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DYNAMICS OF MULTI-PENDULUM SYSTEMS WITH FRACTIONAL ORDER CREEP ELEMENTS

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A survey as a short review of author's research results in area of dynamics of hybrid systems and analytical dynamics of discrete material particle system containing creep elements described by fractional order derivatives, is presented. Free vibrations of a multi-pendulum system intercoupled by standard light elements and different properties are considered. The corresponding system of an ordinary fractional order as well as integro-differential equations, described dynamics of the multi-pendulum system, are derived and analytically solved. For the case of one pendulum and two pendulum systems containing standard light creep elements with the stress-strain constitutive relation expressed by a fractional order derivative, ordinary differential equations are analytically solved. From the analytical solutions, for the case of the homogeneous two-pendulum system, it is visible that free vibrations under arbitrary initial conditions contain three modes, one pure periodic and two aperiodic expressed by time series expansions. The obtained analytical solution modes are numerically analysed and characteristic vibration modes for different kinetic parameters are graphically presented.

Key words: coupled subsystems, coupled dynamics, multi-pendulum system

1. Introduction

Fast development of material science (Rzhanitsin, 1949; Savin and Ruschisky, 1976) and experimental mechanics as well as of methods of numerical analysis led to the creation of different models of real material bodies (Enelund, 1996) and methods for studying dynamics and processes which happen in them during the traveling of a disturbance through deformable bodies.

The interest in the study of coupled systems (Hedrih, 1999, 2003a, 2004b,c, 2005a,b, 2006a,b,e,f, 2007a,c) as new qualitative systems has grown exponentially over the last few years because of the theoretical challenges involved in the investigation of such systems with hereditary elements (Goroško and Hedrih, 2001, 2007a,b; Goroshko and Puchko, 1977) with stress-strain constitutive relations expressed by integral forms as well as with creep elements with the stress-strain constitutive relation described by fractional order derivatives(Hedrih, 2002a,b, 2003b, 2004c, 2005c, 2006c; Hedrih and Filipovski, 2002).

In the References, monographs by Rzhanitsin (1949) and Savin and Ruschisky (1976) as well as by Goroško and Hedrih (2001, 2007a,b) and paper by Goroshko and Puchko (1977), different approaches to creation of real body models are given. In basic, these approaches contain physical discretization of a body or mathematical discretisation from partial to ordinary differential equations. One such an approach is represented by a model of a discrete system of material particles (Hedrih, 2001, 2003a, 2004b, 2006a), which are connected by certain ties. The number of the particles then increased to create a continuum, their motion and deformable wave propagation is described by partial differential equations. And then, due to impossibility of solving them analytically, an approximation method is used for that purpose. The book by Enelund (1996) contains the same applications with elements of fractional calculus in structural dynamics.

In the monograph by Gorošhko and Hedrih (2001), analytical dynamics of discrete hereditary systems and corresponding solutions was first published as an integral theory such kind of systems. In Goroško and Hedrih (2001, 2007a,b) and Goroshko and Puchko (1977) as well as in the cited monograph, a standard light hereditary element is defined and used as a connecting or coupling rheological element between the material particles of the system.

A series of References by Hedrih (2002b, 2006c, 2007b,e), Hedrih and Filipovski (2002) and [10] present numerous results of research on the properties of vibrations of rods, plates, belts made of different materials. Also, in Hedrih (1999, 2001, 2003a, 2004b, 2005a,b, 2006a,b,e,f, 2007a,c) a series of coupled subsystems and hybrid systems with different material properties or different properties of the standard light elements as discrete as well as distributed coupling elements between deformable bodies, or discrete and continuum subsystems (see Hedrih, 2005a, 2006b) are investigated. In a series of the author's work (Hedrih, 2003b, 2005c, 2006c, 2007b,e), results of research on vibrations of deformable bodies with creep properties described by the stress strain constitutive relation expressed by a fractional order derivative, are presented.

All these engineering problems are, also, mathematical problems and are described by partial differential equations with integral or fractional order derivative terms which can be discretised into a problem of solving of a system of ordinary differential or integro-differential or fractional order differential equations.

In the last decade, an interest to the applied fractional calculus for description of material properties rose. Papers by Enelund (1996), Gorenflo and Mainardi (2000) are recommended as the primary mathematical literature containing the basics of the fractional calculus.

Hedrih (2006a) studied modes of homogeneous chain signals for different kinds of homogenous connections between material mass particles in the chain and different chain boundary conditions. A finite number of coupled fractional order differential equations of creep vibrations of connected multi-mass particles into a homogeneous chain system has been derived. The mass particles were connected by standard creeping light elements and the constitutive relations of the stress-strain state were expressed through terms of the fractional order derivatives. The analytical solution to the system of coupled fractional order differential equations of the corresponding dynamical free creep processes was obtained by using Laplace's transform method and trigonometrical method (see Rašković, 1965). By using inverse Laplace's transform, time series functions as particular mode components of the solution were found. By using those component visualizations, analysis of the dynamical creep component processes in mass particle displacements were done. Also, analysis and a comparison between signals in the corresponding homogeneous chains with ideal elastic or visco-elastic standard light elements between the material particles were pointed out.

2. Light standard elements

The *basic elements* of a discrete material system with interconnections between material particles as well as the mathematical multi-pendulum system considered in this paper, are:

- *Material particle* with mass m_k having one degree of freedom, defined by the following independent and generalised angular coordinate φ_k, for $k = 1, 2, \ldots, n$.
- *Light standard coupling element* of negligible mass in the form of an axially stressed rod without bending, having the ability to resist deformation under static and dynamic conditions. The constitutive relation between the restitution force P and elongation x can be written down in the form $f_{psr}(P, \dot{P}, x, \dot{x}, x_t^{\alpha}, \mathsf{D}, \mathsf{D}_t^{\alpha}, \mathsf{J}, n, c, \widetilde{c}, \mu, \alpha, c_{\alpha}, T, U, \ldots) = 0$, where

D, D_t^α and J are fractional order differential and integral operators (Goroško and Hedrih, 2001; Hedrih, 2006a), which find their justification in experimental verifications of material behaviour (Goroško and Hedrih, 2001; Rzhanitsin, 1949; Savin and Ruschisky, 1976), while n, c, $\widetilde{c}$, μ, c_α, α, ... are material constants, which are also experimentally determined.

For every single standard coupling light element of negligible mass, we shall define a specific stress-strain constitutive relation-law of dynamics. This means that we will define the stress-strain constitutive relation as determinants of forces and/or changes of forces with distances between two constrained-coupled material particles and with changes of the distances in time, with accuracy up to constants which depend on the accuracy of their determination through an experiment.

The accuracy of those constants, forces and elongations would depend not only on the nature of an object, but also on the knowledge of very complex stress-strain relations to be dealt with (see Goroško and Hedrih, 2001; Hedrih, 2006a). In this paper, we shall use three such light standard constraint-coupling elements, and they will be:

- *Light standard ideally elastic coupling element* for which the stress-strain relation for the restitution force as a function of element axial elongation is given by a linear relation of the form

$$P = -cy \tag{2.1}$$

  as well as given by a nonlinear relation of the form

$$P = -cy - \widehat{c}y^3 \tag{2.2}$$

  where c is the rigidity coefficient or elasticity coefficient for the linear, and $\widehat{c} = \varepsilon\chi c$ for the nonlinear functional stress-strain constitutive relation between the force and rheological coordinate of axial deformation of the standard elastic element. In a natural state, non-stressed by a force and undeformed, the force and deformation of such an element are equal to zero.

- *Light standard hereditary constraint element* for which the stress-strain constitutive relation for the restitution force as a function of element elongation (rheological coordinate) is given:

  — in a differential form

$$\mathsf{D}P = \mathsf{C}y \qquad \text{or} \qquad n_h\dot{P}(t) + P(t) = n_h c_h y(t) + \widetilde{c}_h y(t) \tag{2.3}$$

where the following differential operators are introduced

$$\mathsf{D} = n_h \frac{d}{dt} + 1 \qquad\qquad \mathsf{C} = n_h c_h \frac{d}{dt} + \widetilde{c}_h \tag{2.4}$$

and n_h is the relaxation time and c_h, $\widetilde{c}_h$ are rigidity coefficients – momentary and prolonged.

— in an integral form

$$P(t) = c_h \Big[y(t) - \int\limits_0^t \mathsf{R}(t-\tau) y(\tau)\, d\tau \Big] \tag{2.5}$$

where

$$\mathsf{R}(t-\tau) = \frac{c_h - \widetilde{c}_h}{n_h c_h} \exp\Big[-\frac{1}{n_n}(t-\tau) \Big] \tag{2.6}$$

is the relaxation kernel (or resolvent).

— in an integral form

$$y(t) = \frac{1}{c_h} \Big[P(t) + \int\limits_0^t \mathsf{K}(t-\tau) P(\tau)\, d\tau \Big] \tag{2.7}$$

where

$$\mathsf{K}(t-\tau) = \frac{c_h - \widetilde{c}_h}{n_h c_h} \exp\Big[-\frac{\widetilde{c}_h}{n_h c_h}(t-\tau) \Big] \tag{2.8}$$

is kernel of rheology (or retardation).

