

TRAVELING WAVES IN A THIN LAYER COMPOSED OF NONLINEAR HYPERELASTIC ZAHORSKI'S MATERIAL

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The propagation of nonlinear travelling waves in the Zahorski elastic layer is investigated. By the averaging process over the width of the layer, we obtain a system of partial differential equations in one-dimensional space and time. A technique of phase planes is used to study the wave processes. Based on the phase trajectory method, we can make an interpretation of conditions of propagation of the nonlinear travelling wave and can establish the existence conditions under which the phase plane contains physically acceptable solutions.

Key words: travelling waves, phase plane, discontinuous surface, hyper-elastic materials

1. Introduction

We can treat a layer as a one-dimensional elastic structure with one scalar variable representing transverse symmetric motion. However, the finished transverse dimensions carry the weight during propagation of elastic waves. In the simplest description, one scalar variable can be used to describe effects of finite transverse dimensions in the elastic layer that undergoes longitudinal and symmetrical transverse motion only.

General equations describing motion of an incompressible nonlinear elastic medium and symmetric lateral motion of the elastic layer are presented in Section 2. In this section, we derive equations of motion by averaging the equations of elasticity across the layer. The travelling waves are described in Section 3. We obtained a solution for the travelling wave propagating with speed V in the direction of the coordinate X_1 depending on one parameter only. In Section 4, we use phase plane methods to classify different solutions

for travelling waves that are possible. Some of the solutions to the differential equations do not correspond to physically acceptable waves propagating in the layer, and so additional restrictions must be imposed from the physical problem. We explore such restrictions in Section 5. We are able to establish conditions for the existence of physically acceptable solutions as represented by individual paths in the phase plane. Finally, in Section 6 we present numerical analysis for travelling waves in the layer composed of the Zahorski material.

2. Basic equations

Motion of a continuum is given by a set of functions (Truesdell and Toupin, 1960)

$$x_i = x_i(X_\alpha, t) \quad i, \alpha = 1, 2, 3 \quad (2.1)$$

We consider an elastic layer which occupies the material region $X_1 > 0$ (Fig. 1) in the direction of the axis X_1 . At the frontal area of the layer $X_1 = 0$, the boundary conditions for deformations are given (Fu and Scott, 1989). We assume that motion described by equation (2.2) undergoes without imposing additional contact forces on the lateral planes of the layer $X_2 = \pm h$ (Coleman and Newman, 1990; Wright, 1981).

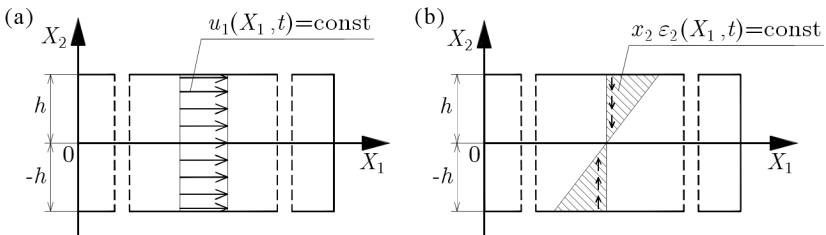


Fig. 1. Motion of the layer; (a) main motion in the longitudinal direction, (b) secondary motion in the transverse direction

The symmetric motion of the considered travelling wave is described by the equations

$$x_1 = X_1 + u_1(X_1, t) \quad x_2 = X_2 + X_2 \varepsilon_2(X_1, t) \quad x_3 = X_3 \quad (2.2)$$

The strain ε_1 , the gradient of the transversal strain κ and speed of the particle of the medium ν_1 and ν_2 in both directions of the layer are equal, respectively

$$\begin{aligned} \varepsilon_1 &= u_{1,1} & \kappa &= \varepsilon_{2,1} \\ \nu_1 &= \dot{x}_1 = \dot{u}_1(X_1, t) & \nu_2 &= \dot{x}_2 = X_2 \dot{\varepsilon}_2(X_1, t) \end{aligned} \quad (2.3)$$

For assumed motion (2.2), the left Cauchy-Green tensor and the deformation gradient have the form

$$\mathbf{B} = \begin{bmatrix} \overset{\circ}{\varepsilon}_1^2 & \overset{\circ}{\varepsilon}_1 X_2 \kappa & 0 \\ \overset{\circ}{\varepsilon}_1 X_2 \kappa & (X_2 \kappa)^2 + \overset{\circ}{\varepsilon}_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} \overset{\circ}{\varepsilon}_1 & 0 & 0 \\ X_2 \kappa & \overset{\circ}{\varepsilon}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.4)$$

where $\overset{\circ}{\varepsilon}_1 = 1 + \varepsilon_1$ and $\overset{\circ}{\varepsilon}_2 = 1 + \varepsilon_2$.

The invariants of the deformation \mathbf{B} are

$$I_1 = I_2 = \overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_2^2 + (X_2 \kappa)^2 + 1 \quad I_3 = 1 \quad (2.5)$$

For an incompressible material, there is identity $\det \mathbf{F} = 1$, then for the considered material

$$\overset{\circ}{\varepsilon}_1 \overset{\circ}{\varepsilon}_2 = 1 \quad (2.6)$$

We assume that the layer is made of the Zahorski material characterised by the strain-energy function

$$W(I_1, I_2) = \mu[C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(I_1^2 - 9)] \quad (2.7)$$

where C_1 , C_2 and C_3 are elastic constants.

According to Wesołowski (1972a,b) or Dai (2001), the Cauchy tensor has the form

$$\mathbf{T} = -q\mathbf{I} + 2\mu[(C_1 + 2C_3 I_1)\mathbf{B} - C_2 \mathbf{B}^{-1}] \quad (2.8)$$

where q is an arbitrary hydrostatic pressure.

The first Piola-Kirchhoff stress tensor may be expressed by the Cauchy tensor \mathbf{T}

$$\mathbf{T}_R = \mathbf{T} \mathbf{F}^{-\top} \quad (2.9)$$

and its non-zero components are given by

$$\begin{aligned} T_{R11} &= -\overset{\circ}{\varepsilon}_2 [q + 2\mu C_2 (X_2 \kappa)^2] + 2\mu \overset{\circ}{\varepsilon}_1 (C_1 + 2C_3 I_1 - C_2 \overset{\circ}{\varepsilon}_2^4) \\ T_{R12} &= 2\mu C_2 X_2 \kappa [\overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_2^2 + (X_2 \kappa)^2] + q X_2 \kappa \\ T_{R21} &= 2\mu X_2 \kappa (C_1 + 2C_3 I_1 + C_2) \\ T_{R22} &= -\overset{\circ}{\varepsilon}_1 [q + 2\mu C_2 (X_2 \kappa)^2] + 2\mu \overset{\circ}{\varepsilon}_2 (C_1 + 2C_3 I_1 - C_2 \overset{\circ}{\varepsilon}_1^4) \\ T_{R33} &= -q + 2\mu (C_1 + 2C_3 I_1 - C_2) \end{aligned} \quad (2.10)$$

For deformation gradient $(2.4)_2$, the equations of motion are reduced to a system of equations for plane strain deformation

$$\begin{aligned} T_{R11,1} + T_{R12,2} &= \rho_R u_{1,tt} & T_{R21,1} + T_{R22,2} &= \rho_R X_2 \varepsilon_{2,tt} \\ T_{R33,3} &= 0 \end{aligned} \quad (2.11)$$

The boundary conditions at the lateral surfaces of the layer $X_2 = \pm h$, have the form

$$T_{R12}(X_1, \pm h, X_3) = T_{R22}(X_1, \pm h, X_3) = 0 \quad (2.12)$$

We employ a procedure, which was described by Wright (1981), consisting in averaging equations of motion $(2.11)_{1,2}$ along the cross-section A of the layer (see Fig. 2).

