

THE ONSET OF CONVECTION IN A ROTATING MULTICOMPONENT FLUID LAYER

JYOTI PRAKASH, VIRENDER SINGH, RAJEEV KUMAR, KULTARAN KUMARI

Department of Mathematics and Statistics, Himachal Pradesh University, Shimla, India

e-mail: jpsmaths67@gmail.com

The onset of convective instability is analysed in a rotating multicomponent fluid layer in which density depends on n stratifying agents (one of them is heat) having different diffusivities. Two problems have been analysed mathematically. In the first problem, a sufficient condition is derived for the validity of the principle of the exchange of stabilities. Further, when the complement of this condition holds good, oscillatory motions of neutral or growing amplitude can exist, and thus it is important to derive upper bounds for the complex growth rate of such motions when at least one of the bounding surfaces is rigid so that exact solutions of the problem in closed form are not obtainable. Thus, as the second problem, bounds for the complex growth rates are also obtained. Above results are uniformly valid for quite general nature of the bounding surfaces.

Keywords: multicomponent convection, principle of exchange of stabilities, oscillatory motions, complex growth rate, concentration Rayleigh number, Lewis number

1. Introduction

When density of a fluid is determined by two stratifying agents, such as heat and salt diffusing at different rates, the fluid at rest can be unstable even if its density increases downward. This convective phenomenon is known as thermosolutal convection or more generally as double diffusive convection. This phenomenon has, now, been extensively studied. For review on the subject of double diffusive convection one may be referred to (Turner, 1973, 1974, 1985; Brandt and Fernando, 1996; Radko, 2013; Sekar *et al.*, 2013).

Although the subject of double diffusive convection is still an important area of research (Sekar *et al.*, 2013; Kellner and Tilgner, 2014; Nield and Kuznetsov, 2011; Schmitt, 2011), there are many fluid systems where more than two components are present (Turner, 1985; Griffiths, 1979b). Examples of such systems include the solidification of molten alloys, Earth core, geothermally heated lakes, sea water, magmas and their laboratory models. The presence of more than one salt in fluid mixtures is very often requested for describing natural phenomena such as contaminant transport, acid rain effects, underground water flow and warming of the stratosphere. The subject of more than two stratifying agents has attracted many researchers (Griffiths, 1979a,b; Pearlstein *et al.*, 1989; Rionero, 2013a,b, 2014; Lopez *et al.*, 1990; Terrones, 1993; Polikakos, 1985; Shivakumara and Naveen Kumar, 2014). In double diffusive convection (Turner, 1974) or, more generally, in multicomponent convection (Turner, 1985; Griffiths, 1979a) instability may occur in two kinds: first in form of steady (or stationary) convection which is called as ‘salt finger’ modes and the second in form of oscillatory motions of growing amplitude (or overstability) which is called as ‘diffusive convection’. When a warm and saltier fluid lies above a cold and fresh fluid then stationary convection is preferred, and when a cold and fresher fluid lies above a warm and saltier fluid then oscillatory motions are preferred. The essence of these researchers is that small concentrations of the third diffusing component with a smaller mass

diffusivity can have a significant effect upon the nature of diffusive instabilities and diffusive convection. The salt finger modes are simultaneously unstable under a wide range of conditions when the density gradients due to components with the greatest and smallest diffusivity are of the same signs even if the overall density stratification is hydrostatically stable. These researchers also notice some fundamental differences between doubly and triply diffusive convection. One is that if the gradients of two of the stratifying agents are held fixed, then three critical values of the Rayleigh number of the third agent are sometimes required to specify the linear stability criteria (in double diffusive convection only one critical Rayleigh number is required). The other difference is that the onset of convection for the case of free boundaries may occur via quassiperiodic bifurcation from the motionless basic state.

Now the triply diffusive convection despite its complexities has also been well studied. But, to the author knowledge, not many investigations have been conducted on stability theory when more than three components are present, which may be, perhaps, due to the complexities involved in mathematical calculations and numerical computations. Some worth researches which may be referred here are due to Terrones and Pearlstein (1989) who derived analytical results for n components and numerical results for $n = 5$ using dynamically free boundary conditions. Later Lopez *et al.* (1990) predicted that the results of triply diffusive convection may be extended to multicomponent convection with n components for rigid surfaces also. Further significant contributions to multicomponent convection are due to Ryzhkov and Shevtsova (2007, 2009) and Ryzhkov (2013).

