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NONLINEAR VIBRATION MODES OF THE DOUBLE TRACKED ROAD VEHICLE

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Free damped oscillations of a double tracked road vehicle with a nonlinear response of the suspension are considered. A 7-DOF nonlinear model is used to analyze dynamics of the suspension with smooth characteristics. Vibration modes of the system and the corresponding skeleton curves are obtained. Nonlinear vibrations of the quarter-car model are considered for the case of non-smooth characteristics of the shock absorber.

Key words: vehicle suspension, nonlinear normal modes, smooth and non-smooth characteristics

1. Introduction

VEstimation of the vehicle dynamical state is essential for all designing automotive systems that enhance safety and handling characteristics. Many works have been carried out on the dynamic response and the dynamics control with linear vehicle models. However, substantially, the system is nonlinear because it consists of elastic components with nonlinear characteristics. A vehicle can be modelled as a complex multi-body dynamic system. The degree of complexity depends on the aim of modelling. As a rule, the simplest models of vehicles are considered, in particular, the "quarter-car" model (two-DOF system) for studying heave motion (Robson, 1979; Williams, 1997; Haroon *et al.*, 2005), or "half-car" model (four-DOF system) for studying the heave and pitch motions (Hrovat, 1991; Vetturi *et al.*, 1996; Marzbanrad *et al.*, 2004).

A 7-DOF model of the vertical and axial vehicle dynamics is considered here for the case of an independent-solid axle suspension to predict behaviour of the vehicle body and wheels. It is possible to study, by using this model, all principal vehicle motions, namely, heave, pitch and roll (Wong, 1993; Hyo-Jun Kim *et al.*, 2002; Pilipchuk *et al.*, 2006). Correctness of this model is confirmed by comparison with some experiments. It is known that nonlinear effects in the suspension dynamics are important and must be considered if displacements are equal to 0.05-0.1 m, or larger. Here, smooth nonlinear spring characteristics in the front and rear suspensions are taken into account.

Nonlinear normal modes are a generalization of linear normal vibration modes. We can suppose that in many cases, for example after an impact, such regimes are or may be the most important to describe the vehicle nonlinear dynamics, because a transient here is very short in view of the fact that there is a strong dissipation in the system under consideration. Moreover, an approach to the nonlinear normal modes can be successfully used for smooth and for non-smooth characteristics of the car suspension. As we can predetermine, such analysis of nonlinear normal modes for the vehicle model was not made and published previously.

In this paper, the principal 7-DOF model is described in Section 2 at first. In Section 3, normal vibrations are obtained in the linearised model by using a solution to the eigenvalue and eigenvector problem. Then, a concept of the nonlinear normal vibration modes (NNMs) is presented. These NNMs and corresponding skeletons are obtained for the nonlinear 7-DOF system with nonlinear smooth characteristics. In Section 4, nonlinear vibrations of the quarter-car model are considered for the case of a non-smooth characteristic of the shock absorber. Both the piecewise linear characteristic of the absorber and more realistic piecewise cubic shock absorption characteristic are considered. Nonlinear elastic characteristics of the suspension are taken into account in that Section.

Note that analysis of the nonlinear normal modes is crucial for the system because damping is strong here and any transients is a short-term process.

2. Principal model and equations of motion

In order to describe the vertical dynamics of a double-tracked road vehicle, the 7-DOF model (Fig. 1) is used, which is based on works by Wong (1993) and Hyo-Jun Kim *et al.* (2002). The car body is represented as a rigid body. The heave, roll and pitch motions are considered. Here, z is the vertical displacement, α is the pitch angle, β is the roll angle, x_i are the vertical displacements of the ith suspended mass which are equivalent to the wheel, d_1, d_2 are front and rear track widths and l_1, l_2 are front and rear wheel bases.

In this model, the tires are presented as elastic elements with linear characteristics. The suspension is characterised by nonlinear elastic characteristics of the front and rear springs and linear damping characteristics (the shock absorber will be considered later). Typical elastic characteristics are shown in Fig. 2 (some information on such characteristics can be found, for example, in Zhu and Ishitobi (2004)). Elastic forces appearing in the springs $f_1(x)$ and $f_2(x)$ can be correctly approximated by polynomials of the 7-th degree.