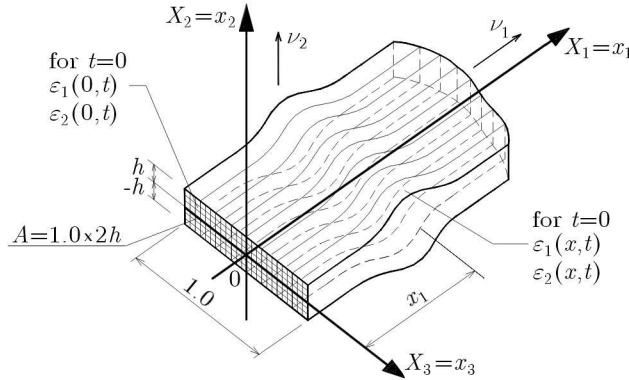


Fig. 2. Propagation of the travelling wave in the layer and cross-section A perpendicular to the axis X_1

We assume that in the averaging process, boundary conditions (2.12) are satisfied. We multiply second equation of motion $(2.11)_2$ by X_2 and average both resulting equations and first equation $(2.11)_1$ over the width of the layer, thus obtaining

$$\begin{aligned} \frac{1}{2h} \int_{-h}^h \frac{\partial T_{R11}}{\partial X_1} dX_2 + \frac{1}{2h} \int_{-h}^h \frac{\partial T_{R12}}{\partial X_2} dX_2 &= \frac{1}{2h} \int_{-h}^h \rho_R \ddot{u}_1 dX_2 \\ \frac{1}{2h} \int_{-h}^h \frac{\partial T_{R21}}{\partial X_1} X_2 dX_2 + \frac{1}{2h} \int_{-h}^h \frac{\partial T_{R22}}{\partial X_2} X_2 dX_2 &= \frac{1}{2h} \int_{-h}^h \rho_R \ddot{\varepsilon}_2 X_2^2 dX_2 \end{aligned} \quad (2.13)$$

Taking into account boundary conditions (2.12), one obtains an averaged equation of motion

$$\begin{aligned} \frac{\partial}{\partial X_1} \left(\frac{1}{2h} \int_{-h}^h T_{R11} dX_2 \right) &= \rho_R \ddot{u}_1 \\ \frac{\partial}{\partial X_1} \left(\frac{1}{2h} \int_{-h}^h X_2 T_{R21} dX_2 \right) - \frac{1}{2h} \int_{-h}^h T_{R22} dX_2 &= \rho_R \ddot{\varepsilon}_2 \frac{h^2}{3} \end{aligned} \quad (2.14)$$

Equations (2.14) are a consequence of the applied average process. For motion (2.11), the cross-section of the layer remains plane, and the normal to the surface of cross-sections overlap the axis X_1 (Fig. 2). The analogical assumption was made in the paper by Braun and Kosiński (1999).

In further analysis, we take the advantage of averaged equation (2.14)₁ and equation (2.11)₂. Substituting the components of Piola-Kirchhoff stress tensor (2.10) into (2.11)₂ and integrating them with respect to X_2 , we obtain the equation of motion in the direction of the axis X_2

$$\begin{aligned} q = \frac{\mu X_2^2}{\varepsilon_1} \left\{ 2\kappa^2 (2C_3 \overset{\circ}{\varepsilon}_2 - C_2 \overset{\circ}{\varepsilon}_1) + \kappa_{,1} \left[C_1 + C_2 + 2C_3 \left(\overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_2^2 + \frac{1}{2} (X_2 \kappa)^2 + 1 \right) \right] \right\} + \\ + 4C_3 \kappa \left(\overset{\circ}{\varepsilon}_1 \varepsilon_{1,1} + \overset{\circ}{\varepsilon}_2 \kappa + \frac{1}{2} X_2^2 \kappa \kappa_{,1} \right) - \frac{1}{2} \nu_o^{-2} \varepsilon_{2,tt} \left\} + \frac{q_1(X_1, t)}{\varepsilon_1} \end{aligned} \quad (2.15)$$

where $q_1 = (X_1, t)$ is an arbitrary function and $\nu_o = \sqrt{\mu/\rho_R}$ is the speed of infinitesimal shear waves.

We determine the function $q_1 = (X_1, t)$ using boundary conditions (2.12) $T_{R22}|_{X_2=\pm h} = 0$ (see Major and Major, 2007). The obtained equation depends on $(X_2)^2$, then both boundary conditions are satisfied.

Finally, for the Zahorski material, (2.15), has the form

$$\begin{aligned} q = \frac{\mu(X_2^2 - h^2)}{\varepsilon_1} \left\{ -2\kappa^2 C_2 \overset{\circ}{\varepsilon}_1 + \kappa_{,1} \left[C_1 + C_2 + 2C_3 \left(\overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_2^2 + \frac{1}{2} \kappa^2 (X_2^2 + h^2) + 1 \right) \right] \right\} + \\ + 4C_3 \kappa \left[\overset{\circ}{\varepsilon}_1 \varepsilon_{1,1} + 2\overset{\circ}{\varepsilon}_2 \kappa + \frac{1}{2} \kappa \kappa_{,1} (X_2^2 + h^2) \right] - \frac{1}{2} \nu_o^{-2} \varepsilon_{2,tt} \left\} + \\ + \frac{2\mu \overset{\circ}{\varepsilon}_2}{\varepsilon_1} \left\{ C_1 + 2C_3 \left[\overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_2^2 + (h\kappa)^2 + 1 \right] - C_2 \overset{\circ}{\varepsilon}_1^2 \left[\overset{\circ}{\varepsilon}_1^2 + (h\kappa)^2 \right] \right\} \end{aligned} \quad (2.16)$$