The establishment of nonoccurrence of any slow oscillatory motions which may be neutral or unstable implies the validity of the principle of the exchange of stabilities (PES). The validity of this principle in stability problems eliminates unsteady terms from linearized perturbation equations which results in notable mathematical simplicity since the transition from stability to instability occurs via a marginal state which is characterized by the vanishing of both real and imaginary parts of the complex time eigenvalue associated with the perturbation. Pellew and Southwell (1940) proved the validity of PES (i.e. occurrence of stationary convection) for the classical Rayleigh-Benard instability problem. Prakash *et al.* (2014a) established such a criterion for the triply diffusive convection problem.

To study the effect of rotation on a multicomponent fluid layer is an interesting topic. Prakash *et al.* (2014b) derived a sufficient condition for the occurrence of stationary convection and upper bounds (Prakash *et al.*, 2015) for the complex growth rate of an arbitrary oscillatory motion of neutral or growing amplitude in rotatory hydrodynamic triply diffusive convection. The further extension of these results to the problem of the onset of convection in a multicomponent fluid layer in the domains of astrophysics and terrestrial physics, wherein the liquid concerned has the property of electrical conduction and the magnetic field and rotation are prevalent, is very much sought after in the present context.

In the present work, we analyse the onset of buoyancy driven convection in a multicomponent fluid layer in the presence of uniform vertical rotation. We generalize the existing results of the rotatory hydrodynamic triply diffusive convection problem concerning the validity of the principle of the exchange of stabilities (Prakash *et al.*, 2014b) and arresting the complex growth rate of oscillatory motion (when it occurs) (Prakash *et al.*, 2015) which are important especially when at least one boundary is rigid so that exact solutions in the closed form are not obtainable. To the authors knowledge, no such results have been obtained so far for the hydrodynamical systems with more than three components. The results derived herein are uniformly valid for any combination of the rigid and free boundaries and the results of doubly diffusive (Banerjee *et al.*, 1981; Gupta *et al.*, 1986) and triply diffusive convection (Prakash *et al.*, 2014a,b,c, 2015) follow as a consequence. Further, the importance of the results obtained herein lies in that these results may be used for any rotatory hydrodynamic multicomponent system where no mathematical calculation or numerical computation is possible.

2. Mathematical formulation and analysis

A viscous finitely heat conducting Boussinesq fluid of infinite horizontal extension is statistically confined between two horizontal boundaries $z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and $T_1 (< T_0)$ and uniform concentrations $S_{10}, S_{20}, \dots, S_{(n-1)0}$ and $S_{11} (< S_{10}), S_{21} (< S_{20}), \dots, S_{(n-1)1} (< S_{(n-1)0})$ in the force field of gravity and in the presence of uniform vertical rotation (as shown in Fig. 1). It is assumed that the cross-diffusion effects of the stratifying agents can be neglected.

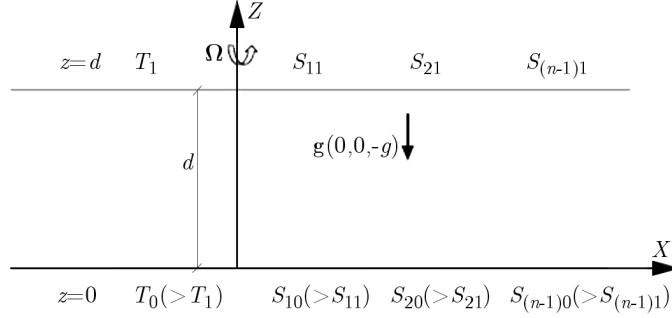


Fig. 1. Physical configuration

The basic equations that govern the motion of the rotatory hydrodynamic multicomponent fluid layer are as follows (Prakash *et al.*, 2014b; Terrones and Pearlstein, 1989)

$$\begin{aligned}
\frac{\partial u_j}{\partial x_j} &= 0 \\
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= -\frac{\partial}{\partial x_i} \left(\frac{P_1}{\rho_0} - \frac{1}{2} |(\mathbf{\Omega} \times \mathbf{r})|^2 \right) + \left(1 + \frac{\delta \rho}{\rho_0} + \frac{\delta \rho_1}{\rho_0} + \frac{\delta \rho_2}{\rho_0} + \dots + \frac{\delta \rho_{n-1}}{\rho_0} \right) X_i \\
&\quad + 2\varepsilon_{ijk} u_j \Omega_k + \nu \nabla^2 u_i \\
\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} &= \kappa \nabla^2 T \\
\frac{\partial S_1}{\partial t} + u_j \frac{\partial S_1}{\partial x_j} &= \kappa_1 \nabla^2 S_1 \\
\frac{\partial S_2}{\partial t} + u_j \frac{\partial S_2}{\partial x_j} &= \kappa_2 \nabla^2 S_2 \\
&\vdots \\
\frac{\partial S_{n-1}}{\partial t} + u_j \frac{\partial S_{n-1}}{\partial x_j} &= \kappa_{n-1} \nabla^2 S_{n-1}
\end{aligned} \tag{2.1}$$

