TREFFTZ METHOD FOR POLYNOMIAL-BASED BOUNDARY IDENTIFICATION IN TWO-DIMENSIONAL LAPLACIAN PROBLEMS

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The paper addresses a two-dimensional boundary identification (reconstruction) problem in steady-state heat conduction. Having found the solution to the Laplace equation by super-positioning T-complete functions, the unknown boundary of a plane region is approximated by polynomials of an increasing degree. The provided examples indicate that sufficient accuracy can be obtained with a use of polynomials of a relatively low degree, which allows avoidance of large systems of nonlinear equations. Numerical simulations for assessing the performance of the proposed algorithm show better than 1% accuracy after a few iterations and very low sensitivity to small data errors.

Keywords: inverse geometry problem, boundary identification, Trefftz method, Laplace equation

1. Introduction

Inverse problems appear in many areas of engineering research. They contain, among others, problems of partly unknown geometry where we seek for the shape and location of the boundary of a considered domain (or a part of the boundary) and try to determine it from the information available at the known portion of the boundary. A number of these problems, referred to as inverse geometry problems (IGPs), arise in engineering practice. The intensive study of IGPs in the past decades has resulted in many propositions of efficient solution algorithms. Although purely computational aspects are crucial to many publications in this area, researchers usually indicate possible industrial applications of their theoretical results. Illustrative examples could be non-destructive detection of defects like voids, cracks or inclusions (Cheng and Chang, 2003a,b; Karageorghis *et al.*, 2014), locating the 1150°C isotherm in a blast furnace hearth (Gonzalez and Goldschmit, 2006), or non-destructive evaluation of corrosion (Lesnic *et al.*, 2002; Mera and Lesnic, 2005; Liu, 2008) – to name but a few.

The problem of numerical identification (reconstruction) of geometrical characteristics of the studied object can be solved by a variety of computational methods. For example, Hsieh and Kassab (1986) presented a general solution method which led to a nonlinear algebraic equation, or a nonlinear ordinary differential equation solved numerically by the Newton-Raphson method. Cheng and Chang (2003a,b) used a computational technique based on the conjugate-gradient method for the recovery of inner voids within a solid body, assuming different kinds of data prescribed on the outer surface. Huang and Liu (2010) estimated interfacial configurations in a 3D composite domain by the conjugate gradient method and commercial package CFD-ACE+. Lesnic *et al.* (2002) treated an inverse geometric problem for the Laplace equation by the boundary element method in conjunction with the Tikhonov first-order regularization. Mera *et al.* (2004) modelled the same problem as variational and minimized a defect functional by a real coded genetic algorithm combined with the function specification method. Kazemzadeh-Parsi and Daneshmand (2009) applied the smoothed fixed grid finite element method for the problem

of cavity detection. Liu *et al.* (2008) proposed a new algorithm, named the regularized integral equation method, which employed Lavrentiev regularization and a second kind Fredholm integral equation to ultimately attain the unknown boundary from a nonlinear algebraic equation by the bisection method.

A good reason for the choice of a method is the claim to avoid mesh generation. One could point to several numerical schemes which meet this requirement and perform well in inverse problems, among them the radial basis functions method and the method of fundamental solutions (MFS), both enjoying growing interest recently, as well as the Trefftz method (TM) dating back to the 1920s (Trefftz, 1926).

MFS has been extensively used in scientific computation and simulation over the last two decades. A comprehensive survey of its application to inverse problems (including inverse geometric problems) can be found in Karageorghis *et al.* (2011a,b).

TM and MFS, which are equivalent for Laplace and biharmonic equations (Chen *et al.*, 2007), share the same concept of expressing the solution to a differential equation by a linear combination of basis functions satisfying the governing equation. The coefficients of the bases (T-complete functions) have to be selected to make the solution satisfy, in some sense, all imposed conditions. A broad discussion of the method together with a tutorial on the applications can be found in Li *et al.* (2008). Clearly, TM suits for solving different kinds of inverse problems (Ciałkowski and Grysa, 2010).

