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MODELING OF UNDERACTUATED MECHANICAL SYSTEMS IN PARTLY SPECIFIED MOTION

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This paper deals with the inverse dynamics problem of underactuated mechanical systems subjected to execute partly specified motions. The modeling methodology focuses on a special class of differentially flat systems, represented by a group of relevant technical examples. The governing equations are obtained as index three differential-algebraic equations, and a simple numerical code for their solution is reported. The solution comprises both the dynamic analysis of the underactuated systems in partly specified motion and the synthesis of control that assures realization of such motion.

Key words: underactuated systems, inverse dynamics, differential flatness, servo-constraints

1. Introduction

Underactuated mechanical systems are mechanical systems with fewer control inputs than degrees of freedom (Spong, 1997), $m < n$, and as such $k = n - m$ degrees of freedom cannot be directly actuated. The determination of a control input strategy that forces an underactuated system to complete a given set of specified motion tasks is a challenging problem, and a solution to the inverse dynamics problem at hand is possible only if the motion is defined by as many system outputs as the number of control inputs – the motion is *partly specified*. The situation is thus substantially different from the fully actuated case, $m = n$, which requires also that the motion of the system must be fully specified.

The controllability of underactuated systems in partly specified motions is closely related to the concept of *differential flatness* (Fliess *et al.*, 1995; Rouchon, 2005), denoted that all the system state variables and control inputs can be algebraically expressed in terms of desired outputs and their time derivatives up to a certain order. The relationships between the imposed outputs and the required control inputs provide a basis for the synthesis of control laws. Such analytical algebraic relations are usually difficult to obtain, however. The equivalent/alternative formulation reported in this paper is much more straightforward and applicable.

2. Relevant technical examples

The modeling methodology is developed for a group of relevant technical examples of underactuated mechanical systems in partly specified motion, creating a class of differentially flat systems (Blajer and Kołodziejczyk, 2004). A simple representative of the group is a two-mass system $(n = 2)$ shown in Fig. 1, where desired motion $s(t)$ of the mass m_2 is actuated by the force F applied to the mass m_1, and as such $m = k = 1 = n/2$. Another example of qualitatively the same type is the position control of a manipulator with elastic joints (Spong, 1987). For robot models with compliance lumped between rigid links and actuators (Fig. 2), the number of degrees of freedom is $n = 2m$, related to m links and m actuator rotors, $\boldsymbol{q} = [q_1, \ldots, q_m]^\top$ and $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_m]^\top$. The specified motion $\boldsymbol{q}_d(t)$ of links is actuated by motor torques $\boldsymbol{\tau} = [\tau_1, \ldots, \tau_m]^\top$ applied to the actuator rotors, and as such $m = k = n/2$.

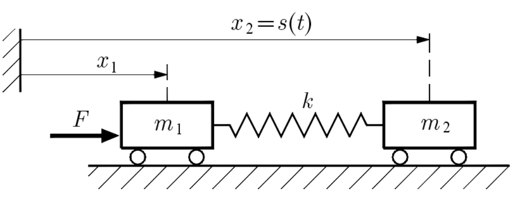

Fig. 1. A two-mass system

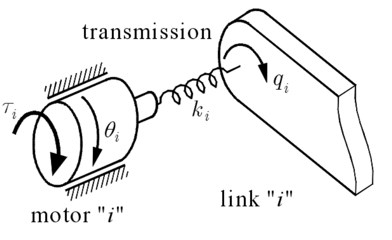

Fig. 2. A sketch of a flexible joint

The overhead trolley crane and the rotary crane seen in Figures 3 and 4 are both five-degree-of-freedom systems, $\boldsymbol{q} = [s_1, s_2, l, \theta_1, \theta_2]^\top$ and $\boldsymbol{q} = [\varphi, s, l, \theta_1, \theta_2]^\top$, respectively, the three desired outputs are specified in time load coordinates $\boldsymbol{r}_d(t) = [x_d(t), y_d(t), z_d(t)]^\top$, and the respective three control inputs are $\boldsymbol{u} = [F_1, F_2, M_w]^\top$ and $\boldsymbol{u} = [M_b, F, M_w]^\top$; see also Blajer and Kołodziejczyk (2007) for more details. We have thus $n = 5$, $m = 3$ and $k = n - m = 2$.