where ρ is density; t is time; x_j ($j = 1, 2, 3$) are cartesian coordinates x, y, z ; u_j ($j = 1, 2, 3$) are velocity components; X_i ($i = 1, 2, 3$) are components of the external force in the x, y, z directions, respectively; P_1 is pressure; μ is viscosity; $\mathbf{\Omega}$ is angular velocity; T is temperature, κ is the coefficient of thermal diffusivity; S_1, S_2, \dots, S_{n-1} are $n - 1$ concentrations and $\kappa_1, \kappa_2, \dots, \kappa_{n-1}$ are respectively the coefficients of mass diffusivity of S_1, S_2, \dots, S_{n-1} with $\kappa_1 > \kappa_2 > \dots > \kappa_{n-1}$.

The above basic equations must be supplemented by the equation of state

$$\rho = \rho_0 [1 + \alpha(T_0 - T_1) - \alpha_1(S_{10} - S_{11}) - \alpha_2(S_{20} - S_{21}) - \dots - \alpha_{n-1}(S_{(n-1)0} - S_{(n-1)1})] \tag{2.2}$$

where $\alpha, \alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are respectively the coefficients of volume expansion due to temperature variation and concentration variations for $n - 1$ concentration components S_1, S_2, \dots, S_{n-1} .

ρ_0 is the value of ρ at $z = 0$. $\nu = \mu/\rho_0$ is kinematic viscosity and $\frac{P_1}{\rho_0} - \frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2$ is hydrostatic pressure.

The basic state is assumed to be stationary, and the standard linear stability analysis procedure as outlined in the studies of Prakash *et al.* (2014b) is followed to obtain the following non-dimensional stability equations

$$\begin{aligned} (D^2 - a^2)\left(D^2 - a^2 - \frac{\sigma}{\text{Pr}}\right)w &= \text{Ra}a^2\theta - R_1a^2\phi_1 - R_2a^2\phi_2 - \dots - R_{n-1}a^2\phi_{n-1} + \text{Ta}D\zeta \\ (D^2 - a^2 - \sigma)\theta &= -w \\ \left(D^2 - a^2 - \frac{\sigma}{\text{Le}_1}\right)\phi_1 &= -\frac{w}{\text{Le}_1} \\ \left(D^2 - a^2 - \frac{\sigma}{\text{Le}_2}\right)\phi_2 &= -\frac{w}{\text{Le}_2} \\ &\vdots \\ \left(D^2 - a^2 - \frac{\sigma}{\text{Le}_{n-1}}\right)\phi_{n-1} &= -\frac{w}{\text{Le}_{n-1}} \end{aligned} \quad (2.3)$$

and

$$\left(D^2 - a^2 - \frac{\sigma}{\text{Pr}}\right)\zeta = -Dw \quad (2.4)$$

respectively.

Equations (2.3) and (2.4) are to be solved using the following appropriate boundary conditions:

— $w = 0 = \theta = \phi_1 = \phi_2 = \dots = \phi_{n-1}$ on both the horizontal boundaries which are at