There have been quite a few Trefftz-type approaches to a boundary identification problem. Karageorghis *et al.* (2014) proposed collocation TM for reconstruction of the void(s) whose centre(s) is (are) unknown, solving the proceeding non-linear least squares problem with a use of a suitable predefined routine in MATLAB. The collocation Trefftz method, modified by Liu (2007) and further referred as to modified collocation Trefftz method (MCTM), was applied by Fan *et al.* (2012) to a Laplacian problem to infer the spatial position of the unknown part of the boundary from the Dirichlet condition. The authors adopted the exponentially convergent scalar homotopy algorithm (ECSHA) to solve the resulting system of nonlinear equations. A similar problem, but with the Neumann condition on the unknown part of the boundary, was solved by Fan and Chan (2011) using the same technique. A combination of MCTM and ECSHA proved successful also in boundary detection problems for a modified Helmholtz equation (Chan and Fan, 2011); Fan *et al.*, 2014) and biharmonic equation (Chan and Fan, 2013).

In this study, we propose a fast converging computational algorithm based on TM to solve a two-dimensional boundary identification problem for the Laplace equation (excluding, however, highly complicated domains). In the first stage of computationm we approximate the solution to the Laplace equation with a linear combination of harmonic polynomials so that it satisfies the prescribed conditions in the least-squares sense. The reconstruction of an unknown boundary is through the adjustment of parameters of an approximating polynomial curve. It is demonstrated by numerical examples that relatively low degree polynomials could provide, at a small computational cost, sufficient accuracy of boundary identification. Moreover, computation with simulated errors exhibited very low sensitivity to small disturbances of the data. The approach allows immediate extension to problems governed by other linear differential equations.

2. Boundary reconstruction problem for the Laplace equation

Consider a two-dimensional inverse problem in a domain Ω whose boundary $\partial\Omega$ consists of two segments: Γ, which is known, and γ whose shape and position is not known and, therefore, being detected. The governing equation is

$$\Delta T = 0 \qquad \text{in} \quad \Omega \tag{2.1}$$

where Δ is Laplace operator, and it satisfies corresponding conditions prescribed on the domain boundary. The problem could be regarded as a steady-state heat conduction problem. To compensate for the missing information about the geometry of the boundary, an overspecified condition will be imposed on the known part of the boundary. Hence, we have

$$T = T_\Gamma \qquad \text{and} \qquad \frac{\partial T}{\partial n} = q_\Gamma \qquad \text{on} \quad \Gamma \tag{2.2}$$

where $\partial / \partial n$ denotes differentiation along the outward normal n. On the remaining part of the boundary, we assume

$$\alpha T + \beta \frac{\partial T}{\partial n} = g_\gamma \qquad \text{on} \quad \gamma \tag{2.3}$$

where α and β denote arbitrary constants which are not zero simultaneously. In other words, a condition of Dirichlet, Neumann or Robin type can be imposed at the boundary γ.

Rather than discuss the problem in full generality, let us look at a particular situation when the problem domain Ω is an area under the curve γ. It can always be assumed that in appropriate coordinates x varies from 0 to 1, so γ can be described by an explicit equation

$$y = f(x) \qquad x \in \langle 0, 1 \rangle \tag{2.4}$$

For this particular case, the problem specified by equations (2.1)-(2.3) is presented in Fig. 1.
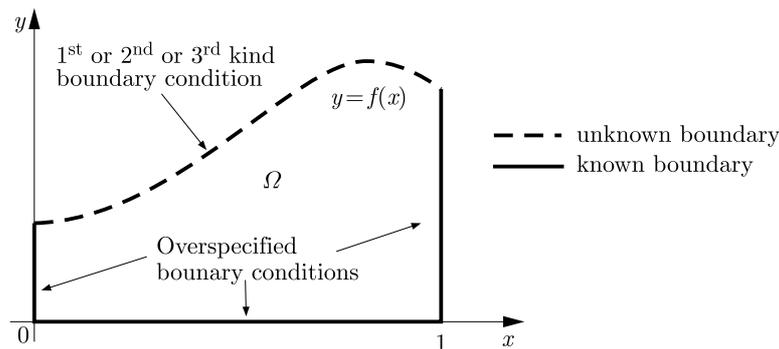
Fig. 1. A diagram of the boundary identification problem

Both boundary curves Γ (known) and γ (unknown), meet at points $(0, y_0)$ and $(1, y_1)$, hence, we can write

$$f(0) = y_0 \qquad f(1) = y_1 \tag{2.5}$$

As stated before, the aim of the study is to determine the shape and location of the unknown part of the boundary γ.