- *Light standard creep coupling element* for which the stress-strain constitutive relation for the restitution force as a function of element elongation (rheological coordinate) is given by a fractional order derivative term (see Hedrih, 2006a) in the form

$$P(t) = -\{c_0 x(t) + c_\alpha \mathsf{D}_t^\alpha [x(t)]\} \tag{2.9}$$

where $\mathsf{D}_t^\alpha[\cdot]$ is the fractional order differential operator of the α-th derivative with respect to time t in the following form

$$\mathsf{D}_t^\alpha [x(t)] = \frac{d^\alpha x(t)}{dt^\alpha} = x^{(\alpha)}(t) = \frac{1}{\Gamma}(1-\alpha) \frac{d}{dt} \int\limits_0^t \frac{x(\tau)}{(t-\tau)^\alpha}\, d\tau \tag{2.10}$$

where c_0, c_α are rigidity coefficients – momentary and prolonged ones, and α is a rational number between 0 and 1, $0 < \alpha < 1$, depending on the material properties found experimentally.

3. Multi-pendulum system with intercoupling by standard light elements

In this paper, we shall define a discrete continuum mathematical pendulum chain as a system of material particles intercoupled by the *light standard coupling elements* (*elastic*, *hereditary* or *creep*), presented in Figures 1 and 2, and which are, in the natural state, on the defined interdistances (when the coupling elements are unstressed and without prehistory as well as without memory before the initial moment of the system motion).

The discrete continuum multi-pendulum chain system is an *ideally elastic* chain if its material particles are interconnected by light standard ideally elastic coupling elements with the stress-strain constitutive relation expressed by (2.1) for the linear or (2.2) for the nonlinear case. The discrete continuum multi-pendulum chain system is a *standard hereditary chain* if its material particles are interconnected by light standard hereditary elements with the stress-strain constitutive relation expressed by one of sets (2.3)-(2.8). The discrete continuum multi-pendulum chain system is a *standard creep chain* if its material particles are interconnected by light standard creep elements with the stress-strain constitutive relation expressed by (2.9) and (2.10).

We shall define a *discrete homogeneous mathematical multi-pendulum chain system* as a system of discrete material particles, same masses, which can rotate along corresponding circular arches with the same radius ℓ and centers on the one horizontal line. All the system is in the vertical plane and in the gravitational field and the material particles are intercoupled by the same type of sets of the parallel standard light elements and on the same distance of the corresponding fixed points of pendula.

The number of degrees of freedom of each of these multi-pendulum chains is equal to the number of material particles in it since we accept the previously defined character of the system.

Further, for a special rheolinear case, we introduce a hypotheses about homogeneity of the discrete continual chain, about small deformations of light standard coupling elements and small displacements of the material particles.

Also, we introduce a hypothesis that the homogenous discrete continuum chain, is in natural, non-stressed state, before the initial moment of motion, i.e. that the light standard coupling elements do not have prehistory nor the memory of the stress-strain state. With these hypotheses, we shall direct our research to the dynamics of chain-like homogenous multi-pendulum systems.

4. Thermo-rheological coupled multi-pendulum system

In Fig. 1, a thermo-rheological system containing a finite number of coupled pendula is presented. We take into consideration the finite number of coupled mathematical pendula presented in Fig. 1 as a system with material particles m_k with the length ℓ_k and with finite numbers (n) of degrees of freedom defined by the generalized coordinates φ_k, $k = 1, 2, \ldots, m$, and the standard light thermo-visco-elastic elements thermo-modified by temperature $T_k(t)$, and the coefficient of thermo-dilatation c_{Tk} coupling the pendula at the distance ℓ_{Tk}, parallelly coupled but temperature isolated, and with the standard light nonlinear springs with coefficients of the linear and nonlinear rigidity respectively denoted by c_k, and $\widehat{c} = \varepsilon\chi_k c_k$, where ε is a small parameter.

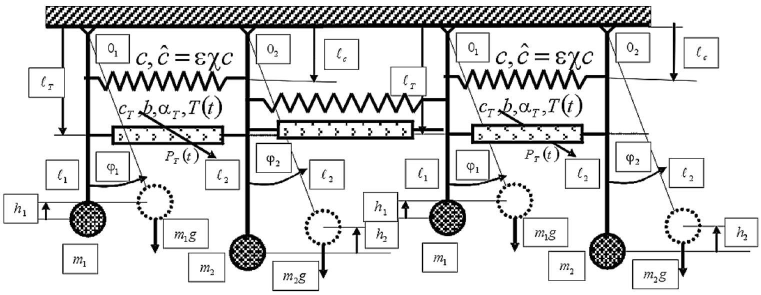

Fig. 1. System with four pendula interconnected by standard light thermo-modified hereditary element and nonlinear springs

Now, we take into account that these standard light thermo-visco-elastic elements, with natural lengths ℓ_{0k} thermo-modified by the temperature $T_k(t)$ are in a dynamical state, and that we do not neglect the thermo-modification of the element strain. We can write that the thermo-dilatation is $\Delta\ell_{0k} = \alpha_{Tk}T_k(t)(\ell_{0k} + x_k)$ and that the constitutive relation of the thermo-elastic stress-strain state is expressed in the following form

$$\begin{aligned} P_{her(k)}(t) &= -c_{T(k)}[\Delta\ell_{0(k)} + \ell_{T(k)}(\varphi_{k+1} - \varphi_k)] = \\ &= -c_{T(k)}\ell_{T(k)}(\varphi_{k+1} - \varphi_k)[1 + \alpha_{T(k)}T_{(k)}(t)] - c_{T(k)}\alpha_{T(k)}\ell_{0(k)}T_{(k)}(t) \end{aligned} \tag{4.1}$$

and the forces of the nonlinear spring between pendula are in the following form

$$P_{nl(k)}(t) = -c_{(k)}[\ell_{c(k)}(\varphi_{k+1} - \varphi_k) + \varepsilon\chi_k\ell_{c(k)}^3(\varphi_{k+1} - \varphi_k)^3] \tag{4.2}$$

and the forces of damping are

$$P_{d(k)}(t) = -b_k\ell_{b(k)}(\dot{\varphi}_{k+1} - \dot{\varphi}_k) \tag{4.3}$$

For the *light standard creep coupling element* between two pendula, the stress-strain relation for the restitution force as a function of the element elongation is given by fractional order derivatives in the form

$$\begin{aligned} P_{k.k+1}(t) &= -\{c_{0k}[\ell_{c(k)}(\varphi_{k+1}(t)-\varphi_k(t))] + c_{\alpha k}\mathsf{D}_t^{\alpha}[\ell_{c(k)}(\varphi_{k+1}(t)-\varphi_k(t))]\} = \\ &= -\{c_{0k}\ell_{c(k)}[\varphi_{k+1}(t)-\varphi_k(t)] + c_{\alpha k}\ell_{c(k)}\langle \mathsf{D}_t^{\alpha}[\varphi_{k+1}(t)] - \mathsf{D}_t^{\alpha}[\varphi_k(t)]\rangle\} \end{aligned} \tag{4.4}$$

where $\mathsf{D}_t^{\alpha}[\cdot]$ is the fractional order differential operator of the α-th derivative with respect to time t in form (2.10), c_{0k}, $c_{\alpha k}$ are rigidity coefficients – momentary and prolonged ones, and α_k is a rational number $0 < \alpha_k < 1$.

For the *light standard hereditary constraint element* between two pendula, the stress-strain relation for the restitution force as a function of the element elongation is given by integral term in the form

$$P_{her(k.k+1)}(t) = c_{hk}\Big[\ell_{c(k)}[\varphi_{k+1}(t)-\varphi_k(t)] - \int_0^t \mathsf{R}_k(t-\tau)\ell_{c(k)}[\varphi_{k+1}(\tau)-\varphi_k(\tau)]\,d\tau\Big] \tag{4.5}$$

where

$$\mathsf{R}_k(t-\tau) = \frac{c_{hk}-\widetilde{c}_{hk}}{n_{hk}c}\exp\Big[-\frac{1}{n_{hk}}(t-\tau)\Big] \tag{4.6}$$

is the relaxation kernel (or resolvent), and where c_{hk}, $\widetilde{c}_{hk}$ are momentary and prolonged rigidity coefficients and n_{hk} is the relaxation time of an element.

5. Governing equations of the multi-pendulum system – general case

Now, we take into account that the pendula are intercoupled by parallel coupled sets of the standard light elements of different properties, as it was presented in the previous part of this paper. Suppose that there are n pendula, and in the equilibrium state the pendulum system is in the vertical position as presented in Fig. 1, i.e. when all generalized coordinates are equal to zero, $\varphi_{k,eq} = 0$.