$$z = 0 \quad \text{and} \quad z = 1 \quad (2.5)$$

— and on rigid boundary

$$Dw = \zeta = 0 \quad (2.6)$$

— or on free boundary

$$D^2w = D\zeta = 0 \quad (2.7)$$

the meaning of the symbols involved in Eqs. (2.3) and (2.4) from the physical point of view are as follows: z is the vertical coordinate, $D = d/dz$ is differentiation w.r.t. z , $a^2 > 0$ is square of the wave number, $\text{Pr} > 0$ is the thermal Prandtl number which is a measure of relative importance of heat conduction and viscosity of the fluid and varies from fluid to fluid. For air $\text{Pr} = 0.7$ (approximately), for water $\text{Pr} = 7$ (approximately), for mercury $\text{Pr} = 0.044$ (approximately) and for glycerine $\text{Pr} = 7250$. The Prandtl number of some fluids (particularly water) depends considerably on temperature. $\text{Le}_1 > 0, \text{Le}_2 > 0, \dots, \text{Le}_{n-1} > 0$ are the Lewis numbers for $n - 1$ concentrations S_1, S_2, \dots, S_{n-1} , respectively, $\text{Ta} > 0$ is the Taylor number, $\text{Ra} > 0$ is the thermal Rayleigh number, $R_1 > 0, R_2 > 0, \dots, R_{n-1} > 0$ are the concentration Rayleigh numbers for the $n - 1$ concentration components. A concentration Rayleigh number is the ratio of the buoyancy forces (which drive free convective transport of solute) to dispersive/viscous forces (which disperse solute and dissipate free convective transport). In the present problem, these have stabilizing effect on the onset of instability. w is vertical velocity, θ is temperature and $\phi_1, \phi_2, \dots, \phi_{n-1}$ are respective concentrations of the $n - 1$ components. $\sigma = \sigma_r + i\sigma_i$ is the complex growth rate where σ_r and σ_i are real constants. For $\sigma_r < 0$, the system is always stable while for $\sigma_r > 0$, the system becomes unstable. When $\sigma = 0$, the system is marginally stable

ensuring the validity of the principle of the exchange of stabilities. When $\sigma_r \geq 0$ and $\sigma_i \neq 0$, the overstability of periodic motion is possible and oscillatory motions of growing or neutral amplitude occur. It may further be noted that equations (2.3) and (2.4) describe an eigenvalue problem for p and govern rotatory hydrodynamic multicomponent convection for quite general nature of the bounding surfaces.

Theorem 1: If $(w, \theta, \phi_1, \phi_2, \dots, \phi_{n-1}, \zeta, \sigma)$, $Ra > 0$, $R_1 > 0$, $R_2 > 0, \dots, R_{n-1} > 0$, $Ta > 0$, $\sigma_r \geq 0$ is a solution to Eqs. (2.3) and (2.4) together with boundary conditions (2.5)-(2.7) and

$$\frac{R_1 Pr}{2Le_1^2 \pi^4} + \frac{R_2 Pr}{2Le_2^2 \pi^4} + \dots + \frac{R_{n-1} Pr}{2Le_{n-1}^2 \pi^4} + \frac{Ta}{\pi^4} \leq 1$$

then $\sigma_i = 0$. In particular,

$$\sigma_r = 0 \Rightarrow \sigma_i = 0 \quad \text{if} \quad \frac{R_1 Pr}{2Le_1^2 \pi^4} + \frac{R_2 Pr}{2Le_2^2 \pi^4} + \dots + \frac{R_{n-1} Pr}{2Le_{n-1}^2 \pi^4} + \frac{Ta}{\pi^4} \leq 1$$

Proof: Multiplying Eq. (2.3)₁ by w^* (the superscript * henceforth denotes complex conjugation) throughout and integrating the resulting equation over the vertical range of z , we have

$$\begin{aligned} \int_0^1 w^* (D^2 - a^2) \left(D^2 - a^2 - \frac{\sigma}{Pr} \right) w \, dz &= Ra a^2 \int_0^1 w^* \theta \, dz - R_1 a^2 \int_0^1 w^* \phi_1 \, dz \\ &- R_2 a^2 \int_0^1 w^* \phi_2 \, dz - \dots - R_{n-1} a^2 \int_0^1 w^* \phi_{n-1} \, dz + Ta \int_0^1 w^* D \zeta \, dz \end{aligned} \tag{2.8}$$

Making use of Eqs. (2.3)₂₋₆ and (2.4), we can write

$$\begin{aligned} \int_0^1 w^* (D^2 - a^2) \left(D^2 - a^2 - \frac{\sigma}{Pr} \right) w \, dz &= -Ra a^2 \int_0^1 \theta (D^2 - a^2 - \sigma^*) \theta^* \, dz \\ &+ R_1 a^2 Le_1 \int_0^1 \phi_1 \left(D^2 - a^2 - \frac{\sigma^*}{Le_1} \right) \phi_1^* \, dz + R_2 a^2 Le_2 \int_0^1 \phi_2 \left(D^2 - a^2 - \frac{\sigma^*}{Le_2} \right) \phi_2^* \, dz + \dots \\ &+ R_{n-1} a^2 Le_{n-1} \int_0^1 \phi_{n-1} \left(D^2 - a^2 - \frac{\sigma^*}{Le_{n-1}} \right) \phi_{n-1}^* \, dz + Ta \int_0^1 \zeta \left(D^2 - a^2 - \frac{\sigma^*}{Pr} \right) \zeta^* \, dz \end{aligned} \tag{2.9}$$