3. Solution methodology

In the first step, we approximate the solution to Laplace equation (2.1) with a combination of the T-complete functions $V_k(x, y)$

$$T(x, y) \approx \Theta(x, y) = \sum_0^K c_k V_k(x, y) \tag{3.1}$$

with coefficients c_k to be found. In the case under consideration, the functions $V_k(x, y)$ which satisfy the Laplace equation will be taken from the set

$$\{1, \text{Re} z^k, \text{Im} z^k, k = 1, 2, \ldots\} \tag{3.2}$$

where $z = x + iy$ is a complex number. These basis functions are termed harmonic polynomials.

Coefficients c_k in expansion (3.1) will be chosen to minimize, along the boundary Γ, squared differences between true values of the function T and of its normal derivative and the corresponding values provided by the model. Hence, we minimize the following functional

$$J(c_0, c_1, \ldots, c_K) = \int_{\Gamma} \left(\sum_0^K c_k V_k \Big|_{\Gamma} - T_{\Gamma} \right)^2 d\Gamma + \int_{\Gamma} \left(\sum_0^K c_k \frac{\partial V_k}{\partial n} \Big|_{\Gamma} - q_{\Gamma} \right)^2 d\Gamma \tag{3.3}$$

The necessary condition for the minimum of $J(c_0, c_1, \ldots, c_K)$ is

$$\frac{\partial J}{\partial c_k} = 0 \qquad k = 0, 1, \ldots, K \tag{3.4}$$

which gives a system of $K + 1$ linear equations with $K + 1$ variables. In order to avoid problems with numerical stability, the coordinates x and y of the points of Ω should range from 0 to 1. To meet this requirement, one can use a proper change of variables when building a mathematical model of the problem. Alternatively, basis functions (3.2) could be modified by taking z/L instead of z, where L denotes the characteristic length. The latter reminds modification of the corresponding T-complete functions in polar coordinates proposed by Liu (2007).

Having found the coefficients c_k, we can now focus our attention directly on identification of the unknown boundary γ assumed to be a graph of the function $f(x)$. The function $f(x)$ can be approximated with a polynomial of degree N, denoted by $P^{(N)}(x)$ or shortly $P(x)$, if there is no danger of misunderstanding. Since $P(x)$ has the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_N x^N \tag{3.5}$$

we only need to determine the coefficients a_n to know the approximate values of $f(x)$ in the whole interval $\langle 0, 1 \rangle$. Conditions (2.6) applied to $P(x)$ give

$$P(0) = y_0 \qquad \text{and} \qquad P(1) = y_1 \tag{3.6}$$

In consequence, any two a_n can be expressed as functions of the remaining coefficients.

Further proceeding must include a prescribed boundary condition (2.3) on the sought curve γ. Since $\Theta(x, y) \approx T(x, y)$ and $P(x) \approx f(x)$, we can rewrite equation (2.3) in terms of the approximated functions as

$$\alpha \Theta(x, P(x)) + \beta \frac{\partial \Theta}{\partial n}(x, P(x)) = g_{\gamma}(x) \tag{3.7}$$

where

$$\frac{\partial \Theta}{\partial n} = \frac{1}{\sqrt{1 + [P'(x)]^2}} \left(-P'(x) \frac{\partial \Theta}{\partial x}(x, P(x)) + \frac{\partial \Theta}{\partial y}(x, P(x)) \right) \tag{3.8}$$

Equation (3.7) formally holds for all $x \in (0, 1)$ but depending on the amount of information available on γ could be given only for $x \in \{x_1, x_2, \ldots, x_M\}$. The polynomial $P(x)$ will be selected so as to minimize the functional

$$J(a_0, a_1, \ldots, a_N) = \int_0^1 \left(\alpha \Theta(x, P(x)) + \beta \frac{\partial \Theta}{\partial n}(x, P(x)) - g_{\gamma}(x) \right)^2 dx \tag{3.9}$$

In the case of discrete data, integration contained in functional (3.9) has to be changed into summation spanning over all x_m.