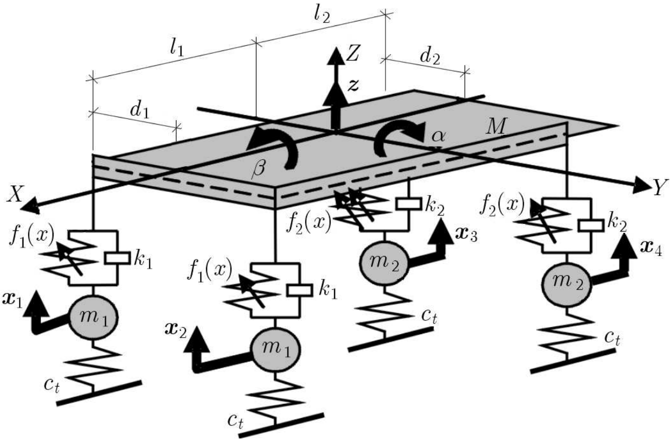

Fig. 1. Model of a double-tracked road vehicle under consideration

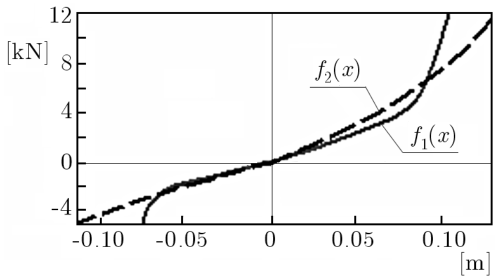

Fig. 2. Typical nonlinear characteristics of the front $f_1(x)$ and rear $f_2(x)$ tracks

One has seven generalized coordinates to describe vibrations of the model. The displacements of the body mass center are described by the vector $\boldsymbol{q} = [z, \alpha, \beta]^\top$, and the displacements of the suspended masses, which are equivalent to the wheels, by the vector $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^\top$. The matrices of the body inertia $\mathbf{M}_C$, of the suspension elements M_S, of the tire stiffness C, and of the damping $\mathbf{K}$ are diagonal. The angular displacements of the body are connected with the displacements of the mass center by the following matrix

$$\mathbf{H} = \begin{bmatrix} 1 & -l_1 & d_1 \\ 1 & -l_1 & -d_1 \\ 1 & l_2 & d_2 \\ 1 & l_2 & -d_2 \end{bmatrix} \tag{2.1}$$

One writes the difference between the displacements of the body and the elements of suspension as $\boldsymbol{U} = \mathbf{H}\boldsymbol{q} - \boldsymbol{x}$, and the difference of velocities as $\boldsymbol{V} = \mathbf{H}\dot{\boldsymbol{q}} - \dot{\boldsymbol{x}}$. A vector of the nonlinear characteristics can be written in the form

$$\boldsymbol{C}_{NL} = [f_1(U_1), f_1(U_2), f_2(U_3), f_2(U_4)]^\top \tag{2.2}$$

where U_i are components of the vector $\boldsymbol{U}$.

Finally, one has the following ODE system (in the matrix form) which describes free nonlinear vibrations of the car

$$\begin{aligned} &\mathbf{M}_C\ddot{\boldsymbol{q}} + \mathbf{H}^\top\boldsymbol{C}_{NL} + \mathbf{H}^\top\mathbf{K}\boldsymbol{V} = \mathbf{0} \\ &M_S\ddot{\boldsymbol{x}} - \boldsymbol{C}_{NL} + C\boldsymbol{x} - \mathbf{K}\boldsymbol{V} = \mathbf{0} \end{aligned} \tag{2.3}$$

Values of the car parameters which were used in the calculations are presented in Table 1.

Table 1. Assumed car parameters

Parameter	Value	Parameter	Value
M	2369 kg	k_2	900 kg/s
I_x	4108 kg m^2	c_t	258000 n/m
I_y	938 kg m^2	l_1	1.459 m
m_1	77 kg	l_2	1.486 m
m_2	108.5 kg	d_1	0.868 m
k_1	700 kg/s	d_2	0.837 m

3. Nonlinear normal modes and transient (smooth characteristics of the suspension)

First, normal vibration modes of the linearised system were obtained by solving the eigenvalue and eigenvector problem. The amplitudes of vibration modes for some concrete parameters of the vehicle are shown in Fig. 3. The frequencies of modes are given at the bottom of this figure. The amplitudes of displacements and angles are presented in m and rad, respectively. Besides, all eigenvectors are normalized. Comparison of motions in time of the full nonlinear system and the corresponding linearised one will be shown later.