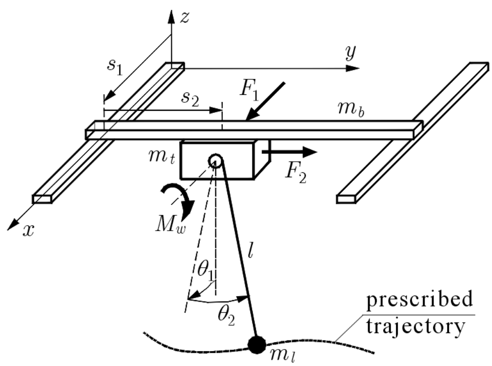

Fig. 3. An overhead trolley crane model

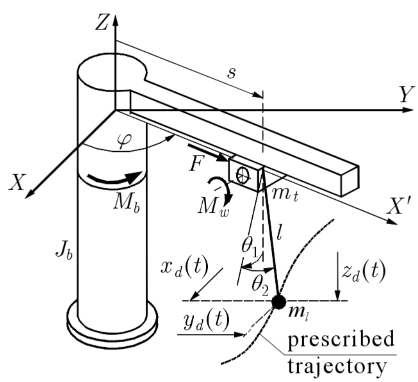

Fig. 4. A rotary crane model

For an aircraft in a prescribed trajectory flight (Blajer *et al.*, 2001), Fig. 5, motion of the six-degree-of-freedom system is prescribed by four demands: the requested trajectory (two specifications), the condition on the airframe attitude with respect to the trajectory, and the desired flight velocity. The control inputs are $\boldsymbol{u} = [\delta_a, \delta_e, \delta_r, T]^\top$. We have then $n = 6$, $m = 4$ and $k = n - m = 2$.

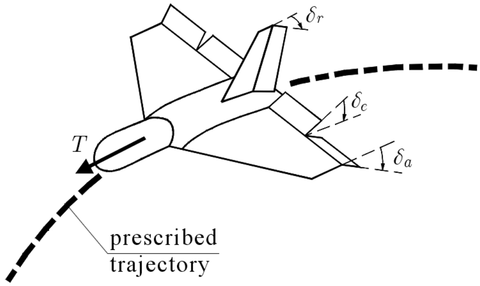

Fig. 5. An aircraft in a prescribed trajectory flight

Finally, for a surface ship tracking a desired path (Fig. 6), the two desired outputs of the three-degree-of-freedom system (Fosse, 1995) are the requested trajectory and the desired velocity variation in time. The two control inputs are $\boldsymbol{u} = [\delta, T]^\top$. We have: $n = 3$, $m = 2$ and $k = n - m = 1$.

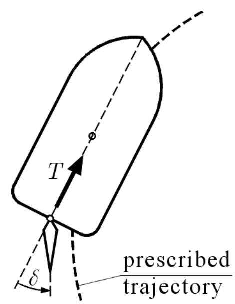

Fig. 6. A surface ship tracking a trajectory

3. Modeling prelimaries

The dynamical systems quoted in Section 2 can be represented by an n-degree-of-freedom system described by n generalized coordinates $\boldsymbol{q} = [q_1, \ldots, q_n]^\top$, and enforced by actuator forces due to m control inputs, $\boldsymbol{u} = [u_1, \ldots, u_m]^\top$, where $m < n$. For the sake of simplicity, let us write dynamic equations of the system in the following generic matrix form

$$\mathbf{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{d}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathbf{B}^\top(\boldsymbol{q})\boldsymbol{u} \tag{3.1}$$

where $\mathbf{M}$ is the $n \times n$ generalized mass matrix related to $\dot{\boldsymbol{q}}$, the n-vectors $\boldsymbol{d}$ and $\boldsymbol{f}$ contain the related dynamic and applied generalized forces, and $\mathbf{B}^\top$