Minimization of $J(a_0, a_1, \ldots, a_N)$ results in solving simultaneous equations

$$\frac{\partial J}{\partial a_n} = 0 \qquad n = 0, 1, \ldots, N \tag{3.10}$$

which are nonlinear. We propose an iterative algorithm for finding the coefficients of the polynomial $P^{(N)}(x)$. The computation always starts with $N = 1$ (i.e. a polynomial of degree 1). Note that conditions (3.6) automatically determine the first approximation $P^{(1)}(x)$. We now check whether boundary condition (3.7) is satisfied. More precisely, we want to estimate the goodness of fit for (3.7). For this purpose, we introduce a measure of inaccuracy defined by

$$Er^{(1)} = \frac{1}{\|g_\gamma\|} \left\| \alpha\Theta\big|_{y=P^{(1)}(x)} + \beta\frac{\partial\Theta}{\partial n}\Big|_{y=P^{(1)}(x)} - g_\gamma \right\| \qquad \|\cdot\| - L^2 \text{ norm} \tag{3.11}$$

assuming either continuous or discrete range of variation of x. In the latter case, both the numerator and the denominator of (3.11) contain the norm of the M-dimensional vector.

If the value of $Er^{(1)}$ is sufficiently close to zero, the computational procedure ends and $P^{(1)}(x)$ is the solution. Otherwise, we increase the polynomial degree by 1 and numerically solve (3.10) for a_2 (since a_0 and a_1 can be expressed as functions of a_2). Numerical root-finding methods require a first guess which we suggest by plotting J versus a_2. Once the coefficients of $P^{(2)}(x)$ are determined, we compute the error $Er^{(2)}$ defined similarly to $Er^{(1)}$. In general, when solving for the coefficients of a polynomial $P^{(N+1)}(x)$, our initial guess are a_n from the previous step and $a_{N+1} = 0$. Having determined $P^{(N+1)}(x)$, we compute $Er^{(N+1)}$. The procedure is continued to achieve the desired accuracy of fulfilment of (3.7), i.e. until $Er^{(N+1)}$ becomes less than the desired level. In geometrical terms, the algorithm allows one to approach the unknown boundary γ with successive polynomial curves. An alternative to the stopping criterion based on $Er^{(N)}$, may well be the rule saying that the iteration stops when the distance between the successive approximations $P^{(N-1)}(x)$ and $P^{(N)}(x)$ is sufficiently small, which gives that either $\|P^{(N)}(x) - P^{(N-1)}(x)\|$ or relative measure $\|P^{(N)}(x) - P^{(N-1)}(x)\| \cdot \|P^{(N-1)}(x)\|^{-1}$ must be less than the user-specified tolerance.

We must emphasise that the proposed method, when compared to the existing approaches using MCTM (Chan and Fan, 2011, 2013; Fan and Chan, 2011; Fan *et al.*, 2012, 2014), differs not only in the representation of the sought boundary (a continuous curve rather than a set of discrete points) but it also uses a different computation scheme. We employ two systems of equations: a linear system to determine the coefficients of T-complete functions and a nonlinear system to adjust the coefficients of a polynomial approximation, the latter having relatively few unknowns. In the cited papers, both coefficients of T-complete functions and the unknown boundary points are obtained from one nonlinear system of equations. In practice, such a system must be large, especially when we use a high-order Trefftz solution for better accuracy and a large number of boundary points for more precise boundary reconstruction. Consequently, the more boundary points we wish to determine, the larger system we obtain, unlike using the present method which allows one to recover infinitely many boundary points at the cost of solving only a small system of equations.