Including (2.16), the left-hand side of equation (2.14)₁ for the Zahorski material is

$$\begin{aligned} \left(\frac{1}{2h} \int_{-h}^h T_{R11} dX_2 \right)_{,1} &= 2\mu \left[(\overset{\circ}{\varepsilon}_1 - \overset{\circ}{\varepsilon}_2^3) \left[C_1 + 2C_3 (\overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_2^2 + (h\kappa)^2 + 1) \right] + \right. \\ &- \frac{4}{3} C_3 \overset{\circ}{\varepsilon}_1 (h\kappa)^2 + C_2 \overset{\circ}{\varepsilon}_2 (\overset{\circ}{\varepsilon}_1^2 - \overset{\circ}{\varepsilon}_2^2 + \frac{2}{3} (h\kappa)^2) - \overset{\circ}{\varepsilon}_2 \frac{h^2}{3} \left\{ 2C_2 \kappa^2 \overset{\circ}{\varepsilon}_1 - \frac{18}{5} C_3 (h\kappa)^2 \kappa_{,1} + \right. \\ &\left. \left. + \frac{1}{2} \nu_o^{-2} \varepsilon_{2,tt} - \kappa_{,1} \left[C_1 + 2C_3 (\overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_2^2 + 1) + C_2 \right] - 4C_3 \kappa (\overset{\circ}{\varepsilon}_1 \varepsilon_{1,1} + 2\overset{\circ}{\varepsilon}_2 \kappa) \right\} \right]_{,1} \end{aligned} \quad (2.17)$$

After transformation and differentiation, we obtain from (2.6)

$$\kappa = \varepsilon_{2,1} = -\varepsilon_{1,1} \overset{\circ}{\varepsilon}_1^{-2} \quad \kappa_{,1} = 2\varepsilon_{1,1}^2 \overset{\circ}{\varepsilon}_1^{-3} - \varepsilon_{1,11} \overset{\circ}{\varepsilon}_1^{-2} \quad (2.18)$$

Including (2.6) and (2.18) in (2.17), we finally obtain an equation which contains the function $\varepsilon_1(X_1, t)$ only

$$\begin{aligned} &\left[(\overset{\circ}{\varepsilon}_1 - \overset{\circ}{\varepsilon}_1^{-3}) \left[C_1 + 2C_3 (\overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_1^{-2} + (h\varepsilon_{1,1})^2 \overset{\circ}{\varepsilon}_1^{-4} + 1) \right] - \frac{4}{3} C_3 \overset{\circ}{\varepsilon}_1^{-3} (h\varepsilon_{1,1})^2 + \right. \\ &- \frac{h^2}{6} \nu_o^{-2} \varepsilon_{2,tt} \overset{\circ}{\varepsilon}_1^{-2} + C_2 (\overset{\circ}{\varepsilon}_1 - \overset{\circ}{\varepsilon}_1^{-3} + \frac{2}{3} (h\varepsilon_{1,1})^2 \overset{\circ}{\varepsilon}_1^{-5}) - \frac{h^2}{3} \left\{ 2C_2 \varepsilon_{1,1}^2 \overset{\circ}{\varepsilon}_1^{-5} + \right. \\ &+ 4C_3 \varepsilon_{1,1}^2 (\overset{\circ}{\varepsilon}_1^{-3} - 2\overset{\circ}{\varepsilon}_1^{-7}) - (2\varepsilon_{1,1}^2 \overset{\circ}{\varepsilon}_1^{-5} - \varepsilon_{1,11} \overset{\circ}{\varepsilon}_1^{-4}) \left[C_1 + C_2 + \right. \\ &\left. \left. + 2C_3 (\overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_1^{-2} + 1) \right] - \frac{18}{5} C_3 (h\varepsilon_{1,1})^2 (2\varepsilon_{1,1}^2 \overset{\circ}{\varepsilon}_1^{-9} - \varepsilon_{1,11} \overset{\circ}{\varepsilon}_1^{-8}) \right\} \right]_{,1} = \frac{1}{2} \nu_o^{-2} u_{1,tt} \end{aligned} \quad (2.19)$$

The above equation, (2.19), is the governing one-dimensional equation describing nonlinear dynamics of the layers.

3. Traveling waves

The phase ξ is defined by

$$\xi = X_1 - Vt \quad (3.1)$$

where V is the speed of propagation of the travelling wave with a constant profile displaced along the axis X_1 . For the travelling wave with any profile, we express motion as a function of one parameter ξ only

$$u_1(X_1, t) = u_1(\xi) \quad \varepsilon_2(X_1, t) = \varepsilon_2(\xi) \quad (3.2)$$

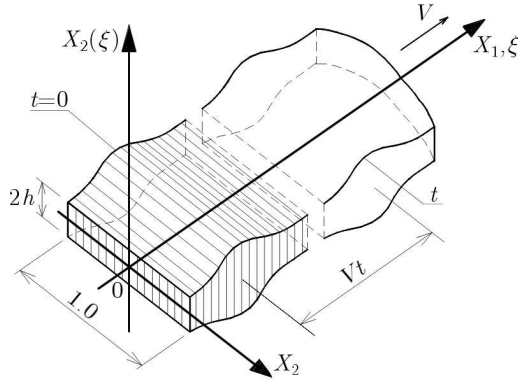


Fig. 3. Propagation of the travelling wave with speed V

Substituting (3.1) and integrating with respect to ξ , equation (2.19) for the travelling wave has the form

$$\begin{aligned}
 & (C_1 + C_2 + 2C_3)(\overset{\circ}{\varepsilon}_1 - \overset{\circ}{\varepsilon}_1^{-3}) + 2C_3(\overset{\circ}{\varepsilon}_1^3 - \overset{\circ}{\varepsilon}_1^{-5}) + \frac{h^2}{6}\nu\overset{\circ}{\varepsilon}_1^{-2} \left[\frac{\varepsilon_1}{\overset{\circ}{\varepsilon}_1} \right]_{,\xi\xi} + \\
 & + \frac{h^2}{3}(C_1 + C_2 + 2C_3)(2\varepsilon_{1,\xi}^2 \overset{\circ}{\varepsilon}_1^{-5} - \varepsilon_{1,\xi\xi} \overset{\circ}{\varepsilon}_1^{-4}) + \frac{2}{3}C_3h^2(\varepsilon_{1,\xi}^2 \overset{\circ}{\varepsilon}_1^{-3} - \varepsilon_{1,\xi\xi} \overset{\circ}{\varepsilon}_1^{-2} + \\
 & + 3\varepsilon_{1,\xi}^2 \overset{\circ}{\varepsilon}_1^{-7} - \varepsilon_{1,\xi\xi} \overset{\circ}{\varepsilon}_1^{-6}) + \frac{3}{5}C_3h^4\varepsilon_{1,\xi}^2(4\varepsilon_{1,\xi}^2 \overset{\circ}{\varepsilon}_1^{-9} - 2\varepsilon_{1,\xi\xi} \overset{\circ}{\varepsilon}_1^{-8}) = \frac{1}{2}\nu\varepsilon_1 + d_1
 \end{aligned} \tag{3.3}$$

where $\nu = V^2/\nu_0^2$ and d_1 is a constant of integration.