is an $n \times m$ matrix of coefficients that relate the control inputs $\boldsymbol{u}$ and the actuating generalized force vector $\boldsymbol{f}_u = -\mathbf{B}^\top \boldsymbol{u}$. In fact, the dynamic equations for aircrafts and ships are usually introduced as $\mathbf{M}\dot{\boldsymbol{v}} + \boldsymbol{d} = \boldsymbol{f} - \mathbf{B}^\top \boldsymbol{u}$, and then supplemented with the kinematic equation $\dot{\boldsymbol{q}} = \mathbf{A}\boldsymbol{v}$; see Blajer *et al.* (2001) and Fossen (1995) for more details. However, the subsequent methodology introduced for dynamic formulation (3.1) is valid for more general formulations as well.

The system described in (3.1) is subjected to m motion specifications (performance goals) formulated as m desired system outputs $\boldsymbol{\gamma} = [\gamma_1, \dots, \gamma_m]^\top$, which, expressed in terms of $\boldsymbol{q}$, lead to *servo-constraints* (called also *active* or *program constraints*) on the system (Blajer and Kołodziejczyk, 2004; Gutowski, 1971). The constraint equations, in the original and twice-differentiated with respect to time forms, are

$$
\begin{aligned}
\boldsymbol{c}(\boldsymbol{q}, t) &\equiv \boldsymbol{\Phi}(\boldsymbol{q}) - \boldsymbol{\gamma}(t) = \mathbf{0} \\
\ddot{\boldsymbol{c}} &\equiv \mathbf{C}(\boldsymbol{q})\ddot{\boldsymbol{q}} - \boldsymbol{\xi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \mathbf{0}
\end{aligned}
\tag{3.2}
$$

where $\mathbf{C} = \partial\boldsymbol{\Phi}/\partial\boldsymbol{q}$ is an $m \times n$ constraint matrix, $\boldsymbol{\xi} = \ddot{\boldsymbol{\gamma}} - \dot{\mathbf{C}}\dot{\boldsymbol{q}}$ is an m-vector of constraint-induced accelerations. Evidently, equation $(3.2)_1$ is mathematically equivalent to m rheonomic holonomic constraints on the system, $\boldsymbol{c}(\boldsymbol{q}, t) = \mathbf{0}$, and $\boldsymbol{\gamma}(t) = \mathbf{0}$ can be interpreted as drift in time of manifold $\boldsymbol{\Phi}(\boldsymbol{q}) = \mathbf{0}$ in the linear space related to $\dot{\boldsymbol{q}}$ (Blajer, 2001). The actuating generalized force $\boldsymbol{f}_u = -\mathbf{B}^\top \boldsymbol{u}$ can be then interpreted as the generalized reaction force of servo-constraints, called the *control reaction* (Blajer and Kołodziejczyk, 2004) in the sequel. The realization of servo-constraints $(3.2)_1$ by applying the active control force $\boldsymbol{f}_u = -\mathbf{B}^\top \boldsymbol{u}$ (control inputs $\boldsymbol{u}$) is, however, qualitatively different from the realization of *contact* (*passive*) *constraints* caused by joints, hard surfaces, rigid links, slip-less rolling contacts, etc. Namely, assumed equation $(3.2)_1$ represents contact constraints, the actuating force $\boldsymbol{f}_u = -\mathbf{B}^\top \boldsymbol{u}$ in (3.1) would be replaced by the passive constraint reaction force $\boldsymbol{f}_c = -\mathbf{C}^\top \boldsymbol{\lambda}$, where $\mathbf{C}$ is the constraint matrix as defined in $(3.2)_1$. While the reaction $\boldsymbol{f}_c$ of (ideal) passive constraints is orthogonal to the instantaneous manifold of contact constraints, the actuating force $\boldsymbol{f}_u$ may be arbitrary directed with respect to the manifold of servo-constraints, and, in the extreme, may be tangent to the manifold (for more geometrical insight the reader is referred to Blajer (2001), Blajer and Kołodziejczyk (2004)). More specifically, denoted: $\mathcal{N}$ – linear n-space of the system related to $\dot{\boldsymbol{q}}$, $\mathcal{C}$ – constrained (specified) m-subspace spanned by constraint gradients represented in $\mathcal{N}$ as rows in $\mathbf{C}$, $\mathcal{B}$ – controlled m-subspace spanned by vectors represented in $\mathcal{N}$ as rows in $\mathbf{B}$, for all the dynamical systems reported in Section 2, the controllability (differential flatness) of this class of underactuated mechanical systems in partly specified motion is warranted by