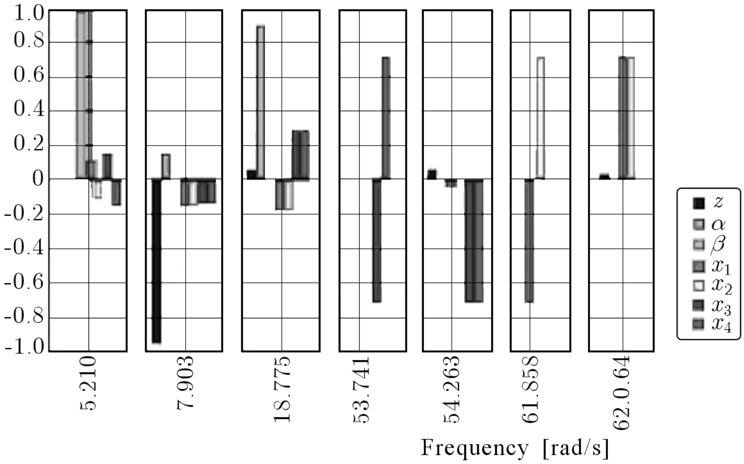

Fig. 3. Amplitudes of seven linear normal modes

The nonlinear normal vibrations modes (NNMs) are a generalization of the normal vibrations of linear systems. In the normal mode, a finite-dimensional system behaves like a conservative one having a single degree of freedom, and all position coordinates can be analytically parameterised by any of them (Rosenberg, 1966; Mikhlin, 1996; Vakakis *et al.*, 1996). In general, NNMs trajectories in a configuration space are curvilinear instead of straight lines as for linear systems.

In Shaw and Pierre (1991, 1993), the authors reformulated the concept of NNMs for a general class of nonlinear discrete oscillators. The analysis is based on computation of invariant manifolds of motion on which the NNMs take place. Some mechanical applications of this approach are presented in Shaw and Pierre (1991, 1993) and other publications by the same authors.

To use this approach, original ODE system (2.3) must be presented in the standard form

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{y} \qquad \frac{d\boldsymbol{y}}{dt} = f(\boldsymbol{x}, \boldsymbol{y}) \tag{3.1}$$

where $\boldsymbol{x} = [x_1, \dots, x_N]^\top$ is the vector of generalized coordinates, $\boldsymbol{y} = [y_1, \dots, y_N]^\top$ is the vector of generalized velocities, and $\boldsymbol{f} = [f_1, \dots, f_N]^\top$ is the vector of forces. One chooses a couple of new independent variables

(u, v), where u is some dominant generalized coordinate, and v is the corresponding generalized velocity. By the Shaw-Pierre approach, the nonlinear normal mode is such a regime when all generalized coordinates and velocities are univalent functions of the selected couple of variables. Choosing the selected couple of variables as the coordinate and velocity with the index 1, one writes the nonlinear normal mode of the form

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_N \\ y_N \end{bmatrix} = \begin{bmatrix} u \\ v \\ X_2(u, v) \\ Y_2(u, v) \\ \vdots \\ X_N(u, v) \\ X_N(u, v) \end{bmatrix} \tag{3.2}$$

Computing derivatives of all variables in system (3.2), and taking into account that $u = u(t)$ and $v = v(t)$, then substituting the obtained expressions to system (3.1), one obtains the following system of partial derivative equations $(i = 1, \ldots, N)$

$$\begin{aligned} \frac{\partial X_i}{\partial u} v + \frac{\partial X_i}{\partial v} f_1(x, y) &= Y_i(u, v) \\ \frac{\partial Y_i}{\partial u} v + \frac{\partial Y_i}{\partial v} f_1(x, y) &= f_i(x, y) \end{aligned} \tag{3.3}$$