Multiplying (3.3) mutually by $\varepsilon_{1,\xi}$, we integrate it once more to obtain

$$\begin{aligned}
 & \frac{1}{2}(C_1 + C_2 + 2C_3)(\overset{\circ}{\varepsilon}_1^2 + \overset{\circ}{\varepsilon}_1^{-2}) + \frac{1}{2}C_3(\overset{\circ}{\varepsilon}_1^4 + \overset{\circ}{\varepsilon}_1^{-4}) + \frac{h^2}{12}\nu\varepsilon_{1,\xi}^2 \overset{\circ}{\varepsilon}_1^{-4} + \\
 & - \frac{h^2}{6}(C_1 + C_2 + 2C_3)\varepsilon_{1,\xi}^2 \overset{\circ}{\varepsilon}_1^{-4} - \frac{1}{3}C_3h^2\varepsilon_{1,\xi}^2(\overset{\circ}{\varepsilon}_1^{-2} + \overset{\circ}{\varepsilon}_1^{-6}) + \\
 & - \frac{3}{10}C_3h^4\varepsilon_{1,\xi}^4 \overset{\circ}{\varepsilon}_1^{-8} = \frac{1}{4}\nu\varepsilon_1^2 + d_1\varepsilon_1 + d_2
 \end{aligned} \tag{3.4}$$

where d_2 is another constant of integration.

This equation gives a solution for the travelling wave propagating with the speed V in the X_1 direction and depending on one parameter ξ (3.1) only.

If the constant $\nu = \rho V^2/\mu$ and the constants of integration d_1 and d_2 are known, we could find the solution.

For the constant $C_3 = 0$, we obtain the equation of motion for the Mooney-Rivlin material (compare Major and Major, 2007).

4. Phase plane analysis of propagation of the travelling wave in the layer

By constructing phase portraits of the solution in the $(\varepsilon_1, \varepsilon_1, \xi)$ plane, we can make an interpretation of the conditions of propagation of nonlinear travelling waves (Dai, 2001; Major and Major, 2006).

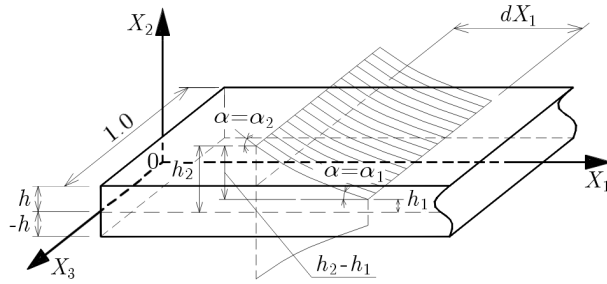


Fig. 4. Deformation of the layer during propagation of the travelling wave

For the considered travelling wave we can make an approximation of equation (3.4) for a small slope angle surface of the layer to the axis X_1 (see Fig. 4)

$$\tan \alpha \approx \alpha = \lim_{\Delta X_1 \rightarrow 0} \frac{h_2 - h_1}{\Delta X_1} = \lim_{\Delta X_1 \rightarrow 0} \frac{\Delta \varepsilon_2}{\Delta X_1} h = \frac{d\varepsilon_2}{dX_1} h = \varepsilon_{2,1} h \quad (4.1)$$

according to (2.3)₂ $\varepsilon_{2,1} = \kappa$, then $\tan \alpha \approx \kappa h$. Assumed that there is a little modification to the surface of the layer slope to the axis X_1 , we have

$$|\kappa h| \ll 1 \quad \text{or} \quad |h\varepsilon_{1,1}\varepsilon_1^{\circ-2}| \ll 1 \quad (4.2)$$

The last equality follows from (2.18)₁.

First, we introduce the dimensionless variables

$$\begin{aligned} b_1 &= \frac{2C_1}{\nu - 2C_1} & b_2 &= \frac{2C_2}{\nu - 2C_1} & b_3 &= \frac{2C_3}{\nu - 2C_1} \\ D_1 &= \frac{2d_1}{\nu - 2C_1} & D_2 &= \frac{2d_2}{\nu - 2C_1} \end{aligned} \quad (4.3)$$

Multiplying equations of motion (3.4) by $4/(\nu - 2C_1)$ and substituting (4.3), we obtain an approximate form

$$\begin{aligned}
& (b_1 + b_2 + 2b_3) \left(\varepsilon_1^{\circ 2} + \varepsilon_1^{\circ -2} \right) + b_3 \left(\varepsilon_1^{\circ 4} + \varepsilon_1^{\circ -4} \right) + \frac{h^2}{3} \varepsilon_{1,\xi}^2 \varepsilon_1^{\circ -4} (1 - b_2) + \\
& - \frac{2}{3} b_3 h^2 \varepsilon_{1,\xi}^2 \left(\varepsilon_1^{\circ -2} + \varepsilon_1^{\circ -4} + \varepsilon_1^{\circ -6} \right) - \frac{3}{5} b_3 h^4 \varepsilon_{1,\xi}^4 \varepsilon_1^{\circ -8} = \varepsilon_1^2 (1 + b_1) + 2D_1 \varepsilon_1 + 2D_2
\end{aligned} \tag{4.4}$$

According to (4.2)₂, the expression $h^4 \varepsilon_{1,\xi}^4 \varepsilon_1^{\circ -8}$ is infinitesimal superior in rank compared to the rest of term of equation (4.4).

Now we introduce the following transformation

$$\zeta = \frac{\sqrt{3}}{h} \xi \tag{4.5}$$

Apart from the scaling factor, ζ is just the current configuration coordinate X_1 in terms of the phase ξ , and (4.4) takes the form

$$\varepsilon_{1,\zeta}^2 = F(\varepsilon_1, D_2) \tag{4.6}$$

where

$$\begin{aligned}
& F(\varepsilon_1, D_2) = \\
& = \frac{\varepsilon_1^6 \left[2D_1 \varepsilon_1 + 2D_2 + \varepsilon_1^2 (1 + b_1) \right] - (b_1 + b_2 + 2b_3) \left(\varepsilon_1^{\circ 8} + \varepsilon_1^{\circ 4} \right) + b_3 \left(\varepsilon_1^{\circ 10} + \varepsilon_1^{\circ 2} \right)}{(1 - b_2) \varepsilon_1^{\circ 2} - 2b_3 \left(\varepsilon_1^{\circ 4} + \varepsilon_1^{\circ 2} + 1 \right)}
\end{aligned} \tag{4.7}$$

D_2 is an argument of F in above equations (4.6) and (4.7) because different curves in the phase plane correspond to different values of D_2 .

More precisely, the parameters b_1 , b_2 , b_3 and D_1 uniquely determine a portrait, and then D_2 determines the curves in that portrait.