$$\mathcal{C} \cup \mathcal{B} = \mathcal{N} \quad \Longleftrightarrow \quad \mathrm{rank}\,([\mathbf{C}^\top \vdots \mathbf{B}^\top]^\top) = \max = n \quad \Longrightarrow \quad 2m \geqslant n \tag{3.3}$$

where $\mathbf{Q} = [\mathbf{C}^\top \vdots \mathbf{B}^\top]^\top$ is a $2m \times n$ matrix. According to (3.3), the $\mathcal{C}$ and $\mathcal{B}$ subspaces cover thus the whole space $\mathcal{N}$ (see Fig. 7), and then

$$\mathcal{C} \cap \mathcal{B} = \mathcal{P} \quad \Longrightarrow \quad \mathrm{rank}\,(\mathbf{C}\mathbf{M}^{-1}\mathbf{B}^\top) = p \quad \Longrightarrow \quad p = 2m - n \tag{3.4}$$

where $\mathbf{P} = \mathbf{C}\mathbf{M}^{-1}\mathbf{B}^\top$ is a $m \times m$ matrix. The deficiency in rank of $\mathbf{P}$, $p < m$, shows that only p desired outputs can directly be governed by the available control inputs, while the other $k = m - p = n - m$ outputs are actuated indirectly through the dynamical couplings, referred respectively to *orthogonal* and *tangent realization* of servo constraints (Blajer and Kołodziejczyk, 2004). For the systems in Figs. 1 and 2 we have $2m = n$ and $p = 0$ (pure tangent realization of servo-constraints), and for the other systems in Figs. 3-6, $2m = n + 1$ and $p = 1$ (mixed orthogonal-tangent realization of servo-constraints).

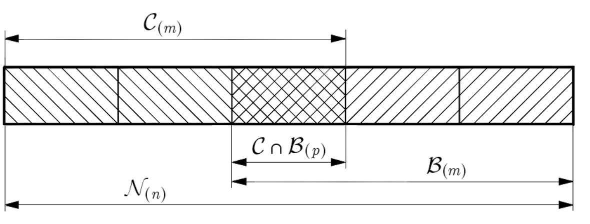

Fig. 7. Illustration of $\mathcal{C}$ and $\mathcal{B}$ subspaces in $\mathcal{N}$

4. Differential flatness formulation

The initial governing equations for the inverse dynamics problem of an underactuated system in partly specified motion are

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{v} \\ \mathbf{M}(\boldsymbol{q})\dot{\boldsymbol{v}} + \boldsymbol{d}(\boldsymbol{q},\boldsymbol{v}) &= \boldsymbol{f}(\boldsymbol{q},\boldsymbol{v}) - \mathbf{B}^\top(\boldsymbol{q})\boldsymbol{u} \\ \mathbf{0} &= \boldsymbol{\Phi}(\boldsymbol{q}) - \boldsymbol{\gamma}(t) \end{aligned} \tag{4.1}$$

which form $2n+m$ differential-algebraic equations (DAEs) in $2n$ states $\boldsymbol{x} = [\boldsymbol{q}^\top, \boldsymbol{v}^\top]^\top$, and m control inputs $\boldsymbol{u}$. An important characteristic of a DAE system is its *index*, which is a measure of singularity of the DAEs and determines difficulty in their numerical integration (Ascher and Petzold, 1998). The index of DAEs (4.1) is equal to five (Blajer and Kołodziejczyk, 2004), which produces nuisances in their numerical treatment (most DAE solvers are designed for index-one DAEs (Ascher and Petzold, 1998)).