The generalized forces corresponding to the generalized coordinates φ_k between two pendula are

$$\begin{aligned} Q_{her(k)}(t) &= P_{her(k,k+1)}(t)\ell_{T(k)} = \\ &= -c_{T(k)}\ell_{T(k)}^2(\varphi_{k+1}-\varphi_k)[1+\alpha_{T(k)}T_{(k)}(t)] - c_{T(k)}\alpha_{T(k)}\ell_{T(k)}\ell_{0(k)}T_{(k)}(t) \end{aligned}$$

$$
\begin{aligned}
&Q_{nl(k1)}(t) = P_{nl(k,k+1)}(t)\ell_{c(k)} = -c_{(k)}\ell_{c(k)}[\ell_{c(k)}(\varphi_{k+1}-\varphi_k) + \\
&\quad +\varepsilon\chi_k\ell_{c(k)}^3(\varphi_{k+1}-\varphi_k)^3] \\
&Q_{d(k)}(t) = P_{d(k,k+1)}(t)\ell_{b(k)} = -b_k\ell_{b(k)}^2(\dot{\varphi}_{k+1}-\dot{\varphi}_k) \\
&Q_{cr(k)}(t) = P_{cr(k.k+1)}(t)\ell_{0(k)} = \\
&\quad = -\ell_{0(k)}^2\{c_{0k}[\varphi_{k+1}(t)-\varphi_k(t)] + c_{\alpha k}\langle \mathsf{D}_t^\alpha[\varphi_{k+1}(t)] - \mathsf{D}_t^\alpha[\varphi_k(t)]\rangle\} \\
&Q_{her(k.)}(t) = P_{her(k.k+1)}(t)\ell_{h(k)} = c_{hk}\ell_{h(k)}^2\Big[[\varphi_{k+1}(t)-\varphi_k(t)] + \\
&\quad -\int\limits_0^t \mathsf{R}_k(t-\tau)[\varphi_{k+1}(\tau)-\varphi_k(\tau)]\,d\tau\Big] \\
&Q_{g,k} = \frac{\partial E_{p\varphi n}}{\partial \varphi_k} = m_k g\ell_k \sin\varphi_k \approx m_k g\ell_k\Big(\frac{\varphi_k}{1!} - \frac{\varphi_k^3}{3!} + \frac{\varphi_k^5}{5!} - \frac{\varphi_k^7}{7!} + \frac{\varphi_k^9}{9!} + \ldots\Big)
\end{aligned}
\tag{5.1}
$$

The system of governing differential equations of the thermo-rheological coupled multi-pendulum system presented in Fig. 1, is in the following form

$$
\begin{aligned}
m_k\ell_k^2\ddot{\varphi}_k = &-c_{Tk-1}\ell_{Tk-1}^2(\varphi_k-\varphi_{k-1})[1+\alpha_{Tk-1}T_{k-1}(t)] + \\
&-c_{Tk-1}\alpha_{Tk-1}\ell_{0l-1}\ell_{Tk-1}T_{k-1}(t) - b_{k-1}\ell_{bk-1}^2(\dot{\varphi}_k-\dot{\varphi}_{k-1}) + \\
&-m_k g\ell_k\Big(\frac{\varphi_k}{1!} - \frac{\varphi_k^3}{3!} + \frac{\varphi_k^5}{5!} - \frac{\varphi_k^7}{7!} + \frac{\varphi_k^9}{9!} + \ldots\Big) + \\
&-c_{k-1}\ell_{ck-1}[\ell_{k-1}(\varphi_k-\varphi_{k-1}) + \varepsilon\chi_{k-1}\ell_{ck-1}^3(\varphi_k-\varphi_{k-1})^3] + \\
&+c_{Tk}\ell_{Tk}^2(\varphi_{k+1}-\varphi_k)[1+\alpha_{Tk}T_k(t)] + c_{Tk}\alpha_{Tk}\ell_{0k}\ell_{Tk}T_k(t) + \\
&+b_k\ell_{bk}^2(\dot{\varphi}_{k+1}-\dot{\varphi}_k) + c_k\ell_{ck}[\ell_{ck}(\varphi_{k+1}-\varphi_k) + \varepsilon\chi_k\ell_{ck}^3(\varphi_{k+1}-\varphi_k)^3] + \\
&+\ell_{c(k)}\{c_{0k}\ell_{c(k)}[\varphi_{k+1}(t)-\varphi_k(t)] + c_{\alpha k}\ell_{c(k)}\langle \mathsf{D}_t^\alpha[\varphi_{k+1}(t)] - \mathsf{D}_t^\alpha[\varphi_k(t)]\rangle\} + \\
&-\ell_{c(k-1)}\{c_{0k-1}\ell_{c(k-1)}[\varphi_k(t)-\varphi_{k-1}(t)] + \\
&+c_{\alpha k-1}\ell_{c(k-1)}\langle \mathsf{D}_t^\alpha[\varphi_k(t)] - \mathsf{D}_t^\alpha[\varphi_{k-1}(t)]\rangle\} + c_{hk}\ell_{c(k)}\Big[\ell_{c(k)}[\varphi_{k+1}(t)-\varphi_k(t)] + \\
&-\int\limits_0^t \mathsf{R}_k(t-\tau)\ell_{c(k)}[\varphi_{k+1}(\tau)-\varphi_k(\tau)]\,d\tau\Big] + \\
&-c_{hk-1}\ell_{c(k-1)}\Big[\ell_{c(k-1)}[\varphi_k(t)-\varphi_{k-1}(t)] + \\
&-\int\limits_0^t \mathsf{R}_{k=1}(t-\tau)\ell_{c(k-1)}[\varphi_k(\tau)-\varphi_{k-1}(\tau)]\,d\tau\Big]
\end{aligned}
\tag{5.2}
$$

where $k = 1, 2, \ldots, n$, $\varphi_0 = 0$, and $\varphi_{n+1} = 0$.

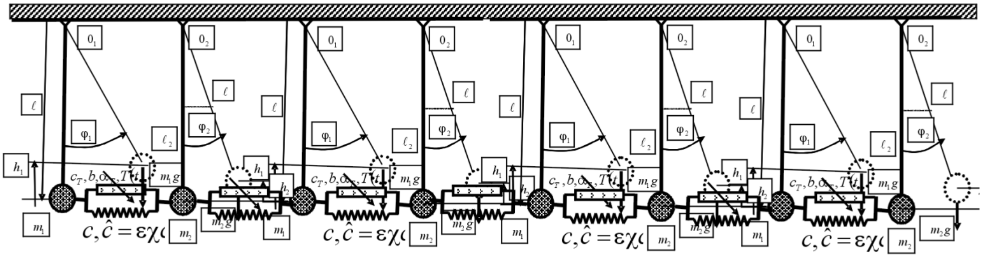

Fig. 2. System with "chain" pendula interconnected by standard light thermo-modified hereditary elements and creep elements

A secial case is a homogeneous multi-pendulum system shown in Fig. 2 for all equal lengths ℓ and the same coupling sets with parallelly coupled standard light elements between the pendula.

After introducing the following notations

$$\omega_0^2=\frac{c}{m}\qquad \omega_{0T}^2=\frac{c_T}{m}\qquad \widetilde{\omega}_0^2=\frac{g}{\ell}\qquad \gamma=\alpha_T T_0$$
$$2\delta=\frac{b}{m}\qquad h_0=\frac{\alpha_T\ell_0 T_0}{\ell}\qquad \widetilde{T}(t)=\frac{1}{T_0}T(t)\qquad \ddot{\chi}=\chi\ell^2$$
$$\omega_{00}^2=\frac{c_0}{m}\qquad \omega_{0\alpha}^2=\frac{c_\alpha}{m}\qquad \omega_{0h}^2=\frac{c_h}{m}$$

the previous system of equations can be transformed into the following

$$\begin{aligned}
&\ddot{\varphi}_1+\widetilde{\omega}_0^2\varphi_1-\omega_0^2(\varphi_2-\varphi_1)-\omega_{00}^2(\varphi_2-\varphi_1)-\omega_{0T}^2(\varphi_2-\varphi_1)[1+\gamma\widetilde{T}(t)]+\\
&\quad-2\delta(\dot{\varphi}_2-\dot{\varphi}_1)=\omega_{0T}^2h_0\widetilde{T}(t)-\widetilde{\omega}_0^2\Big(-\frac{\varphi_1^3}{3!}+\frac{\varphi_1^5}{5!}-\frac{\varphi_1^7}{7!}+\frac{\varphi_1^9}{9!}+\ldots\Big)+\\
&\quad+\varepsilon\omega_0^2\widetilde{\chi}(\varphi_2-\varphi_1)^3+\omega_{0\alpha}^2\{\mathsf{D}_t^\alpha[\varphi_2(t)-\mathsf{D}_t^\alpha[\varphi_1(t)]\}+\\
&\quad+\omega_{0h}^2\Big[[\varphi_2(t)-\varphi_1(t)]-\int_0^t\mathsf{R}(t-\tau)[\varphi_2(\tau)-\varphi_1(\tau)]\,d\tau\Big]
\end{aligned}$$

...