4. Numerical examples

In this Section, some examples will be shown to test the developed theoretical method. Although the boundary detection problem originates from realistic applications, a number of studies proposing efficient solution algorithms, among them those based on MCTM which are cited in

the previous paragraph, validate the proposed computational methods on numerical simulations (synthetic data). Likewise, the present method will be tested on data coming from numerical simulations. The results will be shown for the function $T(x,y) = \mathrm{e}^{2x}\cos(2y)$, which is an analytical solution to the Laplace equation in the domain Ω defined by inequalities

$$0 \leqslant x \leqslant 1 \qquad \text{and} \qquad 0 \leqslant y \leqslant f(x) \tag{4.1}$$

where the graph of $y = f(x)$ represents the unknown boundary γ.

In expansion (3.1), we took $K = 14$, and the values of T_γ, q_γ and g_γ are specified at $M = 49$ distinct points. The resulting simultaneous nonlinear equations are solved using the Levenberg-Marquardt method.

For quantitative evaluation of the final results, we introduce ε, a non-negative number defined by

$$\varepsilon = \frac{\|P(x) - f(x)\|}{\|f(x)\|} \tag{4.2}$$

which can be interpreted as a mean error which is made when replacing the original boundary γ with its polynomial approximation $P(x)$.

Example 1

The unknown boundary is inferred from the Dirichlet condition on γ. We take $f(x) = (1 + x - \sin 4x)\mathrm{e}^{-x}$. In Fig. 2, the true boundary is compared with its polynomial approximation.
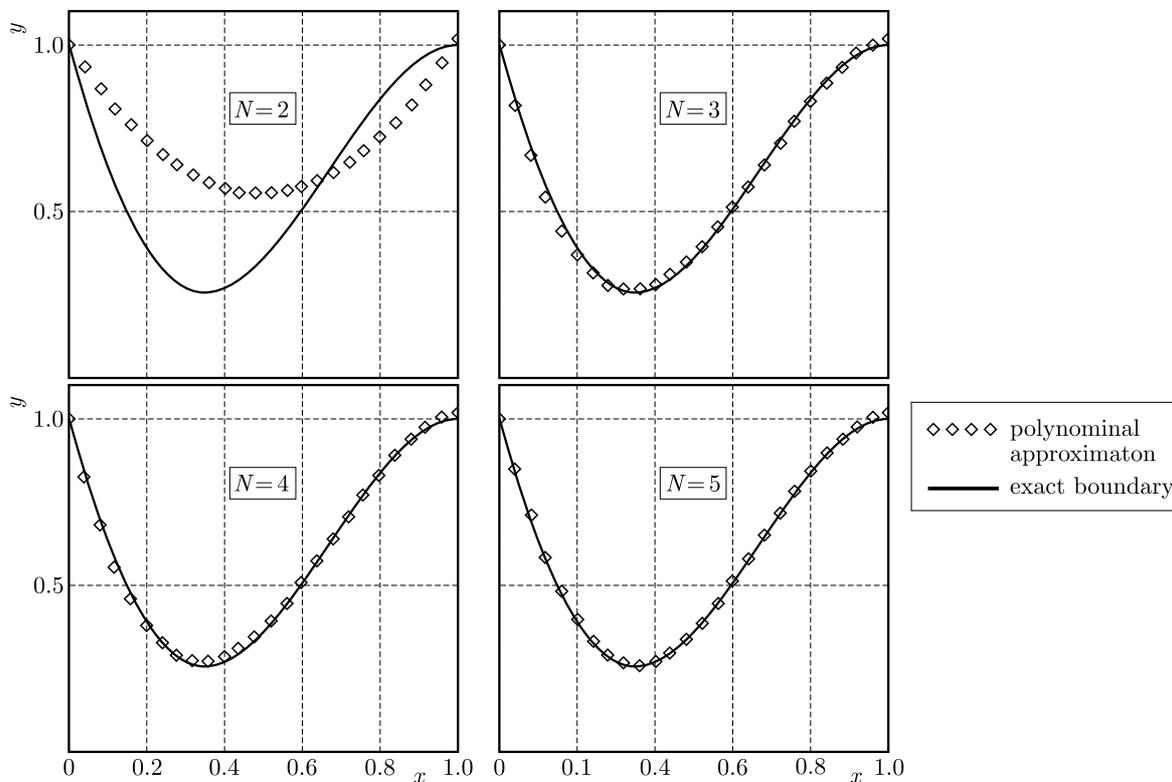
Fig. 2. Boundary shape identification by polynomials of degree $N = 2, 3, 4, 5$

Table 1 gives information about the accuracy of the presented approximations.