One can present the solution to the system in the form of a power series with respect to the new independent variables u and v

$$\begin{aligned} x_i = X_i(u, v) &= a_{1i} u + a_{2i} v + a_{3i} u^2 + a_{4i} uv + a_{5i} v^2 + \ldots \\ y_i = Y_i(u, v) &= b_{1i} u + b_{2i} v + b_{3i} u^2 + b_{4i} uv + b_{5i} v^2 + \ldots \end{aligned} \tag{3.4}$$

Series (3.4) are introduced to equations (3.3), then the coefficients of independent variables of the same degree, are equated. So, a system of recurrent algebraic equations can be written. The coefficients of series (3.4) can be determined from these equations, and as a result, the corresponding nonlinear normal mode is obtained. This procedure permits one to obtain seven NNMs of the system under consideration. Few surfaces, which characterise the first NNM, and the corresponding trajectories of motion on these surfaces for some assumed coordinates are shown in Fig. 4. Here the coordinate z describing the vertical displacement is chosen as the independent variable u.

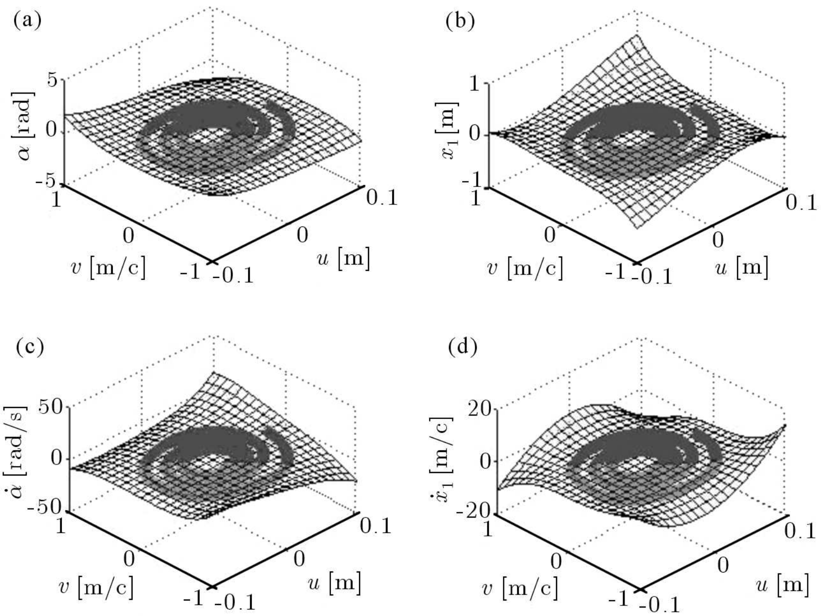

Fig. 4. First vibration mode of the nonlinear system (independent coordinates are $u = z$ and $v = \dot{z}$; initial values: $z = 0.075$, and $\dot{z} = 0$); change of the angle α (a), displacement x_1 (b), angle velocity $\dot{\alpha}$ (c), displacement velocity $\dot{x}_i$ (d) depending on the variables u and v

If the NNM in form (3.4) is obtained, these series are introduced to equations of motion, and the functions $u = u(t)$ and $v = v(t)$ can be obtained too. Numerical calculations show good exactness of the obtained analytical results. To construct skeletons of the NNMs, a harmonic linearisation method together with a continuation procedure are used. Skeletons of the nonlinear system are shown in Fig. 5.

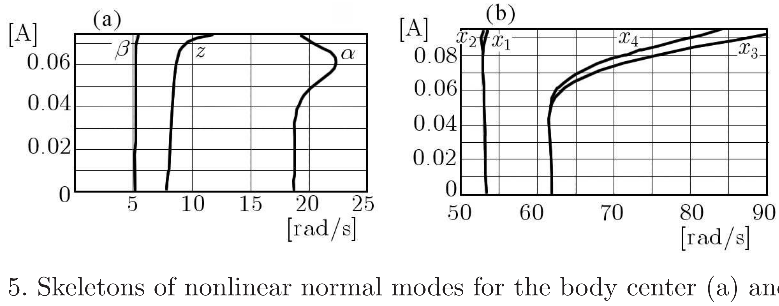

Fig. 5. Skeletons of nonlinear normal modes for the body center (a) and for unsprung masses (b)

Transient and normal vibration modes of the linearised and nonlinear systems for different initial conditions of the car body are shown in Figs. 6-8.