$$\begin{aligned}
&\ddot{\varphi}_k+\widetilde{\omega}_0^2\varphi_k+\omega_0^2(\varphi_k-\varphi_{k-1})+\omega_{00}^2(\varphi_k-\varphi_{k-1})-\omega_{00}^2(\varphi_{k+1}-\varphi_k)+\\
&\quad-\omega_0^2(\varphi_{k+1}-\varphi_k)+\omega_{0T}^2(\varphi_k-\varphi_{k-1})[1+\gamma\widetilde{T}(t)]+\\
&\quad-\omega_{0T}^2(\varphi_{k+1}-\varphi_k)[1+\gamma\widetilde{T}(t)]+2\delta(\dot{\varphi}_k-\dot{\varphi}_{k-1})-2\delta(\dot{\varphi}_{k+1}-\dot{\varphi}_k)=\\
&\quad=-\omega_{0T}^2h_0\widetilde{T}(t)+\omega_{0T}^2h_0\widetilde{T}(t)++\widetilde{\omega}_0^2\Big(\frac{\varphi_k^3}{3!}-\frac{\varphi_k^5}{5!}+\frac{\varphi_k^7}{7!}-\frac{\varphi_k^9}{9!}+\ldots\Big)+\\
&\quad-\varepsilon\omega_0^2\widetilde{\chi}(\varphi_k-\varphi_{k-1})^3+\varepsilon\omega_0^2\widetilde{\chi}(\varphi_{k+1}-\varphi_k)^3-\omega_{0\alpha}^2\{\mathsf{D}_t^\alpha[\varphi_k(t)]-\mathsf{D}_t^\alpha[\varphi_{k-1}(t)]\}+
\end{aligned}$$

$$
\begin{aligned}
&+\omega_{0\alpha}^2\{\mathsf{D}_t^\alpha[\varphi_{k+1}(t)]-\mathsf{D}_t^\alpha[\varphi_k(t)]\}+ \\
&+\omega_{0h}^2\ell\Big[[\varphi_{k+1}(t)-\varphi_k(t)]-\int\limits_0^t \mathsf{R}(t-\tau)[\varphi_{k+1}(\tau)-\varphi_k(\tau)]\,d\tau\Big]+ \\
&-\omega_{0h}^2\Big[[\varphi_k(t)-\varphi_{k-1}(t)]-\int\limits_0^t \mathsf{R}(t-\tau)[\varphi_k(\tau)-\varphi_{k-1}(\tau)]\,d\tau\Big]
\end{aligned}
\tag{5.3}
$$

$$\ldots$$

$$
\begin{aligned}
&\ddot{\varphi}_n+\widetilde{\omega}_0^2\varphi_n+\omega_0^2(\varphi_n-\varphi_{n-1})+\omega_{0T}^2(\varphi_n-\varphi_{n-1})[1+\gamma\widetilde{T}(t)]+ \\
&+2\delta(\dot{\varphi}_n-\dot{\varphi}_{n-1})=-\omega_{0T}^2h_0\widetilde{T}(t)+\widetilde{\omega}_0^2\Big(\frac{\varphi_n^3}{3!}-\frac{\varphi_n^5}{5!}+\frac{\varphi_n^7}{7!}-\frac{\varphi_n^9}{9!}+\ldots\Big)+ \\
&-\varepsilon\omega_0^2\widetilde{\chi}(\varphi_n-\varphi_{n-1})^3-\omega_{0\alpha}^2\{\mathsf{D}_t^\alpha[\varphi_n(t)]-\mathsf{D}_t^\alpha[\varphi_{n-1}(t)]\}+ \\
&-\omega_{0h}^2\Big[[\varphi_n(t)-\varphi_{n-1}(t)]-\int\limits_0^t \mathsf{R}(t-\tau)[\varphi_n(\tau)-\varphi_{m-1}(\tau)]\,d\tau\Big]
\end{aligned}
$$

The basic linear ordinary differential equations of the previous system for the homogeneous case are in the following form

$$
\begin{aligned}
&\ddot{\varphi}_1+(\widetilde{\omega}_0^2+\omega_0^2+\omega_{00}^2+\omega_{0h}^2+\omega_{0T}^2)\varphi_1-(\omega_0^2+\omega_{00}^2+\omega_{0h}^2+\omega_{0T}^2)\varphi_2=0 \\
&\ldots \\
&\ddot{\varphi}_k-(\omega_0^2+\omega_{00}^2+\omega_{0h}^2+\omega_{0T}^2)\varphi_{k-1}+(\widetilde{\omega}_0^2+2\omega_0^2+2\omega_{00}^2+2\omega_{0h}^2+2\omega_{0T}^2)\varphi_k+ \\
&\qquad-(\omega_0^2+\omega_{00}^2+\omega_{0h}^2+\omega_{0T}^2)\varphi_{k+1}=0 \\
&\ldots \\
&\ddot{\varphi}_n-(\omega_0^2+\omega_{00}^2+\omega_{0h}^2+\omega_{0T}^2)\varphi+(\widetilde{\omega}_0^2+\omega_0^2+\omega_{00}^2+\omega_{0h}^2+\omega_{0T}^2)\varphi_n=0
\end{aligned}
\tag{5.4}
$$

By introducing the following notations

$$
\omega_{0cT}^2=(\omega_0^2+\omega_{00}^2+\omega_{0h}^2+\omega_{0T}^2) \qquad\qquad u=\frac{\widetilde{\omega}^2-\widetilde{\omega}_0^2}{\widetilde{\omega}_{0cT}^2}
\tag{5.5}
$$

formally for obtaining eigen amplitude vectors of previous system (5.4), it is possible to write matrix equations (see Hedrih, 2004b, 2006a)

$$
(\widetilde{\mathbf{C}}-u\widetilde{\mathbf{A}})\boldsymbol{A}=\begin{bmatrix}
1-u & -1 & 0 & 0 & 0 \\
-1 & 2-u & -1 & 0 & 0 \\
0 & -1 & 2-u & -1 & 0 \\
0 & 0 & -1 & 2-u & -1 \\
0 & 0 & 0 & -1 & 1-u
\end{bmatrix}
\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{bmatrix}=0
\tag{5.6}
$$

and by the use of the trigonometrical method (see Rašković, 1965; Hedrih, 2006a), for a free homogeneous coupled pendulum system, one obtains the eigen numbers in the form

$$u_s = \frac{\omega_s^2 - \widetilde{\omega}_0^2}{\widetilde{\omega}_{0cT}^2} = 4\sin^2\frac{s\pi}{2n} \qquad s = 0, 1, 2, \dots, n-1 \tag{5.7}$$

with the eigen frequencies

$$\omega_s = \sqrt{\widetilde{\omega}_0^2 + 4(\omega_{0c}^2 + \omega_{0T}^2)\sin^2\frac{s\pi}{2n}} \qquad s = 0, 1, 2, \dots, n-1 \tag{5.8}$$

For a three-pendulum system the eigen-frequencies are

$$\omega_s^2 = \widetilde{\omega}_{0cT}^2 u_s + \widetilde{\omega}_0^2 = \left\{ \begin{array}{c} \widetilde{\omega}_0^2 \\ \widetilde{\omega}_{0cT}^2 + \widetilde{\omega}_0^2 \\ 3\widetilde{\omega}_{0cT}^2 + \widetilde{\omega}_0^2 \end{array} \right. = \left\{ \begin{array}{c} \widetilde{\omega}_0^2 \\ \omega_{0T}^2 + \omega_0^2 + \widetilde{\omega}_0^2 + \omega_{00}^2 + \omega_{0h}^2 \\ 3\omega_{0T}^2 + 3\omega_{0c}^2 + 3\omega_{00}^2 + 3\omega_{0h}^2 + \widetilde{\omega}_0^2 \end{array} \right. \tag{5.9}$$

and the solution for a free three-pendulum system is in the form

$$\begin{aligned} \varphi_1(t) &= C_1\cos(\omega_1 t + \alpha_1) + C_2\cos(\omega_2 t + \alpha_2) + C_3\cos(\omega_3 t + \alpha_3) \\ \varphi_2(t) &= C_1\cos(\omega_1 t + \alpha_1) - 2C_3\cos(\omega_3 t + \alpha_3) \\ \varphi_3(t) &= C_1\cos(\omega_1 t + \alpha_1) - C_2\cos(\omega_2 t + \alpha_2) + C_3\cos(\omega_3 t + \alpha_3) \end{aligned} \tag{5.10}$$

where C_1, C_2, C_3, α_1, α_2 and α_3 are constants.

By the use of different methods of constants variation in the previous solution to the linear basic system with equations (5.4) corresponding to the obtained system of governing equations (5.3) with additional restrictions to the system parameters, it is possible to obtain some partial approximation of the solution. Also, it is possible to study different dynamical properties of the system as well as some phenomena in the rheolinear, thermo-visco-elastic, nonlinear, hereditary or creep properties of the multi-pendulum system defined in our paper.

In the beginning, we consider solutions of the system with one (Fig. 3a), and two-pendulum system (Fig. 3b) with only one simpler set of the parallelly coupled standard light elements.

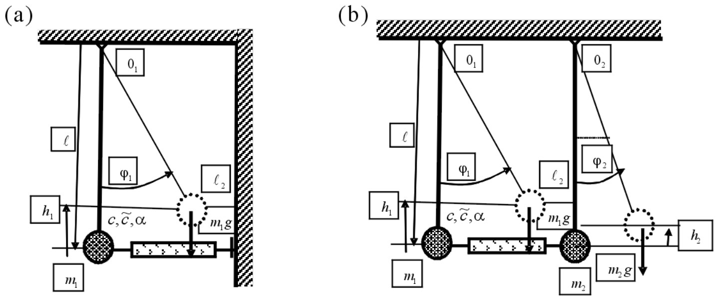

Fig. 3. System with one pendulum coupled (a) and with two pendula intercoupled (b) by standard light creep element

6. Analytical solution to governing equations of the pendulum system for special cases

6.1. One pendulum oscillator

In the case of **one pendulum** (Fig. 3a), coupled of a fixed point by a set of parallel standard light elements: nonlinear spring, hereditary, creep and thermo-modified by temperature $T(t)$, a differential equation of motion is given in the following form

$$\begin{aligned}&\ddot{\varphi}_1 + (\widetilde{\omega}_0^2 + \omega_0^2 + \omega_{00}^2 + \omega_{0h}^2)\varphi_1 + \omega_{0T}^2[1 + \gamma\widetilde{T}(t)]\varphi_1 + 2\delta\dot{\varphi}_1 = \\ &= \omega_{0T}^2 h_0 \widetilde{T}(t) + \widetilde{\omega}_0^2\Big(\frac{\varphi_1^3}{3!}\Big) - \varepsilon\omega_0^2\widetilde{\chi}\varphi_1^3 - \omega_{0\alpha}^2 \mathsf{D}_t^\alpha[\varphi_1(t)] + \omega_{0h}^2\int\limits_0^t \mathsf{R}(t-\tau)\varphi_1(t)\;d\tau\end{aligned} \tag{6.1}$$

A solution to a separate case of the previous governing equation, can be obtained through the following simpler tasks:
— *Hereditary one pendulum oscillator* (Fig. 3a) governed by

$$\ddot{\varphi}_1 + (\widetilde{\omega}_0^2 + \omega_{0h}^2)\varphi_{11} = \omega_{0h}^2\int\limits_0^t \mathsf{R}(t-\tau)\varphi_1(t)\;d\tau \tag{6.2}$$

where

$$\mathsf{R}(t-\tau) = \frac{c_h - \widetilde{c}_h}{n_c}\exp\Big[-\frac{1}{n}(t-\tau)\Big]$$

which is analogous to the problem solved approximately, and the solution is presented in Goroško and Hedrih (2001) and also in Goroško and Hedrih (2007a,b).

