(t_1+t_2+t_3)^3 - (t_1+t_2)^3]\cos 2\theta_3\Big\} \\
&= \frac{dV_1}{dt_1} - 2t_1^2(\cos 2\theta_1 - \cos 2\theta_2) \\
\frac{dV_1}{dt_3} &= \frac{2}{3}\frac{d}{dt_3}\Big\{t_1^3 \cos 2\theta_1 + [(t_1+t_2)^3 - t_1^3]\cos 2\theta_2 + [(t_1+t_2+t_3)^3 - (t_1+t_2)^3]\cos 2\theta_3\Big\} \\
&= 2(t_1+t_2+t_3)^2 \cos 2\theta_3 = \frac{dV_1}{dt_2} - 2(t_1+t_2)^2(\cos 2\theta_2 - \cos 2\theta_3)
\end{aligned}
\tag{4.6}
$$

Similar recurrent relations may be formulated for V_3 derivative – the only difference being an angle θ factor 2 is to be replaced by 4.

4.2. General relations

Following the above simple case, general relations for dV_i/dt_k, $i = 0, 1, 3$, $k = 1, \dots, N$ terms are formulated.

Set of equations (4.4) remains valid for every ply in the laminate, so

$$
\frac{dV_0}{dt_k} = \frac{1}{2}h^2 \qquad \text{for} \quad k = 1, 2, \dots, N \qquad \Leftrightarrow \qquad \frac{dV_0}{d\mathbf{t}} = \Big[\frac{h^2}{2}, \dots, \frac{h^2}{2}\Big]^{\rm T}
\tag{4.7}
$$

Relations (4.5) and (4.6) for the integral V_1 derivatives may be simplified and generalised taking into account the initial assumption $t_1 = t_2 = t_3 = t$

$$
\begin{aligned}
&\frac{dV_1}{dt_1} = 2[t^2\cos 2\theta_1 + (4-1)t^2\cos 2\theta_2 + (9-4)t^2\cos 2\theta_3] = 2t^2\sum_{k=1}^{3}[k^2-(k-1)^2]\cos 2\theta_k \\
&\frac{dV_1}{dt_2} = \frac{dV_1}{dt_1} - 2t^2(\cos 2\theta_1 - \cos 2\theta_2) \qquad \frac{dV_1}{dt_3} = \frac{dV_1}{dt_2} - 2\cdot 4t^2(\cos 2\theta_2 - \cos 2\theta_3)
\end{aligned}
\tag{4.8}
$$

Finally, one arrives at the recurrent dependence for individual terms of $dV_1/d\mathbf{t}$ and $dV_3/d\mathbf{t}$ vectors

$$
\begin{aligned}
&\left.\begin{aligned}
\frac{dV_1}{dt_k} &= \frac{dV_1}{dt_{k-1}} - 2(k-1)^2t^2(\cos 2\theta_{k-1} - \cos 2\theta_k) \\
\frac{dV_3}{dt_k} &= \frac{dV_3}{dt_{k-1}} - 2(k-1)^2t^2(\cos 4\theta_{k-1} - \cos 4\theta_k)
\end{aligned}\right\} \qquad \text{for} \quad k = 2,\dots,N/2 \\
&\frac{dV_1}{dt_1} = 2t^2\sum_{k=1}^{N/2}[k^2-(k-1)^2]\cos 2\theta_k \qquad \frac{dV_3}{dt_1} = 2t^2\sum_{k=1}^{N/2}[k^2-(k-1)^2]\cos 4\theta_k
\end{aligned}
\tag{4.9}
$$

5. Numerical examples

To illustrate the proposed method of incorporating thicknesses uncertainties into a design code, examples of uniaxial and biaxial plate compression are presented. Both of them come from the paper by Haftka and Walsh (1992), where they were discussed for nominal design values only.

Computations are performed for the plate consisting of $N = 16$ plies in total, each of the equal thickness $t = 0.127\,\text{mm}$. The axial stress resultant $N_x = 175\,\text{N/m}$ is constant. Material properties for the graphite-epoxy laminate are considered: $E_1 = 128.0\,\text{GPa}$, $E_2 = 13.0\,\text{GPa}$, $G_{12} = 6.4\,\text{GPa}$ and $\nu_{12} = 0.3$.