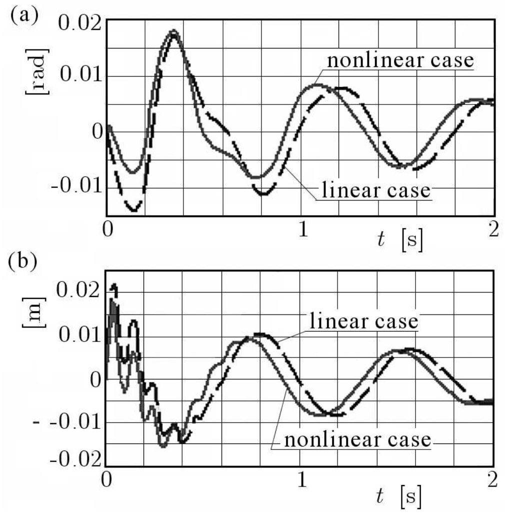

Fig. 6. Comparison of transients for linearised and nonlinear systems for the initial $z = 100\,\mathrm{mm}$; (a) pitch angle α, (b) displacement of the unsprung mass x_1

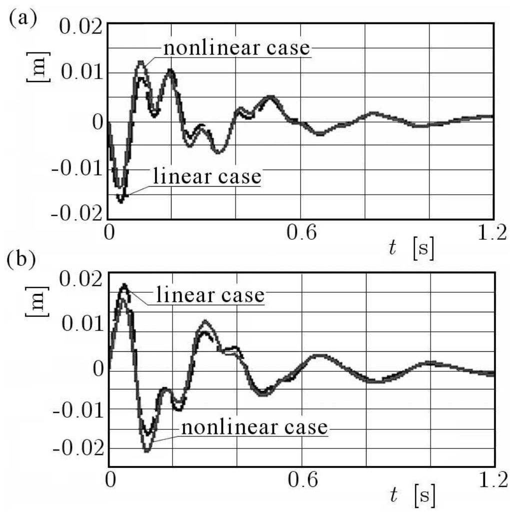

Fig. 7. Displacement of the unsprung mass x_1 (a) and x_3 (b). Comparison of transients for linearised and nonlinear systems for the initial $\alpha = 3.5°$

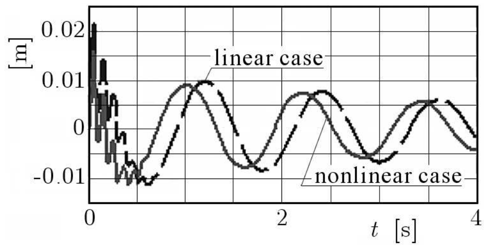

Fig. 8. Displacement of the unsprung mass x_1. Comparison of transients for linearised and nonlinear systems for the initial $\beta = 6°$

One can observe that in the case of large initial displacements, the vibration frequency of the unsprung masses quickly lowers and all nonlinear vibration modes tend to low-frequency vibration modes of the car body.

4. Nonlinear normal modes and transients (shock absorber, non-smooth characteristic)

4.1. Quarter-car model. Equations of motion

To investigate the suspension dynamics taking into account a non-smooth characteristic of the shock absorber, the quarter-car model is considered (Fig. 9).

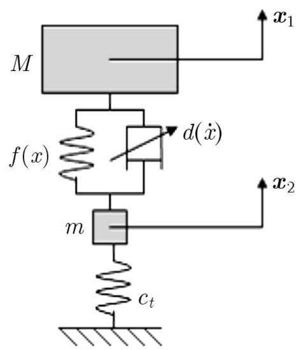

Fig. 9. Quarter-car model

The equations of motion for the quarter-car model are

$$
\begin{aligned}
&M\ddot{x}_1 + f(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) = 0 \\
&m\ddot{x}_2 + f(x_2 - x_1) + d(\dot{x}_2 - \dot{x}_1) + c_t x_2 = 0
\end{aligned}
\tag{4.1}
$$

where $f(x)$ is a stiffness function, $d(\dot{x})$ is a piecewise damping function of the suspension, namely

$$d(\dot{x}) = \begin{cases} d_1(\dot{x}) & \text{for} \quad \dot{x}_1 - \dot{x}_2 < 0 \\ d_2(\dot{x}) & \text{for} \quad \dot{x}_1 - \dot{x}_2 \geqslant 0 \end{cases} \tag{4.2}$$

Values of the car parameters, which were used in the calculations, are given in Table 2.