Integrating the various terms of Eq. (2.9) by parts for an appropriate number of times and utilizing boundary conditions (2.5)-(2.7), we obtain

$$\begin{aligned} &\int_0^1 (|D^2 w|^2 + 2a^2 |Dw|^2 + a^4 |w|^2) + \frac{\sigma}{Pr} \int_0^1 (|Dw|^2 + a^2 |w|^2) \, dz \\ &= Ra a^2 \int_0^1 (|D\theta|^2 + a^2 |\theta|^2 + \sigma^* |\theta|^2) \, dz - R_1 a^2 Le_1 \int_0^1 (|D\phi_1|^2 + a^2 |\phi_1|^2 + \frac{\sigma^*}{Le_1} |\phi_1|^2) \, dz \\ &- R_2 a^2 Le_2 \int_0^1 (|D\phi_2|^2 + a^2 |\phi_2|^2 + \frac{\sigma^*}{Le_2} |\phi_2|^2) \, dz - \dots \end{aligned} \tag{2.10}$$

$$\begin{aligned}
 & - R_{n-1}a^2Le_{n-1} \int_0^1 (|D\phi_{n-1}|^2 + a^2|\phi_{n-1}|^2 + \frac{\sigma^*}{Le_{n-1}}|\phi_{n-1}|^2) dz \\
 & - Ta \int_0^1 (|D\zeta|^2 + a^2|\zeta|^2 + \frac{\sigma^*}{Pr}|\zeta|^2) dz
 \end{aligned}$$

Equating the imaginary parts of both sides of Eq. (2.10) and cancelling $\sigma_i (\neq 0)$ throughout from the resulting equation, we have

$$\begin{aligned}
 \frac{1}{Pr} \int_0^1 (|Dw|^2 + a^2|w|^2) dz &= -Raa^2 \int_0^1 |\theta|^2 dz + R_1a^2 \int_0^1 |\phi_1|^2 dz \\
 + R_2a^2 \int_0^1 |\phi_2|^2 dz + \dots + R_{n-1}a^2 \int_0^1 |\phi_{n-1}|^2 dz &+ \frac{Ta}{Pr} \int_0^1 |\zeta|^2 dz
 \end{aligned} \tag{2.11}$$

Now, from equation (2.3)₃ we derive that

$$\int_0^1 (D^2 - a^2 - \frac{\sigma}{Le_1})\phi_1 (D^2 - a^2 - \frac{\sigma^*}{Le_1})\phi_1^* dz = \frac{1}{Le_1^2} \int_0^1 |w|^2 dz \tag{2.12}$$

Integrating the various terms on the left hand side of Eq. (2.12) by parts for an appropriate number of times and making use of the boundary conditions on ϕ_1 , it follows that

$$\begin{aligned}
 \int_0^1 (|D^2\phi_1|^2 + 2a^2|D\phi_1|^2 + a^4|\phi_1|^2) dz + \frac{2\sigma_r}{Le_1} \int_0^1 (|D\phi_1|^2 + a^2|\phi_1|^2) dz \\
 + \frac{|\sigma|^2}{Le_1^2} \int_0^1 |\phi_1|^2 dz = \frac{1}{Le_1^2} \int_0^1 |w|^2 dz
 \end{aligned} \tag{2.13}$$

Since $\sigma_r \geq 0$, it follows from Eq. (2.13) that

$$2a^2 \int_0^1 |D\phi_1|^2 dz \leq \frac{1}{Le_1^2} \int_0^1 |w|^2 dz \tag{2.14}$$

Now, since $\phi_1, \phi_2, \dots, \phi_{n-1}$ and w satisfy the boundary conditions $\phi_1(0) = 0 = \phi_1(1)$, $\phi_2(0) = 0 = \phi_2(1), \dots, \phi_{n-1}(0) = 0 = \phi_{n-1}(1)$, $w(0) = 0 = w(1)$, we have from the Rayleigh-Ritz inequality (Schultz, 1973)

$$\begin{aligned}
 \int_0^1 |D\phi_1|^2 dz &\geq \pi^2 \int_0^1 |\phi_1|^2 dz \\
 \int_0^1 |D\phi_2|^2 dz &\geq \pi^2 \int_0^1 |\phi_2|^2 dz \\
 &\vdots \\
 \int_0^1 |D\phi_{n-1}|^2 dz &\geq \pi^2 \int_0^1 |\phi_{n-1}|^2 dz
 \end{aligned} \tag{2.15}$$

and

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz \tag{2.16}$$

respectively.