Both the design statement and limitations imposed on admissible fiber orientations as well result in a combinatorial formulation of the plate optimization task. This class of problems may be solved by any of the integer-programming methods, but such an approach does not guarantee the global optimum to be found. Moreover, since the trigonometric functions appearing in plate analysis are periodic, multiple equivalent solutions are expected for the given problem. Bearing this in mind, and the fact that the total number of all possible combinations is not big – e.g. for a 16 layer laminate it is 4^8 – all further considered examples are solved by a direct enumeration approach.

5.1. Axial compression

For uniaxial compression, no lateral stress resultants N_y are present and the buckling load factor is maximised for various plate aspect ratios a/b. It is known (e.g. from Haftka *et al.* (1990)) that for low aspect ratios the optimum ply angle is $0°$, and for a/b larger than about 1.4–1.5 the optimum solution has $\pm 45°$ plies only. All these observations deal with nominal designs. Therefore, a check is performed to see whether there is any transition range of a/b where the optimum would include both $0°$ and $\pm 45°$ plies. Alternatively, whether there is any plate where for different Δt uncertainties these combined orientations might appear. As demonstrated by a series of performed numerical tests, all solutions for $a/b < 1.445$ cases are a combination of $\pm 45°$ plies only, while for $a/b \geqslant 1.446$ ratios all the plies are $0°$. More detailed results of the analysis are gathered in Table 1.

Table 1. Laminate plate buckling load in uniaxial compression versus considered allowable plies thickness tolerances and a/b aspect ratio. Result notation: $\lambda_{\text{n.s.}(m,n)}$ – where n.s. is the number of equivalent solutions, m, n are buckling mode shape parameters – see Eq. (3.3). Solutions for all $a/b < 1.445$ are a combination of $\pm 45°$ plies, while for $a/b \geqslant 1.446$ all the plies are $0°$

a/b	$\Delta t/t$ [%]				
	0.00	1.00	2.00	3.00	5.00
0.5	$173.949_{70\,(2,1)}$	$164.510_{70\,(2,1)}$	$155.070_{70\,(2,1)}$	$145.631_{70\,(2,1)}$	$126.753_{70\,2,1)}$
1.0	$43.487_{70\,(1,1)}$	$41.127_{70\,(1,1)}$	$38.768_{70\,(1,1)}$	$36.408_{70\,(1,1)}$	$31.688_{70\,(1,1)}$
1.4	$23.798_{70\,(1,1)}$	$22.483_{70\,(1,1)}$	$21.167_{70\,(1,1)}$	$19.851_{70\,(1,1)}$	$17.220_{70\,(1,1)}$
1.44	$22.772_{70\,(1,1)}$	$21.509_{70\,(1,1)}$	$20.246_{70\,(1,1)}$	$18.984_{70\,(1,1)}$	$16.458_{70\,(1,1)}$
1.445	$22.650_{70\,(1,1)}$	$21.394_{70\,(1,1)}$	$20.137_{70\,(1,1)}$	$18.881_{70\,(1,1)}$	$16.368_{70\,(1,1)}$
1.446	$22.643_{1\,(1,1)}$	$21.388_{1\,(1,1)}$	$20.132_{1\,(1,1)}$	$18.876_{1\,(1,1)}$	$16.365_{1\,(1,1)}$
1.45	$22.625_{1\,(1,1)}$	$21.372_{1\,(1,1)}$	$20.119_{1\,(1,1)}$	$18.866_{1\,(1,1)}$	$16.359_{1\,(1,1)}$
2.0	$21.143_{1\,(1,1)}$	$20.096_{1\,(1,1)}$	$19.050_{1\,(1,1)}$	$18.003_{1\,(1,1)}$	$15.910_{1\,(1,1)}$

Looking at Table 1, it seems that if a discussed transition range exists, it is extremely narrow, since even the change in third decimal digit does not alter the optimal solution. Moreover, no changes in the number of equivalent solutions and buckling mode in relation to thickness imperfections are observed.

As a next step, the obtained results are compared with each other. Therefore, the relative decrease $\Delta\lambda_{cr}$ in the buckling load is calculated where the nominal solution is considered as the reference level $\lambda_{cr_{nom}}$, where $\Delta\lambda_{cr} = (|\lambda_{cr} - \lambda_{cr_{nom}}|)/\lambda_{cr_{nom}}$. The results for different tolerance levels $\Delta t/t$ and selected a/b ratios are given in Fig. 2. It is clear that the observed change is linear, since there are no combined $(0°, \pm 45°)$ plies in any of analysed plates. The rate of the buckling load decrease is different, although all the lines stay close to each other. The obtained results indicate a high sensitivity of the optimum solution vs. the assumed lamina thickness tolerance level.