Table 2. Car parameters used in the "quarter-car model"

Parameter	Value
M	592.25 kg
m	108.5 kg
c_t	258000 n/m
$f(x)$	$55000x$

4.2. Piecewise linear characteristics of the absorber

The characteristic of the damping functions for the case of piecewise linear damping is presented in Fig. 10. This is an approximation of a more exact characteristic, which will be considered later. The stiffness characteristic of the suspension is chosen here as linear.

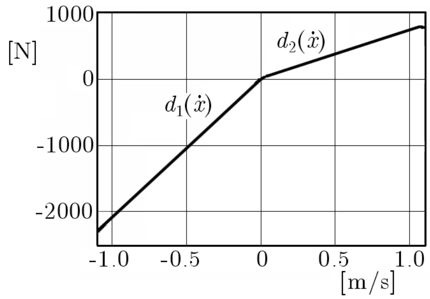

Fig. 10. Piecewise linear shock absorption characteristic of the suspension

Both NNMs obtained by the method described below are shown in Figs. 11 and 12 for some concrete values of the system parameters. One can observe them as motions on places corresponding to the NNMs as well as transients from one place to another one after gap (or "switching") of the piecewise linear damping characteristic. The time representation of these nonlinear normal modes and transients are shown in Figs. 13 and 14.

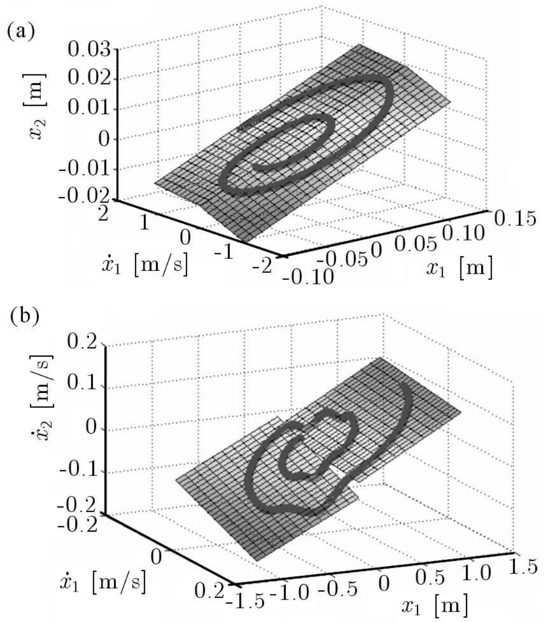

Fig. 11. Change of the displacement x_2 (a) and the displacement velocity $\dot{x}_2$ (depending on u and v). First NNM. Independent variables are: x_1 and the corresponding velocity $\dot{x}_1$ (piecewise linear characteristic)

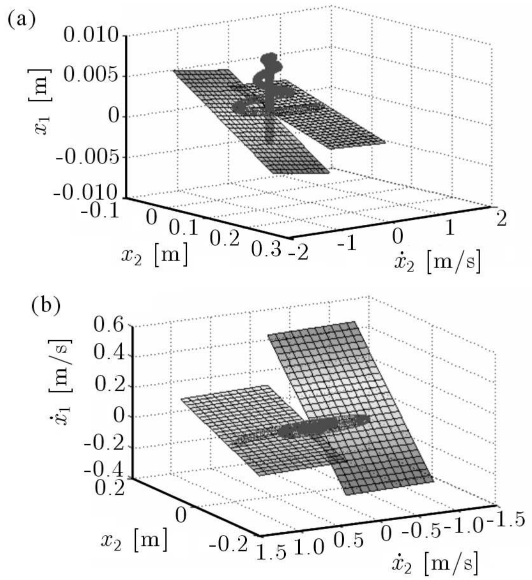

Fig. 12. Change of the displacement x_2 (a) and the displacement velocity $\dot{x}_2$ (depending on u and v). Second NNM. Independent variables are: x_2 and the corresponding velocity $\dot{x}_2$ (piecewise linear characteristic)