We introduce a denotation

$$y = \varepsilon_{1,\zeta} = \sqrt{F(\varepsilon_1, D_2)} \tag{4.8}$$

whose first derivative with respect to ζ is equal

$$y_{,\zeta} = \frac{1}{2} F'(\varepsilon_1, D_2) \quad \text{then} \quad \tan \beta = \frac{dy}{d\varepsilon_1} = \frac{F'}{2y} \tag{4.9}$$

where derivatives of $F(\varepsilon_1, D_2)$ with respect to ε_1 are denoted by prime. This system shows immediately that equilibria in the phase plane satisfy $y = 0$, $F'(\varepsilon_1, D_2) = 0$.

This indicates a specific character of the nonlinear system, which have one or several equilibrium positions, and depends on the function $F(\varepsilon_1, D_2)$ (Dai, 2001).

Equation (4.9)₂ describes a straight line tangent to the trajectory at the function of the phase coordinates (ε_1, y) . The phase points are called ordinary or regular points if the tangent is determinate, however if the tangent is indeterminate, i.e. $dy/d\varepsilon_1 = y_\zeta/\varepsilon_{1,\zeta} \rightarrow 0$ the points are called singular points or equilibrium points.

The phase point is the equilibrium point if

$$\begin{aligned} y = \varepsilon_{1,\zeta} = 0 & \Rightarrow F(\varepsilon_1, D_2) = 0 \\ y_\zeta = 0 & \Rightarrow F'(\varepsilon_1, D_2) = 0 \end{aligned} \quad (4.10)$$

Discharging necessary equilibrium condition (4.10) after substituting (4.7), we have

$$\begin{aligned} \overset{\circ}{\varepsilon}_1^6 [2D_1\varepsilon_1 + 2D_2 + \varepsilon_1^2(1 + b_1)] - (b_1 + b_2 + 2b_3) \left(\overset{\circ}{\varepsilon}_1^8 + \overset{\circ}{\varepsilon}_1^4 \right) - b_3 \left(\overset{\circ}{\varepsilon}_1^{10} + \overset{\circ}{\varepsilon}_1^2 \right) &= 0 \\ 6\overset{\circ}{\varepsilon}_1^5 [2D_1\varepsilon_1 + 2D_2 + \varepsilon_1^2(1 + b_1)] + 2\overset{\circ}{\varepsilon}_1^6 [D_1 + \varepsilon_1(1 + b_1)] + \\ -4(b_1 + b_2 + 2b_3) \left(2\overset{\circ}{\varepsilon}_1^7 + \overset{\circ}{\varepsilon}_1^3 \right) - 2b_3 \left(5\overset{\circ}{\varepsilon}_1^9 + \overset{\circ}{\varepsilon}_1 \right) &= 0 \end{aligned} \quad (4.11)$$

Eliminating D_2 and simplifying, we obtain a polynomial equation

$$\left[\overset{\circ}{\varepsilon}_1^5 [D_1 + \varepsilon_1(1 + b_1)] + 2b_3 \left(1 - \overset{\circ}{\varepsilon}_1^8 \right) + (b_1 + b_2 + 2b_3) \left(\overset{\circ}{\varepsilon}_1^2 - \overset{\circ}{\varepsilon}_1^6 \right) \right] \overset{\circ}{\varepsilon}_1^2 = 0 \quad (4.12)$$

The character of each equilibrium can be found by linearisation of (4.8) and (4.9). If $\varepsilon_1 = \varepsilon_{1e}$, $y = 0$ is a solution to (4.12), according with (4.10) we have

$$F(\varepsilon_{1e}, D_2) = F'(\varepsilon_{1e}, D_2) = 0 \quad (4.13)$$

then close to the equilibrium point

$$y = Y \quad \varepsilon = \varepsilon_{1e} + \Lambda \quad (4.14)$$

where Y and Λ are small perturbations.

Substituting (4.14) into (4.8), we have

$$Y = (\varepsilon_{1e} + \Lambda)_\zeta \quad (4.15)$$

which entails that

$$\Lambda_\zeta = Y \quad (4.16)$$

Similarly, substituting (4.13) into (4.9), we obtain

$$Y_\zeta = \frac{1}{2} F'[(\varepsilon_{1e} + \Lambda), D_2] = \frac{1}{2} F'(\varepsilon_{1e}, D_2) + \frac{1}{2} F''(\varepsilon_{1e}, D_2)(\varepsilon_{1e} + \Lambda - \varepsilon_{1e}) \quad (4.17)$$

According to (4.13) $0.5F'(\varepsilon_{1e}, D_2) = 0$, then

$$Y_\zeta = \frac{1}{2}F''(\varepsilon_{1e}, D_2)\Lambda \tag{4.18}$$

where D_2 is the parameter value representing the equilibrium point (compare Dai, 2001).

From the analysis described by Dai (2001) and Osiński (1980) it follows that if $F''(\varepsilon_{1e}, D_2) < 0$ the singular point (equilibrium point) at the phase plane is a center. Such a point sets a stable state of equilibrium. However, if $F''(\varepsilon_{1e}, D_2) > 0$, the singular point is a saddle and the state of equilibrium is unstable. In the degenerate case, in which $F''(\varepsilon_{1e}, D_2) = 0$, we obtain a cusp point (see Fig. 5). We obtain the curve in the phase plane directly by taking square roots of $F(\varepsilon_1, D_2)$.

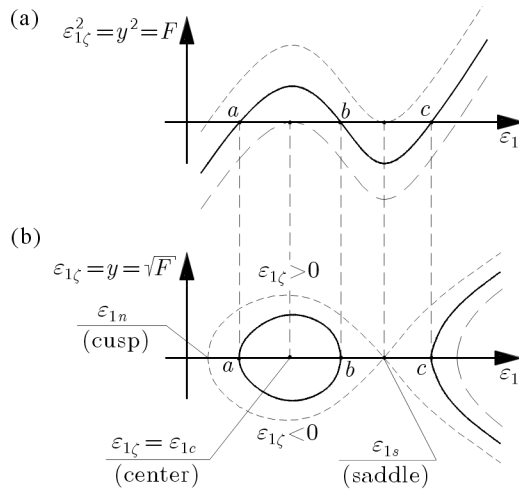


Fig. 5. Graphs of the functions F (a) and phase trajectory y in the phase plane (b) (ε_{1c} – denotes the center point, ε_{1s} – saddle point and ε_{1n} – cusp point)

The foregoing discussion indicates connection between the location and nature of equilibria as well as the form of graphs of $F(\varepsilon_1, D_2)$. The real curves in the phase plane are described by equation (4.7) $y = \pm\sqrt{F(\varepsilon_1, D_2)}$.

5. Discussion about physically acceptable solutions

The phase portrait method allows one to find solutions to differential system (4.6). However, not all curves in the phase plane are interesting for the

physical problem at hand. Our main task consists in characterising such portraits, whose values of b_1 , b_2 , b_3 and D_1 , represent physically meaningful behaviour.