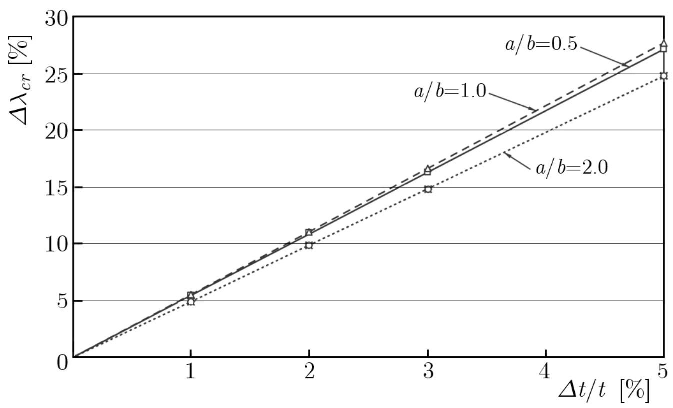

Fig. 2. Relative change of the buckling load factor ($\Delta\lambda_{cr}$ [%]) for different a/b aspect ratios and tolerance levels $\Delta t/t$. The results are calculated with respect to the nominal (ideal) solution

5.2. Biaxial compression

For the purpose of plane stress analysis, the plate dimensions $a = 50.8\,\text{cm}$ and $b = 25.4\,\text{cm}$ ($b/a = 0.5$) are fixed. The lateral loading N_y (see Fig. 1) is varied within a range ($21.875 \leqslant N_y \leqslant 1225\,\text{N/m}$) implying in N_y/N_x ratio changes as $0.125, \ldots, 7.0$. The results of the performed analysis are collected in Table 2 and in Fig. 3, where exemplary optimum configurations for different load ratios and tolerance levels are shown.

Table 2. Results of the buckling load factor λ_{cr} calculations for different levels of lamina thickness t tolerance and different compression loads ratios. Result notation: $\lambda_{\mathrm{n.s.}\,(m,n)}$ – where n.s. is the number of equivalent solutions, m,n are the buckling mode shape parameters

N_y/N_x	$\Delta t/t$ [%]				
	0.00	1.00	2.00	3.00	5.00
0.125	$154.621_{70\,(2,1)}$	$146.231_{70\,(2,1)}$	$137.840_{70\,(2,1)}$	$129.000_{70\,(1,1)}$	$111.752_{20\,(1,1)}$
0.15	$149.000_{20\,(2,1)}$	$140.431_{20\,(1,1)}$	$132.083_{20\,(2,1)}$	$123.939_{20\,(1,1)}$	$106.708_{20\,(1,1)}$
0.20	$137.595_{20\,(2,1)}$	$129.950_{20\,(2,1)}$	$122.277_{20\,(1,1)}$	$114.477_{20\,(1,1)}$	$98.877_{20\,(1,1)}$
0.24	$129.728_{40\,(1,1)}$	$122.472_{40\,(1,1)}$	$115.215_{40\,(1,1)}$	$107.959_{40\,(1,1)}$	$93.445_{40\,(1,1)}$
0.25	$127.946_{6\,(1,1)}$	$120.810_{6\,(1,1)}$	$113.674_{6\,(1,1)}$	$106.480_{6\,(2,1)}$	$92.038_{6\,(2,1)}$
0.50	$94.632_{12\,(1,1)}$	$89.201_{12\,(1,1)}$	$84.004_{24\,(2,1)}$	$78.782_{12\,(1,1)}$	$67.951_{20\,(1,1)}$
1.00	$62.218_{2\,(1,1)}$	$58.564_{6\,(2,1)}$	$55.309_{6\,(2,1)}$	$51.701_{6\,(2,1)}$	$44.825_{2\,(2,1)}$
1.50	$46.355_{2\,(1,1)}$	$43.685_{2\,(1,1)}$	$41.144_{6\,(1,1)}$	$38.565_{4\,(1,1)}$	$33.311_{6\,(1,1)}$
2.00	$36.937_{4\,(2,1)}$	$34.896_{2\,(1,1)}$	$32.679_{2\,(1,1)}$	$30.734_{2\,(1,1)}$	$26.599_{6\,(1,1)}$
2.10	$35.494_{2\,(2,1)}$	$33.411_{2\,(1,1)}$	$31.532_{2\,(1,1)}$	$29.544_{2\,(1,1)}$	$25.499_{2\,(2,1)}$
2.40	$31.760_{2\,(1,1)}$	$29.963_{2\,(1,1)}$	$28.153_{4\,(2,1)}$	$26.408_{2\,(1,1)}$	$22.860_{4\,(2,1)}$
2.50	$30.695_{2\,(2,1)}$	$28.873_{2\,(1,1)}$	$27.258_{2\,(1,1)}$	$25.549_{2\,(1,1)}$	$22.080_{6\,(2,1)}$
2.80	$27.728_{1\,(1,1)}$	$26.313_{2\,(2,1)}$	$24.671_{2\,(1,1)}$	$23.204_{2\,(2,1)}$	$20.074_{2\,(1,1)}$
3.50	$22.552_{1\,(1,1)}$	$21.436_{1\,(1,1)}$	$20.320_{1\,(1,1)}$	$19.105_{2\,(1,1)}$	$16.534_{2\,(1,1)}$
7.00	$11.665_{1\,(1,1)}$	$11.088_{1\,(1,1)}$	$10.510_{1\,(1,1)}$	$9.933_{1\,(1,1)}$	$8.777_{1\,(2,1)}$