In general, the index of a DAE system is the number of times one needs to differentiate it to get a system of differential equations (Ascher and Petzold, 1998). This definition corresponds to the concept of *differential flatness* (Fliess *et al.*, 1995; Rouchon, 2005), i.e. an n-degree-of-freedom underactuated system, $\dim(\boldsymbol{u}) = m < n$, is differentially flat if for a set of (flat) outputs $\boldsymbol{\gamma}$, $\dim(\boldsymbol{\gamma}) = m$, all states and controls of the system can algebraically be determined in terms of $\boldsymbol{\gamma}$ and their time derivatives up to a certain order. More specifically, from DAEs (4.1), one can theoretically obtain

$$\boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{\gamma}, \dot{\boldsymbol{\gamma}}, \dots, \boldsymbol{\gamma}^{r-1}) \qquad \text{and} \qquad \boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{\gamma}, \dot{\boldsymbol{\gamma}}, \dots, \boldsymbol{\gamma}^{r}) \tag{4.2}$$

which can also be interpreted as index-one DAEs. For all the dynamic systems reported in Section 2 we have then $r = 4$ since the value of r is by one smaller than the index of DAEs (4.1). Given relationships (4.2), both the dynamics of an underactuated system in the motion specified by $\boldsymbol{\gamma}$ can be studied, and the requested control that assures realization of the motion can be determined. The problem is, however, that analytical solutions (4.2) to DAEs (4.1) are attainable usually only for simple systems, and are very difficult to derive, if possible at all, for more complex systems of technical relevance.

In order to illustrate the differential flatness concept, let us consider the two-mass system defined in Fig. 1. The respective DAE formulation, (4.1), is

$$\begin{aligned} \dot{x}_1 &= v_1 & m_1\dot{v}_1 &= k(x_2 - x_1 - d) + F \\ \dot{x}_2 &= v_2 & m_2\dot{v}_2 &= -k(x_2 - x_1 - d) \\ 0 &= x_2 - s(t) \end{aligned} \tag{4.3}$$

It can be then deduced that: $\mathbf{M} = \operatorname{diag}(m_1, m_2)$, $\mathbf{C} = [0, 1]$, $\mathbf{B} = [1, 0]$, $\mathbf{P} = \mathbf{C}\mathbf{M}^{-1}\mathbf{B}^\top = [0]$, and $p = 0$ (pure tangent realization of the servo-constraint $c \equiv x_2 - s(t) = 0$ is faced). The DAEs (4.3) can be then manipulated to

$$\begin{aligned} x_1 &= \frac{m_2}{k}\ddot{s} + (s - d) & x_2 &= s \\ v_1 &= \frac{m_2}{k}s^{(3)} + \dot{s} & v_2 &= \dot{s} \\ F &= \frac{m_1 m_2}{k}s^{(4)} + (m_1 + m_2)\ddot{s} \end{aligned} \tag{4.4}$$

which corresponds to (4.2) with $\boldsymbol{x} = [\boldsymbol{q}^\top, \boldsymbol{v}^\top] = [x_1, x_2, v_1, v_2]^\top$, $\boldsymbol{u} = [F]$, and $\boldsymbol{\gamma} = [s]$. As seen $r = 4$, and as such the index of DAEs (4.3) is equal to five. The last equation from (4.4) provides immediately a feedforward control law for the underactuated system in the partly prescribed motion. In accordance with (4.2), the third order $(r - 1 = 3)$ time derivative of $\boldsymbol{\gamma}$ is involved in the determination of the states $\boldsymbol{x}$. More strictly, the third time derivative of $\boldsymbol{\gamma}$ is involved in the determination of velocities $\boldsymbol{v} = [v_1, v_2]^\top$, while the determination of coordinates $\boldsymbol{q} = [x_1, x_2]^\top$ requires only the second order time derivative of $\boldsymbol{\gamma}$.