— *Creep one pendulum oscillator* (Fig. 3a) governed by an ordinary fractional-order differential equation in the form

$$\ddot{\varphi}_1 + (\widetilde{\omega}_0^2 + \omega_{00}^2)\varphi_1 = -\omega_{0\alpha}^2 \mathsf{D}_t^\alpha[\varphi_1(t)] \tag{6.3}$$

In the case when $\alpha \in (0,1)$, we solve previous ordinary fractional-order differential equation (6.3) through Laplace's transformations. After transforming previous ordinary fractional-order differential equation (6.3) with the fractional-order derivative and having in mind that we introduced the notations $\mathsf{L}\{\varphi_1(t)\}$ for Laplace's transformations as well as

$$\mathsf{L}\Big\{\frac{d^\alpha \varphi_1(t)}{dt^\alpha}\Big\} = p^\alpha \mathsf{L}\{\varphi_1(t)\} - \frac{d^{\alpha-1}\varphi_1(t)}{dt^{\alpha-1}}\Big|_{t=0} = p^\alpha \mathsf{L}\{\varphi_1(t)\} \tag{6.4}$$

and also having in mind that we accepted the hypothesis that the initial conditions of the fractional order derivatives of the system are given through the use of: $d^{\alpha-1}\varphi_1(t)/dt^{\alpha-1}|_{t=0} = 0$ and that

$$\mathsf{L}\Big\{\frac{d^2 \varphi_1(t)}{dt^2}\Big\} = p^2 \mathsf{L}\{\varphi_1(t)\} - [p\varphi_{01} + \dot{\varphi}_{01}] \tag{6.5}$$

where φ_{01} and $\dot{\varphi}_{01}$ are the initial conditions of the system we can write the following solution to the equation with unknown Laplace's transform

$$\mathsf{L}\{\varphi_1(t)\} = \frac{p\varphi_{01} + \dot{\varphi}_{01}}{p^2 + \omega_{0\alpha}^2 p^\alpha + \widetilde{\omega}_0^2 + \omega_{00}^2} \tag{6.6}$$

To obtain the inverse to the Laplace transform, we can use the result by Gorenflo and Mainardi (2000) as well as by Hedrih (2006a). For that reason and for the case when $\widetilde{\omega}_0^2 + \omega_{00}^2 \neq 0$, we rewrite the previous expression in the following form

$$\begin{aligned} \mathsf{L}\{\varphi_1(t)\} &= (p\varphi_{01} + \dot{\varphi}_{01})\frac{1}{p^2}\Big[1 + \frac{\omega_{0\alpha}^2}{p^2}\Big(p^\alpha + \frac{\widetilde{\omega}_0^2 + \omega_{00}^2}{\omega_{0\alpha}^2}\Big)\Big]^{-1} = \\ &= \Big(\varphi_{01} + \frac{\dot{\varphi}_{01}}{p}\Big)\frac{1}{p}\Big[1 + \frac{\omega_{0\alpha}^2}{p^2}\Big(p^\alpha + \frac{\widetilde{\omega}_0^2 + \omega_{00}^2}{\omega_{0\alpha}^2}\Big)\Big]^{-1} \end{aligned} \tag{6.7}$$

Then Laplace transform solution (5.7) can be expanded into series by the following way

$$\mathsf{L}\{\varphi_1(t)\} = \Big(\varphi_{01} + \frac{\dot{\varphi}_{01}}{p}\Big)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k \omega_{0\alpha}^{2k}}{p^{2k}}\Big(p^\alpha + \frac{\widetilde{\omega}_0^2 + \omega_{00}^2}{\omega_{0\alpha}^2}\Big)^k \tag{6.8}$$

or

$$\mathsf{L}\{\varphi_1(t)\} = \Big(\varphi_{01} + \frac{\dot{\varphi}_{01}}{p}\Big)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_{0\alpha}^{2k}}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{p^{\alpha j}\omega_{0\alpha}^{2(j-k)}}{(\widetilde{\omega}_0^2+\omega_{00}^2)^j} \tag{6.9}$$

In (6.8), it is assumed that the expansion leads to convergent series. The inverse Laplace transform of the previous transform of solution (6.9) in term-by-term steps is based on the known theorem, and yields the following solution to differential equation (6.3) of the time function in the following form of time series

$$\begin{aligned}\varphi_1(t) &= \mathsf{L}^{-1}\mathsf{L}\{\varphi_1(t)\} = \\ &= \varphi_{01}\sum_{k=0}^{\infty}(-1)^k\omega_{0\alpha}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_{0\alpha}^{2j}t^{-\alpha j}}{(\widetilde{\omega}_0^2+\omega_{00}^2)^j\Gamma(2k+1-\alpha j)} + \\ &+\dot{\varphi}_{01}\sum_{k=0}^{\infty}(-1)^k\omega_{0\alpha}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_{0\alpha}^{-2j}t^{-\alpha j}}{(\widetilde{\omega}_0^2+\omega_{00}^2)^j\Gamma(2k+2-\alpha j)}\end{aligned} \tag{6.10}$$

or

$$\begin{aligned}\varphi_1(t) &= \mathsf{L}^{-1}\mathsf{L}\{\varphi_1(t)\} = \sum_{k=0}^{\infty}(-1)^k\omega_{0\alpha}^{2k}t^{2k}\cdot \\ &\cdot\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_{0\alpha}^{2j}t^{-\alpha j}}{(\widetilde{\omega}_0^2+\omega_{00}^2)^j}\Big[\frac{\varphi_{01}}{\Gamma(2k+1-\alpha j)}+\frac{\dot{\varphi}_{01}t}{\Gamma(2k+2-\alpha j)}\Big]\end{aligned} \tag{6.11}$$

For two special cases of the solution (for $\alpha = 0$ and $\alpha = 1$), we have classical conservative or nonconservative pendulum oscillators.

By using expression (6.10) obtained for the time solution $\varphi_1(t)$ with corresponding particular solutions, we can conclude that the solution contains two particular solutions in the following forms

$$\begin{aligned}T_1(t) &= \sum_{k=0}^{\infty}(-1)^k\omega_{0\alpha}^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_{0\alpha}^{2j}t^{-\alpha j}}{(\widetilde{\omega}_0^2+\omega_{00}^2)^j\Gamma(2k+1-\alpha j)} \\ \widetilde{T}_1(t) &= \sum_{k=0}^{\infty}(-1)^k\omega_{0\alpha}^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_{0\alpha}^{-2j}t^{-\alpha j}}{(\widetilde{\omega}_0^2+\omega_{00}^2)^j\Gamma(2k+2-\alpha j)}\end{aligned} \tag{6.12}$$

which are two vibration "creeping" modes, $T_1(t,\alpha)$ and $\widetilde{T}_1(t,\alpha)$, of the fractional-order dynamical properties of the one-pendulum system. By using these particular solutions, we made a numerical experiment for characteristic cases. The ratio of the pendulum creep system kinetic parameters, coefficient of creeping standard light element with the constitutive relation expressed by the fractional-order derivative and the results are presented in Figures 4 and 5. It is visible that some types of modes are present as in the longitudinal vibrations of the rod with changeable cross-sections and built by a creep material with the stress strain constitutive relation expressed by the fractional-order derivative (see Hedrih, 2004c, 2005c).