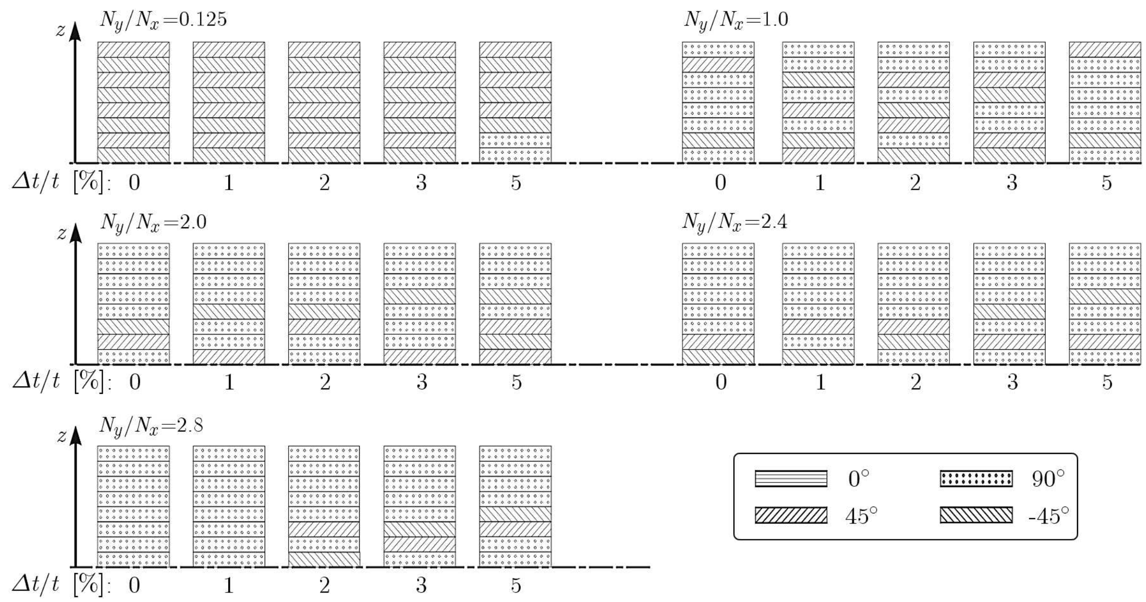

Fig. 3. Optimal ply stacking sequences for different load aspect ratios N_y/N_x and tolerance $\Delta t/t$ levels

Following Table 2, it can be noticed that the change in the thickness tolerance level results in the buckling load decrease; also changes of the plates buckling mode shape and number of equivalent solutions are observed. These are mainly related to the number of ±45° plies present (Fig. 3), since the +45° and –45° plies may be shifted with each other with no impact on the buckling load – cosine function being even.

As arises from Fig. 3, for a very low ratio of biaxial compression ($N_y/N_x = 0.125$) the optimum solution is similar to the uniaxial case presented in Example 1. The difference is that increasing the tolerance Δt level makes the 90° oriented plies to appear and to stay close to the symmetry plane. For intermediate N_y/N_x load ratios, mixed solutions are observed. In the case of nominal solutions composed with mostly $90°$ plies ($N_y/N_x = 2.0, \dots, 2.4$) the increase in the thickness uncertainty level Δt makes the existing $\pm 45°$ layers to shift towards the plate outer surface or even force new $\pm 45°$ layers to appear (i.e. $N_y/N_x = 2.0$ and $\Delta t/t = 0.05$). In such a situation, the already present $\pm 45°$ plies are "not enough" and two additional ones for each side of the symmetry plane are added (to preserve the balance and symmetry constraints the optimizer adds not one but two extra $\pm 45°$ plies).