Utilizing inequalities (2.15)₁ and (2.16) in inequality (2.14), we get

$$a^2 \int_0^1 |\phi_1|^2 dz \leq \frac{1}{2Le_1^2 \pi^4} \int_0^1 |Dw|^2 dz \tag{2.17}$$

In the same manner, we obtain from Eqs. (2.3)₄₋₆ the inequalities

$$\begin{aligned} a^2 \int_0^1 |\phi_2|^2 dz &\leq \frac{1}{2Le_2^2 \pi^4} \int_0^1 |Dw|^2 dz \\ &\vdots \\ a^2 \int_0^1 |\phi_{n-1}|^2 dz &\leq \frac{1}{2Le_{n-1}^2 \pi^4} \int_0^1 |Dw|^2 dz \end{aligned} \tag{2.18}$$

respectively.

Now for the case of rigid boundaries, $\zeta(0) = 0 = \zeta(1)$, again from the Rayleigh-Ritz inequality (Schultz, 1973), we obtain

$$\int_0^1 |D\zeta|^2 dz \geq \pi^2 \int_0^1 |\zeta|^2 dz \tag{2.19}$$

Multiplying Eq. (2.4) by ζ^* and integrating over the vertical range of z , we get from the real part of the final equation

$$\begin{aligned} \int_0^1 (|D\zeta|^2 + a^2|\zeta|^2 + \sigma_r|\zeta|^2) &= \Re \int_0^1 \zeta^* Dw dz \leq \left| \int_0^1 \zeta^* Dw dz \right| \leq \int_0^1 |\zeta^* Dw| dz \\ &\leq \int_0^1 |\zeta^*| |Dw| dz \leq \int_0^1 |\zeta| |Dw| dz \leq \sqrt{\int_0^1 |\zeta|^2 dz} \sqrt{\int_0^1 |Dw|^2 dz} \end{aligned} \tag{2.20}$$

(using Schwartz inequality) which implies that

$$\int_0^1 |D\zeta|^2 dz \leq \sqrt{\int_0^1 |\zeta|^2 dz} \sqrt{\int_0^1 |Dw|^2 dz}$$

and thus using inequality (2.19) for the case of rigid boundaries and the result $\int_0^1 |D\zeta|^2 dz = \pi^2 \int_0^1 |\zeta|^2 dz$ for the case of free boundaries (Banerjee *et al.*, 1995), we obtain

$$\pi^2 \int_0^1 |\zeta|^2 dz \leq \sqrt{\int_0^1 |\zeta|^2 dz} \sqrt{\int_0^1 |Dw|^2 dz}$$

which gives

$$\sqrt{\int_0^1 |\zeta|^2 dz} \leq \frac{1}{\pi^2} \sqrt{\int_0^1 |Dw|^2 dz}$$

which implies that

$$\int_0^1 |\zeta|^2 dz \leq \frac{1}{\pi^4} \int_0^1 |Dw|^2 dz \quad (2.21)$$

Now using inequalities (2.17), (2.18) and (2.21) in Eq. (2.11), we obtain

$$\begin{aligned} & \left[\frac{1}{\text{Pr}} - \left(\frac{R_1}{2\text{Le}_1^2\pi^4} + \frac{R_2}{2\text{Le}_2^2\pi^4} + \dots + \frac{R_{n-1}}{2\text{Le}_{n-1}^2\pi^4} + \frac{\text{Ta}}{\pi^4} \right) \right] \int_0^1 |Dw|^2 dz \\ & + \frac{a^2}{\sigma} \int_0^1 |w|^2 dz + \text{Raa}^2 \int_0^1 |\theta|^2 dz < 0 \end{aligned} \quad (2.22)$$

which clearly implies (for $\sigma_i \neq 0$) that

$$\frac{R_1\text{Pr}}{2\text{Le}_1^2\pi^4} + \frac{R_2\text{Pr}}{2\text{Le}_2^2\pi^4} + \dots + \frac{R_{n-1}\text{Pr}}{2\text{Le}_{n-1}^2\pi^4} + \frac{\text{Ta}}{\pi^4} > 1 \quad (2.23)$$

Hence if $\frac{R_1\text{Pr}}{2\text{Le}_1^2\pi^4} + \frac{R_2\text{Pr}}{2\text{Le}_2^2\pi^4} + \dots + \frac{R_{n-1}\text{Pr}}{2\text{Le}_{n-1}^2\pi^4} + \frac{\text{Ta}}{\pi^4} \leq 1$, then we must have $\sigma_i = 0$.

This proves the theorem.