5. Index-three DAE formulation and solution code

As previously said, differential flatness formulation (4.2) is usually very difficult to attain analytically for more complex systems, if not beyond one's grasp. A much more applicable formulation was proposed in Blajer and Kołodziejczyk (2004), followed the projection of dynamic equations (3.1) into the complementary constrained (specified) $\mathcal{C}$ and unconstrained (unspecified) $\mathcal{D}$ subspaces of $\mathcal{N}$. The constrained m-subspace $\mathcal{C}$ has already been defined as spanned by the vectors represented as rows in the $m \times n$ constraint matrix $\mathbf{C}$. The k-subspace $\mathcal{D}$ $(k = n - m)$ so that $\mathcal{C} \cup \mathcal{D} = \mathcal{N}$ and $\mathcal{C} \cap \mathcal{D} = \mathcal{O}$ can be then defined as spanned by vectors represented as columns in an $n \times k$ full-rank matrix $\mathbf{D}$ such that $\mathbf{CD} = \mathbf{0} \Leftrightarrow \mathbf{D}^\top \mathbf{C}^\top = \mathbf{0}$ (which stands for $\mathcal{C} \cap \mathcal{D} = \mathcal{O}$), and then $\operatorname{rank}([\mathbf{C}^\top \vdots \mathbf{D}^\top]^\top) = n$ (which stands for $\mathcal{C} \cup \mathcal{D} = \mathcal{N}$); for more geometrical insight, the reader is referred to Blajer (2001). For a given matrix $\mathbf{C}$ of full row-rank, its orthogonal complement $\mathbf{D}$ as above can sometimes be guessed or found by inspection (usually for simple systems only) or determined numerically following a range of computer oriented codes. A convenient scheme of the latter code patterns after the coordinate partitioning method (Wehage and Haug, 1982), in which $\mathbf{C}$ is first factorized to $\mathbf{C} = [\mathbf{U} \vdots \mathbf{W}]$, where $\mathbf{U}$ and $\mathbf{W}$ are $m \times k$ and $m \times m$ matrices, respectively, and $\det(\mathbf{W}) \neq 0$. Then, $\mathbf{D} = [\mathbf{I} \vdots (-\mathbf{W}^{-1}\mathbf{U})^\top]^\top$, where $\mathbf{I}$ is the $k \times k$ identity matrix.

The formula of projection of (3.1) into $\mathcal{D}$ and $\mathcal{C}$ is (Blajer, 2001)

$$\begin{bmatrix} \mathbf{D}^\top \\ \mathbf{C}\mathbf{M}^{-1} \end{bmatrix} \Big(\mathbf{M}\dot{\boldsymbol{v}} + \boldsymbol{d} - \boldsymbol{f} + \mathbf{B}^\top \boldsymbol{u} \Big) = \mathbf{0} \tag{5.1}$$