With some approximation, we can assume that in the case of compression or tension of a thin rubber layer, the physically acceptable value ε_1 is in the interval from -0.5 to 0.5 .

According to Theorem 1 from the paper by Dai (2001, p.104), in order that there be a physically acceptable solution we must obtain for the function $F(\varepsilon_1, D_2)$ a center point in the region of the physically acceptable value ε_1 . Supposing that this point exists for $\varepsilon_1 = \varepsilon_{1c}$ (then $F(\varepsilon_{1c}, D_2) = 0$ and $F'(\varepsilon_{1c}, D_2) = 0$), we can find D_1 and D_2 as functions of ε_{1c} , which determine this center

$$D_1 = \frac{(b_1 + b_2 + 2b_3) \left(\overset{\circ}{\varepsilon}_{1c}^6 - \overset{\circ}{\varepsilon}_{1c}^2 \right) + 2b_3 \left(\overset{\circ}{\varepsilon}_{1c}^8 - 1 \right)}{\overset{\circ}{\varepsilon}_{1c}^5} - \varepsilon_{1c}(1 + b_1) \quad (5.1)$$

$$D_2 = \frac{\varepsilon_{1c}(b_1 + b_2 + 2b_3) \left(\overset{\circ}{\varepsilon}_{1c}^2 - \overset{\circ}{\varepsilon}_{1c}^6 \right) + 2b_3 \varepsilon_{1c} \left(1 - \overset{\circ}{\varepsilon}_{1c}^8 \right)}{\overset{\circ}{\varepsilon}_{1c}^5} +$$

$$+ \frac{(b_1 + b_2 + 2b_3) \left(\overset{\circ}{\varepsilon}_{1c}^6 + \overset{\circ}{\varepsilon}_{1c}^2 \right) + b_3 \left(\overset{\circ}{\varepsilon}_{1c}^8 + 1 \right)}{2\overset{\circ}{\varepsilon}_{1c}^4} + \frac{1}{2} \varepsilon_{1c}^2 (1 + b_1)$$

where $\overset{\circ}{\varepsilon}_{1c} = 1 + \varepsilon_{1c}$.

After substituting (5.1) into (4.12), we obtain

$$(\varepsilon_1 - \varepsilon_{1c}) \overset{\circ}{\varepsilon}_1^2 \left\{ \overset{\circ}{\varepsilon}_1^5 (1 + b_1) - \frac{(b_1 + b_2 + 2b_3) \overset{\circ}{\varepsilon}_1^2}{\overset{\circ}{\varepsilon}_{1c}^3} \left(\overset{\circ}{\varepsilon}_1^3 \overset{\circ}{\varepsilon}_{1c}^3 + 3(1 + A) + B \right) + \right. \quad (5.2)$$

$$\left. - \frac{2b_3}{\overset{\circ}{\varepsilon}_1^5} \left[\overset{\circ}{\varepsilon}_1^5 \overset{\circ}{\varepsilon}_{1c}^5 (3(1 + A) + B) + 5(1 + W) + 10(A + B) + Z \right] \right\} = 0$$

where

$$\begin{aligned} A &= \varepsilon_1 + \varepsilon_{1c} & W &= \varepsilon_1^3 + \varepsilon_1^2 \varepsilon_{1c} + \varepsilon_1 \varepsilon_{1c}^2 + \varepsilon_{1c}^3 \\ B &= \varepsilon_1^2 + \varepsilon_1 \varepsilon_{1c} + \varepsilon_{1c}^2 & Z &= \varepsilon_1^4 + \varepsilon_1^3 \varepsilon_{1c} + \varepsilon_1^2 \varepsilon_{1c}^2 + \varepsilon_1 \varepsilon_{1c}^3 + \varepsilon_{1c}^4 \end{aligned} \quad (5.3)$$

The expression in the square brackets of equation (5.2) follows determination of other equilibrium points

$$\begin{aligned} & \varepsilon_1^{\circ 5}(1 + b_1) - \frac{(b_1 + b_2 + 2b_3)\varepsilon_1^{\circ 2}}{\varepsilon_{1c}^{\circ 3}} \left[\varepsilon_1^{\circ 3} \varepsilon_{1c}^{\circ 3} + 3(1 + A) + B \right] + \\ & - \frac{2b_3}{\varepsilon_{1c}^{\circ 5}} \left[\varepsilon_1^{\circ 5} \varepsilon_{1c}^{\circ 5} (3(1 + A) + B) + 5(1 + C) + 10(A + B) + D \right] = 0 \end{aligned} \tag{5.4}$$

Finally, by computing $F''(\varepsilon_{1c}, D_2)$ and substituting (5.1), we find that ε_{1c} will be a center if

$$\frac{1}{2}F''(\varepsilon_{1c}, D_2) = \frac{(1 + b_1)\varepsilon_{1c}^{\circ 6} - (b_1 + b_2 + 2b_3)(3\varepsilon_{1c}^{\circ 2} + \varepsilon_{1c}^{\circ 6}) - 2b_3(3\varepsilon_{1c}^{\circ 8} + 5)}{(1 - b_2)\varepsilon_{1c}^{\circ 2} - 2b_3(\varepsilon_{1c}^{\circ 4} + \varepsilon_{1c}^{\circ 2} + 1)} < 0 \tag{5.5}$$

In order to obtain physically acceptable solutions, we must have $\nu > 2C_1$. Equation (5.5) is not satisfied if $\nu < 2C_1$.

It results from the paper of Dai (2001) that there is a second point except for the point of stable state of equilibrium. It is a point of unstable state of equilibrium – the saddle point.

Since $\nu > 2C_1$, we see from (4.3) that $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$. The difference in the signs of marks of terms in expression (5.4) suggests that there is a positive root, which we assume to be equal $\varepsilon_1 = \varepsilon_{1s}$.

Equation (5.2) takes the form

$$\begin{aligned} & \frac{(\varepsilon_1 - \varepsilon_{1c})(\varepsilon_1 - \varepsilon_{1s})\varepsilon_1^{\circ 2}}{\varepsilon_{1c}^{\circ 2}} \left\{ \frac{(1 - b_2 - 2b_3)\varepsilon_1^{\circ 2}\varepsilon_{1c}^{\circ 2}}{3(1 + A_1) + B_1} \left[8E + 3E^2 + 3(\varepsilon_1^2 A_1 + \varepsilon_{1c}^2 A_2 + \varepsilon_{1s}^2 A) + \right. \right. \\ & + 6(1 + F) + AF + \varepsilon_{1s}^2 B + \varepsilon_1^2 \varepsilon_{1c}^2 + 3(\varepsilon_1 \varepsilon_{1c} + \varepsilon_1 \varepsilon_{1s} + \varepsilon_{1c} \varepsilon_{1s}) \left. \right] + \\ & - 2b_3 \left[\varepsilon_1^{\circ 2} \varepsilon_{1c}^{\circ 2} (3(1 + A) + B)(3(1 + A_2) + B_2) \right] + \\ & + \frac{2b_3}{\varepsilon_{1s}^{\circ 2} (3(1 + A_1) + B_1)} \left[20E + 10E^2 + 7(\varepsilon_1^2 A_1 + \varepsilon_{1c}^2 A_2 + \varepsilon_{1s}^2 A) + \right. \\ & \left. + 2EF + 2G + 15 + 12F + \varepsilon_1 \varepsilon_{1c} B + \varepsilon_1 \varepsilon_{1s} B_2 + \varepsilon_{1c} \varepsilon_{1s} B_1 \right] \left. \right\} = 0 \end{aligned} \tag{5.6}$$