Table 1. Mean error of boundary shape identification for $f(x) = (1 + x - \sin 4x)e^{-x}$

Polynomial degree	$N = 2$	$N = 3$	$N = 4$	$N = 5$
Error ε	32%	2.6%	1.9%	0.1%

Example 2

The computations have been performed for the case of the Neumann condition on γ. Figure 3 shows the results of boundary reconstruction when $f(x) = (1 + x^2)(2 + x^5)^{-1}$.
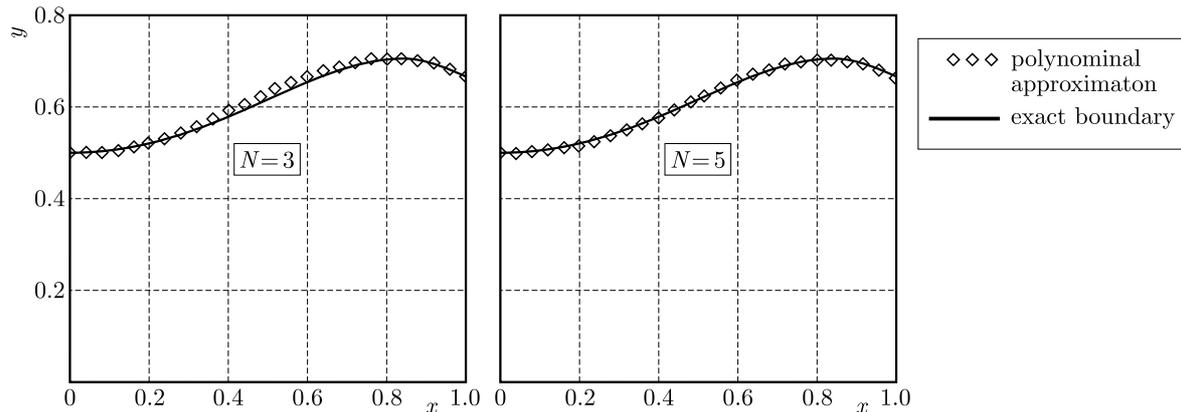
Fig. 3. Boundary shape identification by polynomials of degree $N = 3$ and $N = 5$

The approximation errors ε, concerning the case of the Neumann condition, are listed in Table 2.

Table 2. Mean error of boundary shape identification for $f(x) = (1 + x^2)(2 + x^5)^{-1}$

Polynomial degree	$N = 2$	$N = 3$	$N = 4$	$N = 5$
Error ε	11.9%	1.2%	1.0%	0.4%

Example 3

Finally, we present the results referring to the case of the Robin condition on γ with $\alpha = 1$ and $\beta = -3$ in (2.3). The true boundary γ is a curve $f(x) = (2 + \cos 4x)(x^3 + 2)^{-1}$ and its approximation is presented in Fig. 4.

The approximation errors ε, concerning the case of the Robin boundary condition, are listed in Table 3.

All examples included in this Section prove the effectiveness of the proposed computational procedure for boundary identification. It has been enough to use a 5th degree polynomial to approach the true boundary with an error less than 1%.