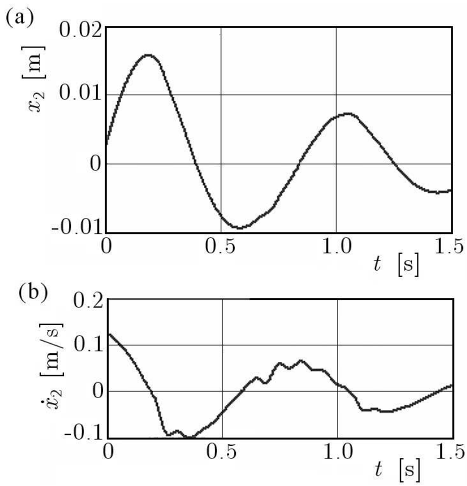

Fig. 13. Change of the displacement x_2 (a) and the displacement velocity $\dot{x}_2$. First NNM and transient

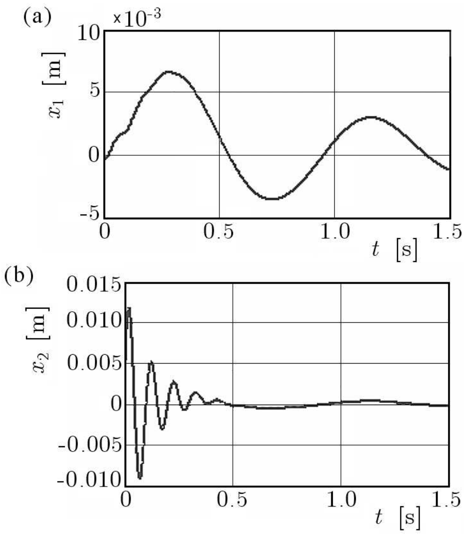

Fig. 14. Change of the displacement x_1 (a) and x_2 (b). Second NNM and transient

More realistic nonlinear suspension characteristics for the stiffness and absorption (for piecewise cubic damping) are displayed in Figs. 15 and 16. There, the car principal parameters correspond to those from Table 2. The corresponding NNMs, obtained in that case, and the transient from one surface to another one after gap (or "switching") in the piecewise cubic damping charac-

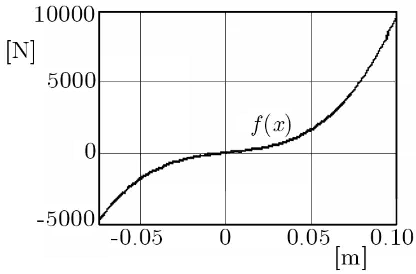

Fig. 15. Nonlinear cubic stiffness characteristic of the suspension

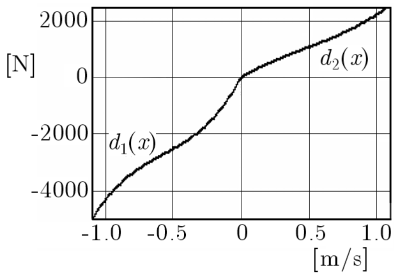

Fig. 16. TPiecewise cubic shock absorption characteristic of the suspension

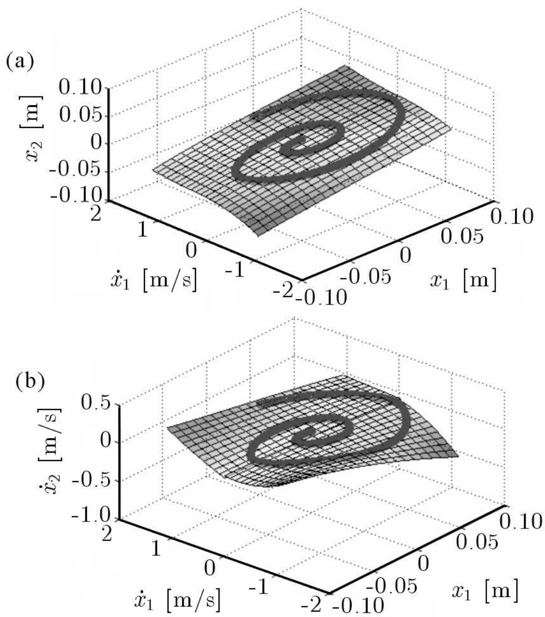

Fig. 17. Change of the displacement x_2 (a) and the displacement velocity $\dot{x}_2$ (depending on u and v). First NNM. Independent variables are x_1 and the corresponding velocity $\dot{x}_1$ (piecewise cubic characteristic)

teristic are shown in Figs. 17 and 18. The first NNM time-representation and transient are given in Fig. 19.