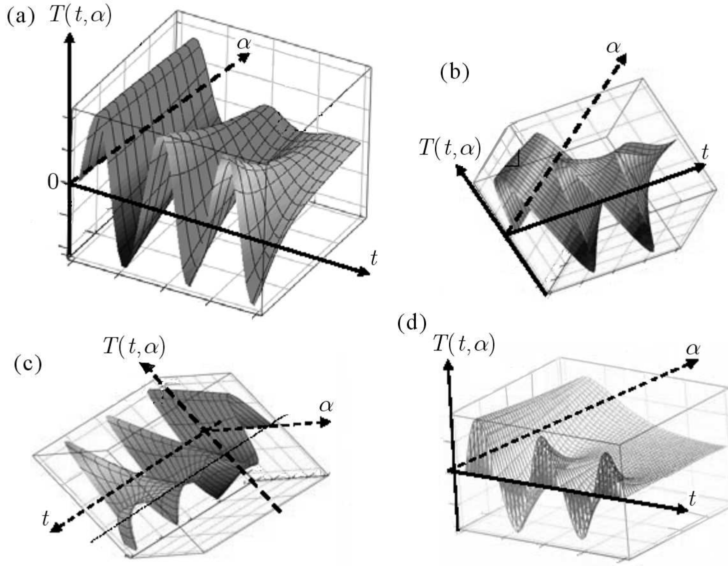

Fig. 4. Time function surfaces of $T_1(t,\alpha)$, $(6.12)_1$, for different kinetic and creep parameters of the-one pendulum system; (a) for $\omega_{0\alpha}^2/(\widetilde{\omega}_0^2+\omega_{00}^2)=1$, (b) for $\omega_{0\alpha}^2/(\widetilde{\omega}_0^2+\omega_{00}^2)=1/16$, (c) for $\omega_{0\alpha}^2/(\widetilde{\omega}_0^2+\omega_{00}^2)=1/9$. (d) for $\omega_{0\alpha}^2/(\widetilde{\omega}_0^2+\omega_{00}^2)=9$

The time functions $T_1(t,\alpha)$ and $\widetilde{T}_1(t,\alpha)$ are surfaces found for different parameters of kinetic and standard light creep elements in the space $(T(t,\alpha),t,\alpha)$ for the interval $0\leqslant\alpha\leqslant 1$.

In Fig. 4, numerical simulations and graphical presentations of the particular solution mode $T_1(t,\alpha)$, $(6.12)_1$, of fractional-differential equation (6.3) for different kinetic parameters are presented.

In Fig. 5, the particular solution $\widetilde{T}_1(t,\alpha)$, $(6.12)_2$, of fractional-differential equation $(6.12)_2$ of the system for different kinetic parameters in the interval $0\leqslant\alpha\leqslant 1$ are given.

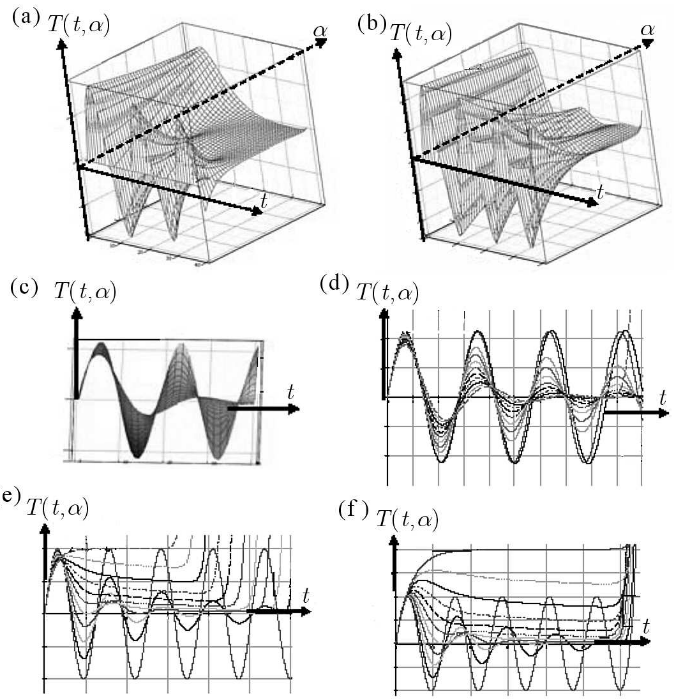

Fig. 5. Time function surfaces of $\widetilde{T}_1(t,\alpha)$, $(6.12)_2$, for different kinetic and creep parameters of the-one pendulum system; (a) for $\omega_{0\alpha}^2/(\widetilde{\omega}_0^2+\omega_{00}^2)=1$, (b) for $\omega_{0\alpha}^2/(\widetilde{\omega}_0^2+\omega_{00}^2)=1/16$, (c) for $\omega_{0\alpha}^2/(\widetilde{\omega}_0^2+\omega_{00}^2)=1/9$. (d) for $\omega_{0\alpha}^2/(\widetilde{\omega}_0^2+\omega_{00}^2)=9$

6.2. Creep double pendulum oscillator

Creep double pendulum oscillator (Fig. 3b), is governed by ordinary fractional-order differential equations in the form

$$\begin{aligned}\ddot{\varphi}_1+\widetilde{\omega}_0^2\varphi_1-\omega_{00}^2(\varphi_2-\varphi_1)&=\omega_{0\alpha}^2\{\mathsf{D}_t^\alpha[\varphi_2(t)]-\mathsf{D}_t^\alpha[\varphi_1(t)]\}\\ \ddot{\varphi}_2+\widetilde{\omega}_0^2\varphi_2+\omega_{00}^2(\varphi_2-\varphi_1)&=-\omega_{0\alpha}^2\{\mathsf{D}_t^\alpha[\varphi_2(t)]-\mathsf{D}_t^\alpha[\varphi_1(t)]\}\end{aligned}\tag{6.13}$$

An analogy with the result presented for the chain system in Hedrih (2006a) is useful to obtain the solution. After applying Laplace's transformations to previous equations (6.13) and having in mind that we introduced the notations $\mathsf{L}\{\varphi_k(t)\}$, $k=1,2$ as well as that

$$\mathsf{L}\Big\{\frac{d^\alpha\varphi_k(t)}{dt^\alpha}\Big\}=p^\alpha\mathsf{L}\{\varphi_k(t)\}-\frac{d^{\alpha-1}\varphi_k(t)}{dt^{\alpha-1}}\Big|_{t=0}=p^\alpha\mathsf{L}\{\varphi_k(t)\}\qquad k=1,2\tag{6.14}$$

and also having in mind that we accepted the hypothesis that the initial conditions of the fractional-order derivatives of the system are given as $(d^{\alpha-1}\varphi_k(t)/dt^{\alpha-1})|_{t=0} = 0$ as well that

$$\mathsf{L}\Big\{\frac{d^2\varphi_k(t)}{dt^2}\Big\} = p^2\mathsf{L}\{\varphi_k(t)\} - [p\varphi_{0k} + \dot{\varphi}_{0k}] \qquad k = 1,2 \tag{6.15}$$

where φ_{0k} and $\dot{\varphi}_{0k}$, $k = 1,2$ are the initial conditions of the double-pendulum system, we can write the following system equations with unknown Laplace's transforms

$$\begin{aligned} (1+v)\mathsf{L}\{\varphi_1(t)\} - \mathsf{L}\{\varphi_2(t)\} &= \frac{p\varphi_{01} + \dot{\varphi}_{01}}{\omega_{0\alpha}^2 p^\alpha + \omega_{00}^2} = h_1 \\ -\mathsf{L}\{\varphi_1(t)\} + (2+v)\mathsf{L}\{\varphi_2(t)\} - \mathsf{L}\{\varphi_3(t)\} &= \frac{p\varphi_{02} + \dot{\varphi}_{02}}{\omega_{0\alpha}^2 p^\alpha + \omega_{00}^2} = h_2 \end{aligned} \tag{6.16}$$

we introduce

$$v = \frac{p^2 + \widetilde{\omega}_0^2}{\omega_{0\alpha}^2 p^\alpha + \omega_{00}^2} \tag{6.17}$$

The determinant of the previous system is

$$\varDelta = \frac{(p^2 + \widetilde{\omega}_0^2 + 2\omega_{0\alpha}^2 p^\alpha + 2\omega_{00}^2)(p^2 + \widetilde{\omega}_0^2)}{(\omega_{0\alpha}^2 p^\alpha + \omega_{00}^2)^2} \tag{6.18}$$

Solutions to system equations (6.16) with respect to $\mathsf{L}\{\varphi_k(t)\}$, $k = 1,2$, i.e. Laplace transforms of fractional-order differential equations (6.13) are in the following forms

$$\begin{aligned} \mathsf{L}\{\varphi_1(t)\} &= \frac{(p\varphi_{01} + \dot{\varphi}_{01})(p^2 + \omega_{0\alpha}^2 p^\alpha + \widetilde{\omega}_0^2 + \omega_{00}^2) + (p\varphi_{02} + \dot{\varphi}_{02})(\omega_{0\alpha}^2 p^\alpha + \omega_{00}^2)}{(p^2 + \widetilde{\omega}_0^2 + 2\omega_{0\alpha}^2 p^\alpha + 2\omega_{00}^2)(p^2 + \widetilde{\omega}_0^2)} \\ \mathsf{L}\{\varphi_2(t)\} &= \frac{(p\varphi_{02} + \dot{\varphi}_{02})(p^2 + \omega_{0\alpha}^2 p^\alpha + \widetilde{\omega}_0^2 + \omega_{00}^2) + (p\varphi_{01} + \dot{\varphi}_{01})(\omega_{0\alpha}^2 p^\alpha + \omega_{00}^2)}{(p^2 + \widetilde{\omega}_0^2 + 2\omega_{0\alpha}^2 p^\alpha + 2\omega_{00}^2)(p^2 + \widetilde{\omega}_0^2)} \end{aligned} \tag{6.19}$$

For special cases of the double-pendulum system initial conditions, when at the initial moment the second pendulum is in the equilibrium position, the solutions are

$$\begin{aligned} \mathsf{L}\{\varphi_1(t)\} &= \frac{(p\varphi_{01} + \dot{\varphi}_{01})(p^2 + \omega_{0\alpha}^2 p^\alpha + \widetilde{\omega}_0^2 + \omega_{00}^2)}{(p^2 + \widetilde{\omega}_0^2 + 2\omega_{0\alpha}^2 p^\alpha + 2\omega_{00}^2)(p^2 + \widetilde{\omega}_0^2)} \\ \mathsf{L}\{\varphi_2(t)\} &= \frac{(p\varphi_{01} + \dot{\varphi}_{01})(\omega_{0\alpha}^2 p^\alpha + \omega_{00}^2)}{(p^2 + \widetilde{\omega}_0^2 + 2\omega_{0\alpha}^2 p^\alpha + 2\omega_{00}^2)(p^2 + \widetilde{\omega}_0^2)} \end{aligned} \tag{6.20}$$