where we have eliminated b_1 using the fact that $\varepsilon_1 = \varepsilon_{1s}$ is a root

$$b_1 = \frac{(1-b_2-2b_3)\varepsilon_{1s}^3\varepsilon_{1c}^3}{3(1+A_1)+B_1} - b_2 - 2b_3\varepsilon_{1s}^3\varepsilon_{1c}^3 - 2b_3 \left(1 + \frac{5(1+W_1) + 10(A_1+B_1) + Z_1}{\varepsilon_{1s}^2\varepsilon_{1c}^2[3(1+A_1)+B_1]} \right) \quad (5.7)$$

where $\varepsilon_{1s} = 1 + \varepsilon_{1s}$ and we used the following variables

$$\begin{aligned} A_1 &= \varepsilon_{1s} + \varepsilon_{1c} & A_2 &= \varepsilon_{1s} + \varepsilon_1 \\ B_1 &= \varepsilon_{1s}^2 + \varepsilon_{1s}\varepsilon_{1c} + \varepsilon_{1c}^2 & B_2 &= \varepsilon_{1s}^2 + \varepsilon_{1s}\varepsilon_1 + \varepsilon_1^2 \\ E &= \varepsilon_1 + \varepsilon_{1s} + \varepsilon_{1c} & F &= \varepsilon_1\varepsilon_{1s}\varepsilon_{1c} \\ G &= \varepsilon_1^3 + \varepsilon_{1s}^3 + \varepsilon_{1c}^3 & W_1 &= \varepsilon_{1s}^3 + \varepsilon_{1s}^2\varepsilon_{1c} + \varepsilon_{1s}\varepsilon_{1c}^2 + \varepsilon_{1c}^3 \\ Z_1 &= \varepsilon_{1s}^4 + \varepsilon_{1s}^3\varepsilon_{1c} + \varepsilon_{1s}^2\varepsilon_{1c}^2 + \varepsilon_{1s}\varepsilon_{1c}^3 + \varepsilon_{1c}^4 \end{aligned} \quad (5.8)$$

however A, B, W, Z are according to (5.3).

Substituting b_1 (see (5.7)) into (5.5), we obtain the following expressions for ε_{1c} and ε_{1s} , respectively

$$\begin{aligned} \frac{1}{2}F''(\varepsilon_{1c}, D_2) &= \frac{\varepsilon_{1c} - \varepsilon_{1s}}{(1-b_2)\varepsilon_{1c}^2 - 2b_3(\varepsilon_{1c}^4 + \varepsilon_{1c}^2 + 1)} \cdot \\ &\cdot \left(\frac{(1-b_2-2b_3)\varepsilon_{1c}^5(6+4\varepsilon_{1c} + \varepsilon_{1c}^2 + 2\varepsilon_{1c}\varepsilon_{1s} + 8\varepsilon_{1s} + 3\varepsilon_{1s}^2)}{3(1+\varepsilon_{1s} + \varepsilon_{1c}) + \varepsilon_{1s}^2 + \varepsilon_{1s}\varepsilon_{1c} + \varepsilon_{1c}^2} + \right. \\ &+ 6b_3\varepsilon_{1c}^5[3(1+\varepsilon_{1s} + \varepsilon_{1c}) + \varepsilon_{1s}^2 + \varepsilon_{1s}\varepsilon_{1c} + \varepsilon_{1c}^2] + \\ &+ \frac{2b_3(15+25\varepsilon_{1c} + 15\varepsilon_{1c}^2 + 3\varepsilon_{1c}^3 + 20\varepsilon_{1c}\varepsilon_{1s} + 6\varepsilon_{1c}^2\varepsilon_{1s})}{\varepsilon_{1s}^2[3(1+\varepsilon_{1s} + \varepsilon_{1c}) + \varepsilon_{1s}^2 + \varepsilon_{1s}\varepsilon_{1c} + \varepsilon_{1c}^2]} + \\ &\left. + \frac{2b_3(4\varepsilon_{1c}\varepsilon_{1s}^2 + 20\varepsilon_{1s} + 10\varepsilon_{1s}^2 + 2\varepsilon_{1s}^3)}{\varepsilon_{1s}^2[3(1+\varepsilon_{1s} + \varepsilon_{1c}) + \varepsilon_{1s}^2 + \varepsilon_{1s}\varepsilon_{1c} + \varepsilon_{1c}^2]} \right) \quad (5.9) \end{aligned}$$

$$\begin{aligned} \frac{1}{2}F''(\varepsilon_{1s}, D_2) &= \frac{\varepsilon_{1s} - \varepsilon_{1c}}{(1-b_2)\varepsilon_{1s}^2 - 2b_3(\varepsilon_{1s}^4 + \varepsilon_{1s}^2 + 1)} \cdot \\ &\cdot \left(\frac{(1-b_2-2b_3)\varepsilon_{1s}^5(6+4\varepsilon_{1s} + \varepsilon_{1s}^2 + 2\varepsilon_{1s}\varepsilon_{1c} + 8\varepsilon_{1c} + 3\varepsilon_{1c}^2)}{3(1+\varepsilon_{1c} + \varepsilon_{1s}) + \varepsilon_{1c}^2 + \varepsilon_{1c}\varepsilon_{1s} + \varepsilon_{1s}^2} + \right. \\ &+ 6b_3\varepsilon_{1s}^5[3(1+\varepsilon_{1c} + \varepsilon_{1s}) + \varepsilon_{1c}^2 + \varepsilon_{1c}\varepsilon_{1s} + \varepsilon_{1s}^2] + \end{aligned}$$

$$\begin{aligned}
& + \frac{2b_3(15 + 25\varepsilon_{1s} + 15\varepsilon_{1s}^2 + 3\varepsilon_{1s}^3 + 20\varepsilon_{1s}\varepsilon_{1c} + 6\varepsilon_{1s}^2\varepsilon_{1c})}{\varepsilon_{1c}^2[3(1 + \varepsilon_{1c} + \varepsilon_{1s}) + \varepsilon_{1c}^2 + \varepsilon_{1c}\varepsilon_{1s} + \varepsilon_{1s}^2]} + \\
& + \frac{2b_3(4\varepsilon_{1s}\varepsilon_{1c}^2 + 20\varepsilon_{1c} + 10\varepsilon_{1c}^2 + 2\varepsilon_{1c}^3)}{\varepsilon_{1c}^2[3(1 + \varepsilon_{1c} + \varepsilon_{1s}) + \varepsilon_{1c}^2 + \varepsilon_{1c}\varepsilon_{1s} + \varepsilon_{1s}^2]} \Big)
\end{aligned}$$

According to conclusions featured at condition (4.18), we can see that if

$$\frac{1}{2}F''(\varepsilon_{1c}, D_2) < 0 \quad \text{or} \quad \frac{1}{2}F''(\varepsilon_{1s}, D_2) > 0 \quad (5.10)$$

we obtain a center point or saddle point in the phase plane, respectively.