5. Stability analysis

The computation results shown in the previous Section have been performed on the exact functions T_γ, q_γ and g_γ. In order to examine the sensitivity of the method to changes in the inputs, one has to introduce random noise to the given functions and then evaluate the impact of such changes on the final boundary detection.
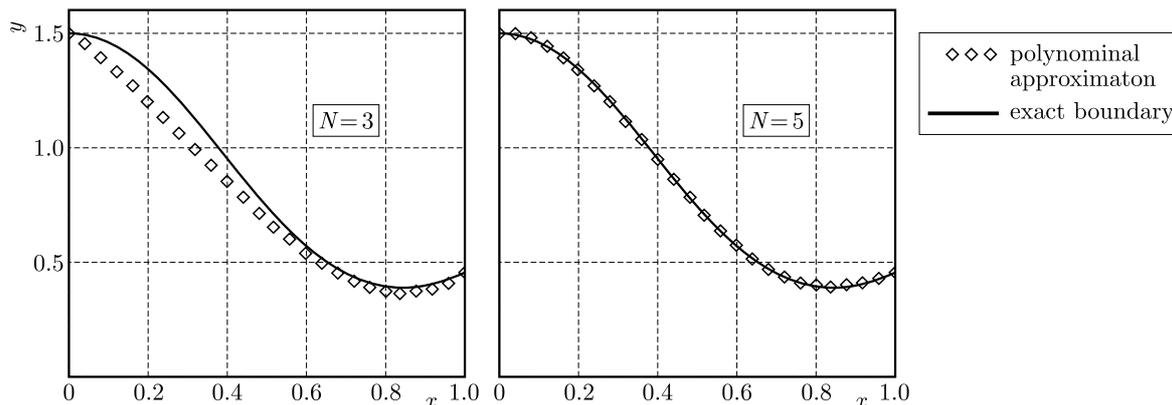
Fig. 4. Boundary shape identification by polynomials of degree $N = 3$ and $N = 5$

Table 3. Mean error of boundary shape identification for $f(x) = (2 + \cos 4x)(x^3 + 2)^{-1}$

Polynomial degree	$N = 2$	$N = 3$	$N = 4$	$N = 5$
Error ε	18.0%	8.6%	1.8%	0.4%

The values of T_γ, q_γ and g_γ are assumed to be given only at M discrete locations. In order to simulate measurement errors, we generate M random numbers having a normal distribution with a mean of 0 and a standard deviation of $\sigma = 0.025$. With such σ, approximately 95% of the simulated errors lie between -5% and 5%.

One can evaluate the influence of the introduced errors on the final results of boundary detection in a variety of ways. For the purpose of stability analysis, we use a measure Δ, defined as

$$\Delta = \frac{\|P_{noisy}(x) - P(x)\|}{\|P(x)\|} \tag{5.1}$$

where $P_{noisy}(x)$ denotes a polynomial approximation of the unknown boundary γ, derived from the noisy data. Since we perform a sequence of computations, each giving a successive approximation to γ, the input data errors might accumulate from step to step. Therefore, it seems reasonable to pay special attention to those solution inaccuracies which occur in the last iteration. Consequently, our discussion of numerical stability is based on the value of Δ calculated for $N = 5$.

For a better insight into the problem, we have recorded Δ in 10 runs of the computational procedure, each with different random noise. The values of Δ in the respective numerical examples are presented in Table 4.

Table 4. Relative changes (Δ) between the solutions derived from exact and disturbed data

	Example 1	Example 2	Example 3
Δ	0.13%-0.84%	0.24%-0.88%	0.06%-0.29%

It turns out that changes in the final results are even smaller than changes in the inputs, as far as percent errors are concerned. A possible explanation is that a polynomial curve, which we fit to a large number of discrete points by least squares, is forced to follow the changes introduced by random errors. Therefore, it should be shifted up at some locations but, on the other hand, shifted down at others. A low-degree polynomial is not very 'flexible', so it remains almost unchanged.

6. Conclusions and final remarks

We proposed a method for the solution of a two-dimensional boundary identification problem governed by the Laplace equation. In contrast to approaches which derive coordinates of unknown boundary points from a large system of nonlinear equations, the proposed algorithm does not require solving large systems and yet it delivers a very accurate reconstruction of the unknown boundary. Its basic advantage is the reduction of the amount of computational work. The provided numerical results exhibit not only high accuracy, but fast convergence of the method. Testing the sensitivity of the algorithm to input data errors showed very low risk of large impact of possible errors in the input data on predicted model outputs. The presented solution procedure can be applied without changes to problems governed by other linear differential equations, provided the appropriate Trefftz functions are known. The only limitation concerns the geometry of a domain, because global Trefftz solutions could provide poor results when used on very complicated domains.

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