The essential content of Theorem 1 from the physical point of view is that for the problem of rotatory hydrodynamic multicomponent convection, an arbitrary neutral or unstable mode of the system is definitely non-oscillatory in character and, in particular, ‘the principle of the exchange of stabilities’ is valid if $\frac{R_1\text{Pr}}{2\text{Le}_1^2\pi^4} + \frac{R_2\text{Pr}}{2\text{Le}_2^2\pi^4} + \dots + \frac{R_{n-1}\text{Pr}}{2\text{Le}_{n-1}^2\pi^4} + \frac{\text{Ta}}{\pi^4} \leq 1$. Further, the above result is uniformly valid for quite general nature of the boundaries.

Special cases: It follows from Theorem 1 that an arbitrary neutral or unstable mode is non oscillatory in character, and in particular PES is valid for:

1. Rayleigh-Benard convection ($R_1 = R_2 = \dots = R_{n-1} = \text{Ta} = 0$) (Pellew and Southwell, 1940)
2. Rotatory Rayleigh-Benard convection ($R_1 = R_2 = \dots = R_{n-1} = 0$) if $\text{Ta}/\pi^4 \leq 1$ (Gupta *et al.*, 1986)
3. Rotatory thermohaline convection ($R_1 > 0, R_2 = \dots = R_{n-1} = 0, T_a > 0$) if $\frac{R_1\text{Pr}}{2\text{Le}_1^2\pi^4} + \frac{\text{Ta}}{\pi^4} \leq 1$ (Gupta *et al.*, 1986)
4. Thermohaline convection ($R_1 > 0, R_2 = \dots = R_{n-1} = \text{Ta} = 0$) if $\frac{R_1\text{Pr}}{2\text{Le}_1^2\pi^4} \leq 1$ (Gupta *et al.*, 1986)
5. Rotatory hydrodynamic triply diffusive convection ($R_1 > 0, R_2 > 0, R_3 = \dots = R_{n-1} = 0, \text{Ta} > 0$) if $\frac{R_1\text{Pr}}{2\text{Le}_1^2\pi^4} + \frac{R_2\text{Pr}}{2\text{Le}_2^2\pi^4} + \frac{\text{Ta}}{\pi^4} \leq 1$ (Prakash *et al.*, 2014b)
6. Triply diffusive convection ($R_1 > 0, R_2 > 0, R_3 = \dots = R_{n-1} = \text{Ta} = 0$) if $\frac{R_1\text{Pr}}{2\text{Le}_1^2\pi^4} + \frac{R_2\text{Pr}}{2\text{Le}_2^2\pi^4} \leq 1$ (Prakash *et al.*, 2014a)

Proceeding in this manner, we can obtain conditions for stationary convection for all configurations with $3, 4, \dots, n - 1$ concentration components, respectively.

Since the complement of the above result implies the occurrence of oscillatory motions, thus it is important to derive the bounds for the complex growth rate of oscillatory motions. We prove the following theorem in this direction.

Theorem 2: If $R_a > 0, R_1 > 0, R_2 > 0, \dots, R_{n-1} > 0, Ta > 0, \sigma_r \geq 0$ and $\sigma_i \neq 0$, then a necessary condition for the existence of a nontrivial solution $(w, \theta, \phi_1, \phi_2, \dots, \phi_{n-1}, \zeta, \sigma)$ to Eqs. (2.3) and (2.) together with boundary conditions (2.5)-(2.7) is that

$$|\sigma| < \max\left\{\sqrt{(R_1 + R_2 + \dots + R_{n-1})Pr}, \sqrt{TaPr}\right\}$$

Proof: Rewriting equation (2.11) for ready reference, we have

$$\begin{aligned} \frac{1}{Pr} \int_0^1 (|Dw|^2 + a^2|w|^2) dz &= -Raa^2 \int_0^1 |\theta|^2 dz + R_1a^2 \int_0^1 |\phi_1|^2 dz \\ &+ R_2a^2 \int_0^1 |\phi_2|^2 dz + \dots + R_{n-1}a^2 \int_0^1 |\phi_{n-1}|^2 dz + \frac{Ta}{Pr} \int_0^1 |\zeta|^2 dz \end{aligned}$$

Now since $\sigma_r \geq 0$, it follows from Eq. (2.13) that

$$\int_0^1 |\phi_1|^2 dz \leq \frac{1}{|\sigma|^2} \int_0^1 |w|^2 dz \tag{2.24}$$

Similarly, from Eqs. (2.3)₄ and (2.3)₆, by adopting the same procedure, we get

$$\begin{aligned} \int_0^1 |\phi_2|^2 dz &\leq \frac{1}{|\sigma|^2} \int_0^1 |w|^2 dz \\ \vdots & \\ \int_0^1 |\phi_{n-1}|^2 dz &\leq \frac{1}{|\sigma|^2} \int_0^1 |w|^2 dz \end{aligned} \tag{2.25}$$

respectively.