A similar phenomenon is observed for plates having only $90°$ plies in nominal design ($N_y/N_x = 2.8$). The increase in thickness uncertainty makes $45°$ plies to appear and next to move apart from the symmetry plane. But for high enough load ratios (e.g. $N_y/N_x = 7.0$ which is not shown in Fig. 3) the robust solution stays the same as the nominal design, and all the layers are $90°$ oriented for nominal design and all considered tolerance levels.

Figure 3 shows also that for mixed plies solutions the increase in N_y/N_x ratio makes the existing $\pm 45°$ plies to move towards the symmetry plane. This phenomenon was previously observed for the nominal designs, and now it is also confirmed for robust ones.

Changes in the buckling load parameter λ_{cr} listed in Table 2 for the examined N_y/N_x ratios are presented in Fig. 4. The depicted relative value $\Delta\lambda_{cr}/\lambda_{cr_{nom}}$ is calculated similarly to Example 1. It is clear that the impact of Δt parameter stays fully linear for uni-ply plates (e.g. $N_y/N_x = 7.0$ with $90°$ layers only) and quasi-linear for plates composed of mixed plies (e.g. $N_y/N_x = 0.125$ or $N_y/N_x = 2.5$ both having $\pm 45°$ and $90°$ fibers). The observed quasi-linearity is explained by the results of Example 1 and Fig. 2, where the slopes of lines for uni-ply plates are different, but still close to each other. Therefore, for mixed solutions observed in the bi-axial case and bearing in mind the fact that the individual plies change their position for different Δt (see also Fig. 3) a noticeable quasi-linearity is observed.

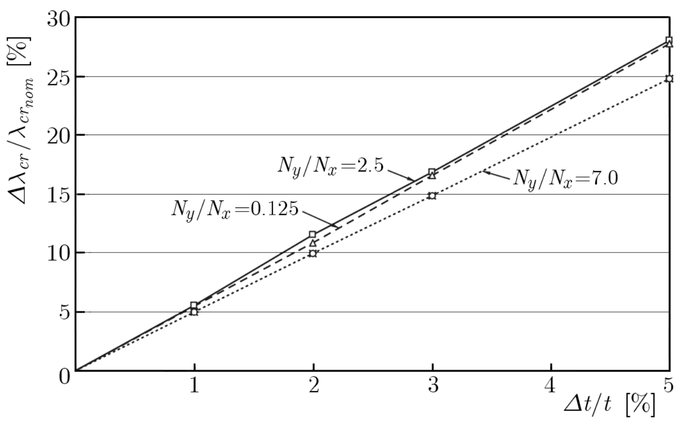

Fig. 4. Relative change of the buckling load factor λ_{cr} for different N_y/N_x load ratios and tolerance level $\Delta t/t$. Calculated with respect to the nominal (ideal) solution

As observed in Fig. 3 most of the final solutions are of different stacking sequence while compared to the nominal design ($\Delta t/t = 0$), so an extra analysis to estimate the benefit of the robust design seems to be justified. Therefore, the question is: what would be the buckling load factor for the nominal solution plate subjected to thicknesses tolerances that are not introduced into the optimization method. This value is denoted in Table 3 by λ'_{cr} and calculated as the buckling load for the nominal optimum solution stacking sequence diminished by possible tolerances and appropriate sensitivities. Next λ'_{cr} is compared to the buckling load amplitude for optimum solutions with tolerances introduced directly into the design algorithm according to the approach presented in this paper (i.e. values given in Table 2). In Fig. 5, a relative change $(\lambda_{cr} - \lambda'_{cr})/\lambda_{cr}$ in the plate buckling load is presented. This corresponds to actual benefit, which is possible to achieve if thickness uncertainties are introduced into the design code, compared to plates designed only for nominal values. It is clear that the best results are achieved for plates of mid N_y/N_x load ratios. For low or high ones (e.g. 0.125 or 7.0), there is no actual benefit of the incorporating thicknesses tolerances in design code. This is because the nominal solutions and robust solutions (the ones that consider tolerances into the design method) are the same and contain only plies of one direction.