Taking into account that the sum and difference between the solutions to (6.16), the Laplace transforms of fractional-order differential equations (6.13) are

$$\begin{aligned}\mathsf{L}\{\xi_1(t)\} &= \mathsf{L}\{\varphi_1(t)+\varphi_2(t)\} = \frac{p\varphi_{01}+\dot{\varphi}_{01}}{p^2+\widetilde{\omega}_0^2} + \frac{p\varphi_{02}+\dot{\varphi}_{02}}{p^2+\widetilde{\omega}_0^2} \\ \mathsf{L}\{\xi_2(t)\} &= \mathsf{L}\{\varphi_1(t)-\varphi_2(t)\} = \frac{(p\varphi_{01}+\dot{\varphi}_{01})-(p\varphi_{02}+\dot{\varphi}_{02})}{p^2+\widetilde{\omega}_0^2+2\omega_{0\alpha}^2p^\alpha+2\omega_{00}^2}\end{aligned} \tag{6.21}$$

The inverse Laplace transform to $\mathsf{L}\{\xi_1(t)\}$ of the sum $\varphi_1(t)+\varphi_2(t)$ of solution $(6.21)_1$ yields the following sum of solutions to the system of differential equations (6.13)

$$\begin{aligned}\xi_1(t) &= \varphi_1(t)+\varphi_2(t) = \mathsf{L}^{-1}\mathsf{L}\{\varphi_1(t)+\varphi_2(t)\} = \\ &= \Big(\frac{\dot{\varphi}_{01}}{\widetilde{\omega}_0}+\frac{\dot{\varphi}_{02}}{\widetilde{\omega}_0}\Big)\sin(\widetilde{\omega}_0t)+(\varphi_{01}+\varphi_{02})\cos(\widetilde{\omega}_0t)\end{aligned} \tag{6.22}$$

The inverse Laplace transform to $\mathsf{L}\{\xi_2(t)\}$ of the difference $\varphi_1(t)-\varphi_2(t)$ of solution $(6.21)_2$ yields the following difference of solutions differential equations (6.13)

$$\xi_2 = \varphi_1(t)-\varphi_2(t) = \mathsf{L}^{-1}\mathsf{L}\{\varphi_1(t)-\varphi_2(t)\} = \mathsf{L}^{-1}\Big\{\frac{p(\varphi_{01}-\varphi_{02})+(\dot{\varphi}_{01}-\dot{\varphi}_{02})}{p^2+2\omega_{0\alpha}^2p^\alpha+\widetilde{\omega}_0^2+2\omega_{00}^2}\Big\} \tag{6.23}$$

We can see that the obtained Laplace transform $\mathsf{L}\{\varphi_1(t)-\varphi_2(t)\}$ is the same as the Laplace transform of the solution for the case of one pendulum with one creep standard light element expressed by (6.6), and then it is possible to use previous expression (6.7) and expansions of series (6.8) and (6.9) as well as (6.10) for obtaining the corresponding solution to the necessary modes for the double-pendulum system by replacing the following parameters

$$\begin{aligned}\varphi_{01} &\to \varphi_{01}-\varphi_{02} & \dot{\varphi}_{01} &\to \dot{\varphi}_{01}-\dot{\varphi}_{02} \\ \omega_{0\alpha}^2 &\to 2\omega_{0\alpha}^2 & \widetilde{\omega}_0^2+\omega_{00}^2 &\to \widetilde{\omega}_0^2+2\omega_{00}^2\end{aligned}$$

In the case when $\widetilde{\omega}_0^2+2\omega_{00}^2 \neq 0$, the Laplace transform can be expanded into following series

$$\begin{aligned}&\mathsf{L}\{\xi_2\} = \mathsf{L}\{\varphi_1(t)-\varphi_2(t)\} = \\ &= \Big(\varphi_{01}-\varphi_{01}+\frac{\dot{\varphi}_{01}-\dot{\varphi}_{02}}{p}\Big)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k2^k\omega_{0\alpha}^{2k}}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{p^{\alpha j}2^{(j-k)}\omega_{0\alpha}^{2(j-k)}}{(\widetilde{\omega}_0^2+2\omega_{00}^2)^j}\end{aligned} \tag{6.24}$$

In (6.24) it is assumed that the expansion leads to convergent series. The inverse Laplace transform of the previous Laplace transform of solution (6.24) in term-by-term steps is based on the known theorem, and yields the following solution to differential equations (6.13) of the time function in the following form of time series

$$\xi_2(t) = \mathsf{L}^{-1}\mathsf{L}\{\varphi_1(t) - \varphi_2(t)\} = \tag{6.25}$$
$$= (\varphi_{01} - \varphi_{02}) \sum_{k=0}^{\infty} (-1)^k 2^k \omega_{0\alpha}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{2^j \omega_{0\alpha}^{2j} t^{-\alpha j}}{(\widetilde{\omega}_0^2 + 2\omega_{00}^2)^j \Gamma(2k+1-\alpha j)} +$$
$$+ (\dot{\varphi}_{01} - \dot{\varphi}_{02}) \sum_{k=0}^{\infty} (-1)^k 2^k \omega_{0\alpha}^{2k} t^{2k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{2^{-j} \omega_{0\alpha}^{-2j} t^{-\alpha j}}{(\widetilde{\omega}_0^2 + 2\omega_{00}^2)^j \Gamma(2k+2-\alpha j)}$$

For two special cases of the solution (for $\alpha = 0$ and $\alpha = 1$), we have classical conservative or nonconservative (Hedrih, 2006d) pendulum oscillators, respectively.

By using expression obtained for the time solution $\xi_2(t)$ with corresponding particular solutions, we can conclude that the solution contains two particular solutions in the following forms

$$T_1(t) = \sum_{k=0}^{\infty} (-1)^k 2^k \omega_{0\alpha}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{2^j \omega_{0\alpha}^{2j} t^{-\alpha j}}{(\widetilde{\omega}_0^2 + 2\omega_{00}^2)^j \Gamma(2k+1-\alpha j)}$$
$$\widetilde{T}_1(t) = \sum_{k=0}^{\infty} (-1)^k 2^k \omega_{0\alpha}^{2k} t^{2k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{2^{-j} \omega_{0\alpha}^{-2j} t^{-\alpha j}}{(\widetilde{\omega}_0^2 + 2\omega_{00}^2)^j \Gamma(2k+2-\alpha j)} \tag{6.26}$$

which are two vibration "creeping" modes, $T_1(t,\alpha)$ and $\widetilde{T}_1(t,\alpha)$, of the fractional one-pendulum system oscillations. By using these particular solutions, we made a numerical experiment for characteristic cases. The ratio of the pendulum creep system kinetic parameters, coefficient of creeping standard light element with the constitutive relation expressed by the fractional order derivative and the results are presented in Figures 4 and 5. It is visible that some types of modes is present as in the longitudinal vibrations of the rod with changeable cross-sections and built by a creep material with the stress strain constitutive relation expressed by the fractional order derivative presented in Hedrih and Filipovski (2002).

Time functions $T_1(t,\alpha)$ and $\widetilde{T}_1(t,\alpha)$ are surfaces found for different kinetic and standard light creep element parameters in the space $(T(t,\alpha), t, \alpha)$ for the interval $0 \leqslant \alpha \leqslant 1$.

In Fig. 4, numerical simulations and graphical presentations of the particular solution mode $T_1(t,\alpha)$ of fractional-differential equation (6.13) for different system kinetic parameters are presented. In Fig. 5, the time particular solution mode, $\widetilde{T}_1(t,\alpha)$, of fractional-differential equation (6.13) of the system for different system kinetic parameters in the interval $0 \leqslant \alpha \leqslant 1$ are shown.