Multiply Eq. (2.4) by its complex conjugate, integrating the resulting equation by parts for an appropriate number of times and using boundary conditions (2.5)-(2.7), we have

$$\begin{aligned} \int_0^1 (|D^2\zeta|^2 + 2a^2|D\zeta|^2 + a^4|\zeta|^2) dz + \frac{2\sigma_r}{Pr} \int_0^1 (|D\zeta|^2 + a^2|\zeta|^2) dz \\ + \frac{|\sigma|^2}{Pr^2} \int_0^1 |\zeta|^2 dz = \int_0^1 |Dw|^2 dz \end{aligned} \tag{2.26}$$

Since, $\sigma_r \geq 0$, it follows from Eq. (2.26) that

$$\int_0^1 |\zeta|^2 dz \leq \frac{Pr^2}{|\sigma|^2} \int_0^1 |Dw|^2 dz \tag{2.27}$$

Now making use of inequalities (2.24), (2.25) and (2.27), in Eq. (2.11), we have

$$\begin{aligned} & \frac{1}{\text{Pr}} \left(1 - \frac{\text{TaPr}^2}{|\sigma|^2}\right) \int_0^1 |Dw|^2 dz + \frac{a^2}{\text{Pr}} \left[1 - \frac{(R_1 + R_2 + \dots + R_{n-1})\text{Pr}}{|\sigma|^2}\right] \int_0^1 |w|^2 dz \\ & + \text{Raa}^2 \int_0^1 |\theta|^2 dz < 0 \end{aligned} \quad (2.28)$$

which clearly implies that

$$|\sigma| < \max\left\{\sqrt{(R_1 + R_2 + \dots + R_{n-1})\text{Pr}}, \sqrt{\text{TaPr}}\right\}$$

This establishes the desired result.

The above theorem may be stated in an equivalent form as: the complex growth rate of an arbitrary, neutral or unstable oscillatory perturbation of growing amplitude in a rotatory hydrodynamic multicomponent fluid layer heated from below must lie inside a semi-circle in the right half of the (p_r, p_i) -plane whose centre is at the origin and radius equals $\max\left\{\sqrt{(R_1 + R_2 + \dots + R_{n-1})\text{Pr}}, \sqrt{\text{TaPr}}\right\}$. Further, it is proved that this result is uniformly valid for quite general nature of the bounding surfaces.

Special cases: The following results may be obtained from Theorem 2 as special cases:

1. For rotatory Rayleigh-Benard convection ($R_1 = 0 = R_2 = \dots = R_{n-1} = 0, \text{Ta} > 0$)

$$|\sigma| < \text{TaPr}$$

(Banerjee *et al.*, 1981)

2. For thermohaline convection ($R_1 > 0, R_2 = \dots = R_{n-1} = \text{Ta} = 0$)

$$|\sigma| < \sqrt{R_1\text{Pr}}$$

(Banerjee *et al.*, 1981)

3. For rotatory Thermohaline convection of the Veronis type (Turner, 1985) ($R_1 > 0, R_2 = \dots = R_{n-1} = 0, \text{Ta} > 0$)

$$|\sigma| < \max\left\{\sqrt{R_1\text{Pr}}, \sqrt{\text{TaPr}}\right\}$$

(Gupta *et al.*, 1983)

4. For triply diffusive convection ($R_1 > 0, R_2 > 0, R_3 = \dots = R_{n-1} = \text{Ta} = 0$)

$$|\sigma| < \sqrt{(R_1 + R_2)\text{Pr}}$$

(Prakash *et al.*, 2014c)

Proceeding in this manner, we can obtain bounds for the complex growth rate for all configurations with $3, 4, \dots, n - 1$ concentration components, respectively.

3. Conclusions

The present analysis generalizes the previous published results for rotatory hydrodynamic singly, doubly and triply diffusive convection. The mathematical analysis carried out here yields a sufficient condition for the validity of the principle of the exchange of stabilities in rotatory hydrodynamic multicomponent convection. Since the complement of this condition implies the occurrence of oscillatory motions, the bounds for the complex growth rate are also obtained as the second problem. It is further proved that the results obtained herein are uniformly applicable for quite general nature of bounding surfaces.

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