The projection into $\mathcal{D}$ yields k differential equations $\mathbf{H}(\boldsymbol{q})\dot{\boldsymbol{v}} = \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{u}, t)$, where $\mathbf{H} = \mathbf{D}^\top \mathbf{M}$ is the $k \times n$ matrix and $\boldsymbol{h} = \mathbf{D}^\top(\boldsymbol{f} - \boldsymbol{d} - \mathbf{B}^\top \boldsymbol{u})$ is the k-vector. The projection into $\mathcal{C}$, after using $(3.2)_2$ for substituting $\mathbf{C}\dot{\boldsymbol{v}} = \mathbf{C}\ddot{\boldsymbol{q}}$ with $\boldsymbol{\xi}$, leads then to $\boldsymbol{\xi} + \mathbf{C}\mathbf{M}^{-1}(\boldsymbol{d} - \boldsymbol{f} + \mathbf{B}^\top \boldsymbol{u}) = \mathbf{0}$, which represents m algebraic equations in the system states $\boldsymbol{x} = [\boldsymbol{q}^\top, \boldsymbol{v}^\top]^\top$ and control inputs $\boldsymbol{u}$, whose symbolic form is $\boldsymbol{b}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{u}, t) = \mathbf{0}$. Since, according to (3.4), $\operatorname{rank}(\mathbf{C}\mathbf{M}^{-1}\mathbf{B}^\top) = p < m$, these m algebraic equations impose only p conditions on $\boldsymbol{u}$, and as such $k = m - p$ restrictions on the system motion, and in particular on the system coordinates $\boldsymbol{q}$, supplementary to m original restrictions $\boldsymbol{c}(\boldsymbol{q}, t) = \mathbf{0}$. The total number of the supplementary and original restrictions on $\boldsymbol{q}$ is thus $k + m = n$, and as such, in this indirect way, the motion of the system is 'fully specified', i.e. $\boldsymbol{q}$ and then $\boldsymbol{v}$ can be determined in terms of $\boldsymbol{\gamma}$ and their time derivatives. More strictly, since $\boldsymbol{\xi}$ involved in $\boldsymbol{b}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{u}, t) = \mathbf{0}$ depends on $\ddot{\gamma}$, $\boldsymbol{q}$ determined this way depends on at most $\ddot{\gamma}$, and $\boldsymbol{v}$ depends on at most $\boldsymbol{\gamma}^{(3)}$. Then, $\boldsymbol{u}$ determined from p independent conditions imposed by $\boldsymbol{b}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{u}, t) = \mathbf{0}$ and k differential equations $\mathbf{H}(\boldsymbol{q})\dot{\boldsymbol{v}} = \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{u}, t)$, $p + k = m$, depends on at most $\boldsymbol{\gamma}^{(4)}$. The conclusions correspond to differential flatness formulation (4.4) for the case study with $p = 0$, though the upper time derivatives $\boldsymbol{\gamma}^{(3)}$ and $\boldsymbol{\gamma}^{(4)}$ are not introduced explicitly in the present formulation. The governing equations are then

$$
\begin{aligned}
&\dot{\boldsymbol{q}} = \boldsymbol{v} && \mathbf{H}(\boldsymbol{q})\dot{\boldsymbol{v}} = \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{u}, t) \\
&\mathbf{0} = \boldsymbol{b}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{u}, t) && \mathbf{0} = \boldsymbol{c}(\boldsymbol{q}, t)
\end{aligned}
\tag{5.2}
$$

which form $n+k+m+m = 2n+m$ index-three DAEs in $n+n+m = 2n+m$ variables $\boldsymbol{q}$, $\boldsymbol{v}$ and $\boldsymbol{u}$, which are just in the middle between initial index-five DAE formulation (4.1) and differential flatness (index-one DAE) formulation (4.2).

The solution to DAEs (5.2) are variations in time of state variables of the system executing the prescribed motion, $\boldsymbol{q}_d(t)$ and $\boldsymbol{v}_d(t)$, and the control $\boldsymbol{u}_d(t)$ that is required to enforce the motion. Due to a specific structure of DAE system (5.2), a simple algorithm based on the Euler backward approximation method can be recommended to solve the DAEs. Namely, given $\boldsymbol{q}_n$ and $\boldsymbol{v}_n$ at time t_n ($\boldsymbol{u}_n$ is not required), the values $\boldsymbol{q}_{n+1}$, $\boldsymbol{v}_{n+1}$ and $\boldsymbol{u}_{n+1}$ at time $t_{n+1} = t_n + \Delta t$, can be found as solution to the following nonlinear equations

$$
\begin{aligned}
&\frac{\boldsymbol{q}_{n+1} - \boldsymbol{q}_n}{\Delta t} - \boldsymbol{v}_{n+1} = \mathbf{0} \\
&\mathbf{H}(\boldsymbol{q}_{n+1})\frac{\boldsymbol{v}_{n+1} - \boldsymbol{v}_n}{\Delta t} - \boldsymbol{h}(\boldsymbol{q}_{n+1}, \boldsymbol{v}_{n+1}, \boldsymbol{u}_{n+1}, t_{n+1}) = \mathbf{0} \\
&\boldsymbol{b}(\boldsymbol{q}_{n+1}, \boldsymbol{v}_{n+1}, \boldsymbol{u}_{n+1}, t_{n+1}) = \mathbf{0} \qquad \boldsymbol{c}(\boldsymbol{q}_{n+1}, t_{n+1}) = \mathbf{0}
\end{aligned}
\tag{5.3}
$$