Table 3. Buckling load factor comparison for plates with (λ_{cr}) and without (λ'_{cr}) thickness uncertainties introduced into the design code

N_y/N_x		$\Delta t/t$ [%]			
		1.00	2.00	3.00	5.00
0.125	λ_{cr}	$146.231_{(2,1)}$	$137.840_{(2,1)}$	$129.000_{(1,1)}$	$111.752_{(1,1)}$
	λ'_{cr}	$146.231_{(2,1)}$	$137.840_{(2,1)}$	$129.000_{(1,1)}$	$110.832_{(1,1)}$
0.50	λ_{cr}	$89.201_{(1,1)}$	$84.004_{(2,1)}$	$78.782_{(1,1)}$	$67.951_{(1,1)}$
	λ'_{cr}	$89.174_{(2,1)}$	$83.515_{(2,1)}$	$77.857_{(2,1)}$	$66.539_{(2,1)}$
1.00	λ_{cr}	$58.564_{(2,1)}$	$55.309_{(2,1)}$	$51.701_{(2,1)}$	$44.825_{(2,1)}$
	λ'_{cr}	$58.438_{(2,1)}$	$54.455_{(2,1)}$	$50.472_{(2,1)}$	$42.506_{(2,1)}$
2.00	λ_{cr}	$34.896_{(1,1)}$	$32.679_{(1,1)}$	$30.734_{(1,1)}$	$26.599_{(1,1)}$
	λ'_{cr}	$34.422_{(2,1)}$	$31.907_{(2,1)}$	$29.392_{(2,1)}$	$24.362_{(2,1)}$
3.50	λ_{cr}	$21.436_{(1,1)}$	$20.320_{(1,1)}$	$19.105_{(1,1)}$	$16.534_{(1,1)}$
	λ'_{cr}	$21.436_{(1,1)}$	$20.320_{(1,1)}$	$18.911_{(2,1)}$	$15.603_{(2,1)}$
7.00	λ_{cr}	$11.088_{(1,1)}$	$10.510_{(1,1)}$	$9.933_{(1,1)}$	$8.777_{(2,1)}$
	λ'_{cr}	$11.088_{(1,1)}$	$10.510_{(1,1)}$	$9.933_{(1,1)}$	$8.777_{(1,1)}$

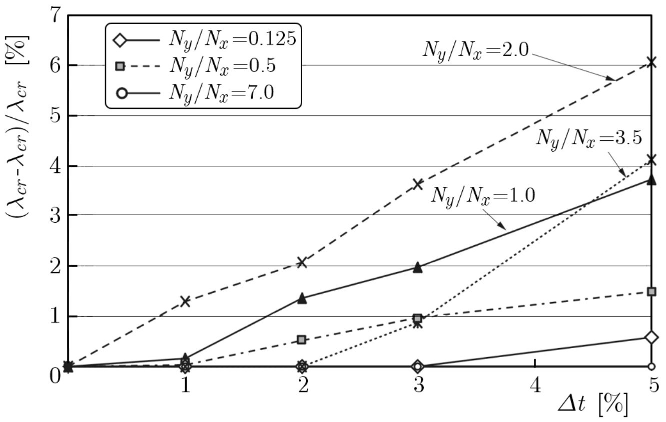

Fig. 5. Relative buckling load change for plates without uncertainties introduced into the design algorithm with respect to robust ones

6. Conclusions

The problem of optimum design of composite laminates considering ply thicknesses tolerances has been addressed by the deterministic approach. Updated formulas for the system performance and cost function problem are derived based on the classical laminated plate theory assumptions. Advantages of the suggested approach with respect to other deterministic methods discussed in "Introduction" section are underlined. The presented idea exploits the assumed tolerances directly into the design optimization code in a relatively simple way, thus computational competitiveness of the method is provided.

The presented theoretical research and further numerical analysis results allow the following specific conclusions to be formulated.

- The algorithm presented in this study offers an efficient and safe approach to incorporate manufacturing tolerances into optimum design. The method accounts for realistic parameter variations, so the design is practical rather than a mathematical abstraction that is of quite limited use in real world.
- Multiple local minima of cost function in composite stacking sequence optimization are confirmed. This applies not only to nominal designs, but also to robust ones. The number of equivalent optima is strictly related to the plate aspect ratio a/b and load ratio N_y/N_x as well.
- Considering ply thicknesses perturbations in the laminate plate optimum design results in a different ply stacking sequence when compared to the solutions of the nominal design problem. This fully justifies the incorporation of thicknesses perturbations in the optimization design algorithm. It is shown that this approach is necessary for mid-N_y/N_x load ratios, but has no practical importance for very low or very high load ratios since the nominal and robust designs in these specific cases are identical.

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