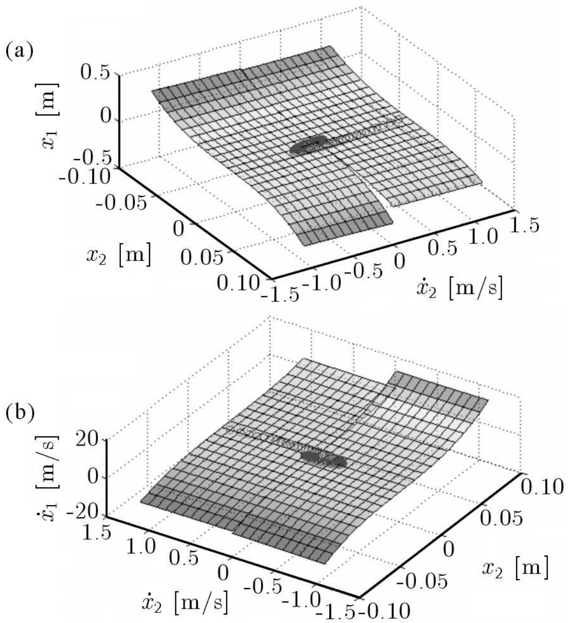

Fig. 18. Change of the displacement x_1 (a) and the displacement velocity $\dot{x}_1$ (depending on u and v). Second NNM. Independent variables are x_2 and the corresponding velocity $\dot{x}_2$ (piecewise cubic system)

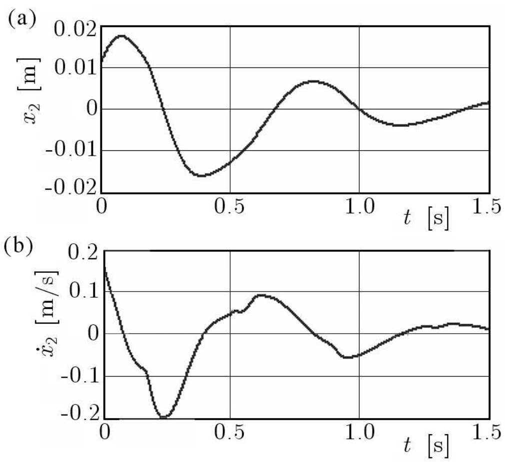

Fig. 19. Change of the displacement x_2 (a) and the displacement velocity $\dot{x}_2$. First NNM and transient

Note that in all cases, the dominant pair of phase variables is chosen as independent coordinates in the analytical representation of NNM.

5. Conclusions

In the paper, nonlinear dynamics of a double-tracked road vehicle with a nonlinear response of the suspension is considered. A 7-DOF nonlinear model is used in the investigation. The concept of nonlinear normal vibration modes was used in the model analysis for the first time. The method of nonlinear normal modes permits one to describe dynamic behaviour of the nonlinear system both for the case of a smooth nonlinear characteristic of the suspension and for the case of a non-smooth characteristic of the shock absorber. In the last case, one can observe a very fast transfer from one nonlinear normal mode to another one after the gap in the non-smooth characteristic. It is possible to conclude that in many cases, the nonlinear normal modes are dominant regimes in the vehicle dynamics.

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Postacie drgań nieliniowych dwuśladowego pojazdu drogowego

Streszczenie

W pracy rozważono tłumione i swobodne drgania dwuśladowego pojazdu drogowego o nieliniowej charakterystyce zawieszenia. Pojazd zamodelowano nieliniowym układem o 7 stopniach swobody, a w analizie jego dynamiki przyjęto gładką charakterystykę zawieszenia. Znaleziono postacie drgań takiego układu i wyznaczono odpowiednie krzywe szkieletowe. Zagadnienie drgań nieliniowych modelu "ćwiartkowego" (o 2 stopniach swobody) rozważono dla przypadku zawieszenia z niegładką charakterystyką amortyzatora.

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