where Δt is the integration time step. This effective simple scheme is of acceptable accuracy for an appropriately small value of Δt. Then, due to the original servo-constraints $\boldsymbol{c}(\boldsymbol{q}, t) = \boldsymbol{0}$ involved in (5.2)/(5.3), supplemented then by $\boldsymbol{b}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{u}, t) = \boldsymbol{0}$, dependent respectively on $\boldsymbol{\gamma}(t)$ and $\ddot{\boldsymbol{\gamma}}(t)$, $\boldsymbol{q}_d(t)$ is determined with a numerical accuracy of solving (5.3), and only $\boldsymbol{v}_d(t)$ and $\boldsymbol{u}_d(t)$ are determined with an error followed the rough backward difference method. The truncation errors do not accumulate in the simulation time, however. The proposed simple solution code leads thus to reasonable and stable solutions. The inverse simulation control $\boldsymbol{u}_d(t)$ can be used as feedforward control law for the system subjected to execute the desired motion. The solution scheme can be then modified to include feedback control to provide stable tracking of the specified outputs $\boldsymbol{\gamma}(t)$ in the presence of possible external perturbations modeling inconsistencies; see Blajer and Kołodziejczyk (2004) for more details.

6. Discussion and conclusions

The formulation developed in this paper relate a class of underactuated mechanical systems in partly specified motion, whose relevant technical examples were reported in Section 2. All of them are differentially flat systems (Fliess *et al.*, 1995; Rouchon, 2005), which assures solvability of this specific inverse dynamics problem (controllability of the system in specified motion). From this point of view, the role of flat systems within the set of underactuated systems is very similar to the role of integrable systems within the set of ordinary differential equation systems (Rouchon, 2005). On the other hand, there is no a general algorithm to decide whether the system is flat or not. The definition that all the states and control inputs of a flat system can be determined in terms of the desired outputs and their time derivatives may be difficult to implement for more complex systems.

Differential flatness formulation (4.2), due to its inherent complexity, is applicable rather to simple systems only. Much more convenient in practical application is index-three DAE formulation (5.2) motivated in this paper. While both the formulations lead to exactly the same (numerical) solutions, the excessive complexity of the differential flatness formulation is completely removed in the present formulation. Of importance is also that, for the considered class of differentially flat systems, the differential flatness formulation involves $\boldsymbol{\gamma}^{(4)}$, while the present formulation requires $\ddot{\boldsymbol{\gamma}}$ only. The analytical solution of the differential flatness formulation is then replaced by simple (and stable) numerical solution code (5.3).

The present formulation has been successfully applied to numerical simulations of many underactuated mechanical systems executing partly specified motions, including the two-mass system (Blajer and Kołodziejczyk, 2004) (reported also briefly in this paper), an aircraft in a prescribed trajectory flight (Blajer *et al.*, 2001), and cranes executing prescribed motions of payloads (Blajer and Kołodziejczyk, 2007). Further developments of this useful approach to the inverse dynamics problem of underactuated systems subjected to execute partly specified motions are foreseen within the frame of the project mentioned below.

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Modelowanie sterowanych układów mechanicznych realizujących ruch programowy niezupełny

Streszczenie

Praca dotyczy zagadnienia symulacji dynamicznej odwrotnej klasy układów mechanicznych, w których liczba sygnałów sterowania, równa liczbie zadanych charakterystyk ruchu (sygnałów wyjściowych), jest mniejsza od liczby stopni swobody, reprezentowanej przez liczne przykłady techniczne. Równania tak zdefiniowanego ruchu programowego niezupełnego otrzymywane są w postaci równań różniczkowo-algebraicznych o indeksie równym trzy. Omawiany jest prosty, efektywny i stabilny algorytm całkowania tych równań. Rozwiązaniem są przebiegi w czasie zmiennych stanu układu w zadanym ruchu programowym oraz sterowania wymaganego dla realizacji tego ruchu.

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