Solution to the normal modes of the system of fractional-order differential equations (6.13) are in the form

$$\begin{aligned}
\xi_1(t) &= \varphi_1(t) + \varphi_2(t) = \mathsf{L}^{-1}\mathsf{L}\{\varphi_1(t) + \varphi_2(t)\} = \\
&= \Big(\frac{\dot{\varphi}_{01}}{\widetilde{\omega}_0} + \frac{\dot{\varphi}_{02}}{\widetilde{\omega}_0}\Big)\sin(\widetilde{\omega}_0 t) + (\varphi_{01} + \varphi_{02})\cos(\widetilde{\omega}_0 t) \\
\xi_2(t) &= \varphi_1(t) - \varphi_2(t) = \\
&= \sum_{k=0}^{\infty}(-1)^k 2^k \omega_{0\alpha}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{2^j \omega_{0\alpha}^{2j} t^{-\alpha j}}{(\widetilde{\omega}_0^2 + 2\omega_{00}^2)^j} \Big[\frac{\varphi_{01} - \varphi_{02}}{\Gamma(2k+1-\alpha j)} + \frac{(\dot{\varphi}_{01} - \dot{\varphi}_{02})t}{\Gamma(2k+2-\alpha j)}\Big]
\end{aligned} \tag{6.27}$$

Then the solutions to equations (6.13) are

$$\begin{aligned}
\varphi_1(t) &= \frac{1}{2}\Big[\Big(\frac{\dot{\varphi}_{01}}{\widetilde{\omega}_0} + \frac{\dot{\varphi}_{02}}{\widetilde{\omega}_0}\Big)\sin(\widetilde{\omega}_0 t) + (\varphi_{01} + \varphi_{02})\cos(\widetilde{\omega}_0 t)\Big] + \\
&+ \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k 2^k \omega_{0\alpha}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{2^j \omega_{0\alpha}^{2j} t^{-\alpha j}}{(\widetilde{\omega}_0^2 + 2\omega_{00}^2)^j} \Big[\frac{\varphi_{01} - \varphi_{02}}{\Gamma(2k+1-\alpha j)} + \frac{(\dot{\varphi}_{01} - \dot{\varphi}_{02})t}{\Gamma(2k+2-\alpha j)}\Big] \\
\varphi_2(t) &= \frac{1}{2}\Big[\Big(\frac{\dot{\varphi}_{01}}{\widetilde{\omega}_0} + \frac{\dot{\varphi}_{02}}{\widetilde{\omega}_0}\Big)\sin(\widetilde{\omega}_0 t) + (\varphi_{01} + \varphi_{02})\cos(\widetilde{\omega}_0 t)\Big] + \\
&- \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k 2^k \omega_{0\alpha}^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{2^j \omega_{0\alpha}^{2j} t^{-\alpha j}}{(\widetilde{\omega}_0^2 + 2\omega_{00}^2)^j} \Big[\frac{\varphi_{01} - \varphi_{02}}{\Gamma(2k+1-\alpha j)} + \frac{(\dot{\varphi}_{01} - \dot{\varphi}_{02})t}{\Gamma(2k+2-\alpha j)}\Big]
\end{aligned} \tag{6.28}$$

6.3. Thermo-visco-elastic double pendulum oscillator excited by random temperature

By using the results by Hedrih (2007e) presented at DSTA 2007, we can add some discussion with respect to the mode of the thermo-visco-elastic double-pendulum system with comparison to the one-pendulum system both excited by random temperature applied to the thermo-elastic standard light element.

For the double-pendulum system, the governing rheolinear equations expressed in terms of normal coordinates of the corresponding linear system can be obtained in the following form

$$\begin{aligned} &\ddot{\xi}_1 + \widetilde{\omega}_0^2 \xi_1 = 0 \\ &\ddot{\xi}_2 + \widetilde{\omega}_0^2 \xi_2 + 2\omega_0^2 \xi_2 + 2\omega_{0T}^2 [1 + \gamma \widetilde{T}(t)] \xi_2 + 4\delta \dot{\xi}_2 = -2\omega_{0T}^2 h_0 \widetilde{T}(t) \end{aligned} \tag{6.29}$$

with two eigen-frequencies $\omega_{1,2}^2 = \widetilde{\omega}_0^2 + \omega_0^2 + \omega_{0T}^2 \mp (\omega_0^2 + \omega_{0T}^2)$ of the linear system.

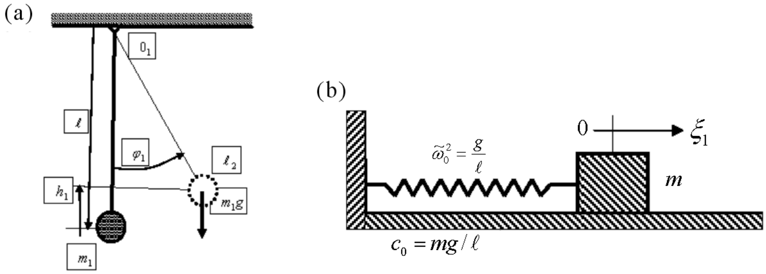

Fig. 6. (a) System with one free pendulum. (b) Thermo-rheological system; partial oscillator 1

For the double pendulum system, the first equation of rheo-nonlinear system (6.29) in the linearised form represents a pure partial harmonic oscillator presented in Fig. 6a and 6b, with the eigen frequency $\omega_1^2 = \widetilde{\omega}_0^2 = g/\ell$ of free one-mode vibrations. This linearised case is when both pendula oscillate with the same frequency, $\widetilde{\omega}_0^2 = g/\ell$, as the decoupled pendula (single mathematical pendula). Then, the standard light thermo-visco-elastic element thermo-modified by temperature $T(t)$ does not influence this normal coordinate composed by sum $\xi_1 = \varphi_1 + \varphi_2$. Along this normal (main) coordinate, the oscillation in the linearised approximation is free, without temperature influence. This is right for all cases of the multi-pendulum systems presented in Fig. 6b.

For the double-pendulum system, the second equation of rheo-nonlinear system (4.4), on the normal coordinate $\xi_2 = \varphi_1 - \varphi_2$ in the linearised form is the Mathieu-Hill equation, and represents mathematical description of the thermo-rheological oscillator presented in Fig. 7a or 7c, with parallelly coupled two light standard thermo-visco-elastic elements thermo-modified by the same temperature $T(t)$ and one linear elastic spring with rigidity $c_0 = mg/\ell$. For the coordinate $\xi_2 = \varphi_1 - \varphi_2$, we can separate two main cases. For both cases, we take into consideration the asymptotic approximation of the amplitude and

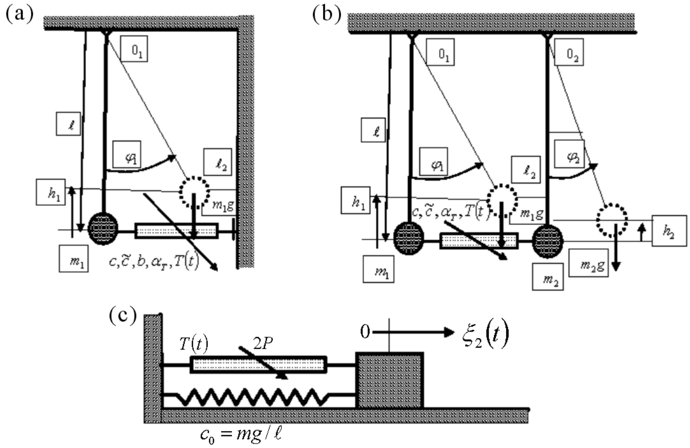

Fig. 7. (a) System with one pendulum coupled by the standard light visco-thermoelastic element. (b) System with two pendula intercoupled by the standard light thermorheological element. (c) Thermo-rheological system; partial oscillator 2

phase of the dynamic process on this coordinate $\xi_2 = \varphi_1 - \varphi_2$ close around, firstly, main resonance when $\Omega \approx \omega_2 = \sqrt{\widetilde{\omega}_0^2 + 2(\omega_0^2 + \omega_{0T}^2)}$ and, secondly, around the parametric resonance when $\Omega \approx 0.5\omega_2 = 0.5\sqrt{\widetilde{\omega}_0^2 + 2(\omega_0^2 + \omega_{0T}^2)}$. Then, we can conclude that along this coordinate under the corresponding kinetic parameters, there can appear, firstly, regimes closest to the main resonant state as well as one main resonant state, and secondly, regimes closest to the parametric resonant state as well as one resonant state under the thermo-visco-elastic temperature single frequency excitation. This second mode has the same character as vibration of the one-pendulum system presented in Fig. 7a. For details, see Hedrih (2007e).

7. Concluding remarks

We can conclude that between multi-pendulum systems and chain dynamical systems there exists a mathematical analogy in descriptions as well as in vibration phenomena depending on the character of standard light coupling elements between pendula or, analogously, between material particles in the chain. Also, there is a mathematical analogy between corresponding modes in a multi-beam system or multi-plate systems with the corresponding character of a light distributed coupling layer between the beams or plates in multi-deformable body systems.

The mathematical description leads to the same ordinary differential equations, or ordinary integro-differential or fractional-order differential equations governing both analogous types of problems.

For a homogeneous sandwich multi-plate, or a multi-beam system, it is possible to identify some analogies between with mechanical multi-material particle chains and multi-pendulum systems with interconnections by standard light elements of different properties.

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Dynamika układów wielowahadłowych z efektem pełzania opisanym elementami ułamkowego rzędu

Streszczenie

W pracy zaprezentowano krótki przegląd rezultatów badań autora nad dynamiką układów hybrydowych i dyskretnych, złożonych z punktów materialnych sprzęgniętych standardowymi elementami odpowiadającymi za pełzanie w materiale i opisywanych pochodną ułamkowego rzędu. Rozważono drgania swobodne układów wielowahadłowych z elementami o różnych właściwościach zdefiniowanych równaniem pomiędzy stanem naprężenia a odkształcenia. Wyprowadzone równania różniczkowo-całkowe ułamkowego rzędu rozwiązano analitycznie. Przedstawiono szczegółowo przypadek układu z pojedynczym wahadłem i układu dwuwahadłowego zawierającego elementy pełzania opisane równaniem konstytutywnym stanu naprężenia i odkształcenia o rzędzie ułamkowym. Na podstawie otrzymanych rozwiązań analitycznych zauważono, że drgania swobodne wykazują charakter okresowy i nieokresowy, przy czym te ostatnie mają dwa różne przebiegi (w tym przypadku rozwiązanie podano w postaci rozwinięć w szeregi potęgowe). Wyniki badań teoretycznych i numerycznych różnego rodzaju drgań przy zmiennych parametrach kinetycznych tych układów przedstawiono graficznie.

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