6. Numerical analysis

The numerical analysis is based on equation (4.6) obtained for the Zahorski material and carried out for the function $F(\varepsilon_1, D_2)$ (see (4.7)).

The constant D_1 depends on ε_1 , and according to (4.12) we have

$$D_1(\varepsilon_1) = \frac{(b_1 + b_2 + 2b_3)(\varepsilon_1^6 - \varepsilon_1^2) + 2b_3(\varepsilon_1^8 - 1)}{\varepsilon_1^5} - \varepsilon_1(1 + b_1) \quad (6.1)$$

Analogously, the constant D_2 (which depends on ε_1 too), according to (4.10)₁ is equal

$$\begin{aligned}
D_2(\varepsilon_1) = & \frac{\varepsilon_1(b_1 + b_2 + 2b_3)(\varepsilon_1^2 - \varepsilon_1^6) + 2b_3\varepsilon_1(1 - \varepsilon_1^8)}{\varepsilon_1^5} + \\
& + \frac{(b_1 + b_2 + 2b_3)(\varepsilon_1^6 + \varepsilon_1^2) + b_3(\varepsilon_1^8 + 1)}{2\varepsilon_1^4} + \frac{1}{2}\varepsilon_1^2(1 + b_1)
\end{aligned} \quad (6.2)$$

Then, we can determine the constants D_1 and D_2 from (6.1) and (6.2), respectively, for a chosen value of ε_1 (in this paper $\varepsilon_1 = 0.5$).

In Fig.6 there are three graphs of the functions $F_i(\varepsilon_1) \equiv F_i(\varepsilon_1, D_2)$, $i = 1, 2, 3$, for constant D_2 calculated according to (6.2) and for $\varepsilon_1 = 0.5$. The functions $y(\varepsilon_1)$ denotes $y_i(\varepsilon_1) \equiv \pm\sqrt{F_i(\varepsilon_1)}$ for $i = 1, 2, 3$.

In the analysis we assumed the rubber density $\rho = 1190 \text{ kg/m}^3$ and the shear modulus $\mu = 1.432 \cdot 10^5 \text{ N/m}^2$. The constants C_1 , C_2 and C_3 are characteristic for the kind of rubber described by Zahorski (1962) and take the

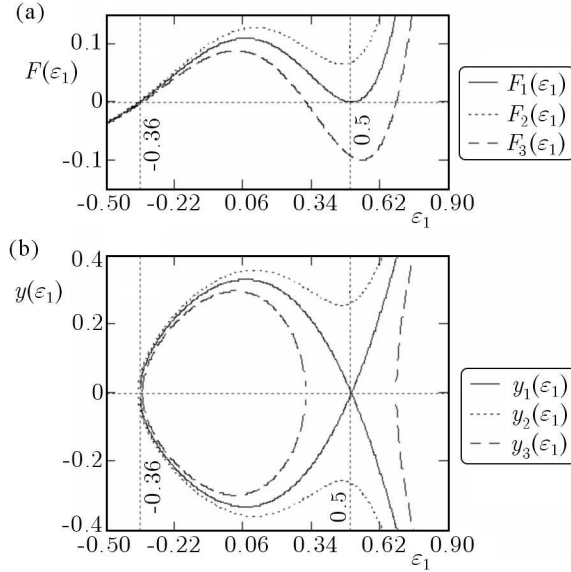


Fig. 6. Graphs for the rubber OKA-1 made of the Zahorski material ($\mu = 1.46 \text{ kG/cm}^2$, $\rho = 1190 \text{ kg/m}^3$) for the speed $V = 25 \text{ m/s}$ and constants $b_1 = 0.203$, $b_2 = 0.029$, $b_3 = 0.022$ and $D_1 = -0.125$ (according to (6.1) for $\varepsilon_1 = 0.5$); (a) distribution of functions: for $F_1(\varepsilon_1)$ the constant $D_2 = 0.342$ (according to (6.2) for $\varepsilon_1 = 0.5$), for $F_2(\varepsilon_1)$ - $D_2 = 0.348$ and for $F_3(\varepsilon_1)$ - $D_2 = 0.335$, respectively, (b) phase trajectory

values $C_1 = 4.299 \cdot 10^4 \text{ N/m}^2$, $C_2 = 0.604 \cdot 10^4 \text{ N/m}^2$ and $C_3 = 0.47 \cdot 10^4 \text{ N/m}^2$. The constants b_1 , b_2 and b_3 were calculated according to (4.3)_{1,2,3}.

Figure 6b shows phase trajectories in the coordinate system $(\varepsilon_1, \zeta = y, \varepsilon_1)$ for the functions $F_i(\varepsilon_1)$, $i = 1, 2, 3$ found from Fig. 6a for the Zahorski material.

The constants D_1 and D_2 for $\varepsilon_1 = 0.5$ are calculated according to (6.1) and (6.2). In Fig. 6, the constant $D_1 = -0.125$ and the constant D_2 calculated from (6.2) is 0.342. The constants $D_2 = 0.348$ and $D_2 = 0.335$ have been established arbitrarily, but here it is fixed at $D_2 = 0.342$.

The center point is obtained for $\varepsilon_1 \cong 0.07$, and the graph is contained within the interval $\varepsilon_1 = \langle -0.36; 0.5 \rangle$ of physically acceptable solutions (see Section 5).

We find that propagation of the travelling wave in the thin layer composed of Zahorski's material is possible for compression and tension. The solution has a periodic character for closed curves in an area limited by the solid line shown in Fig. 6b, and can be a solitary wave for solutions represented by a homoclinic orbit (see the solid line in Fig. 6b).

For the constant $C_3 = 0$, we can receive graphs for the Mooney-Rivlin material. The solutions differ quantitatively (compare Major and Major, 2007).

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Fale biegnące w cienkiej warstwie wykonanej z nieliniowego hipersprężystego materiału Zahorskiego

Streszczenie

W pracy omówiono zagadnienia dotyczące propagacji nieliniowej fali biegnącej w cienkiej sprężystej warstwie wykonanej z materiału Zahorskiego. Metoda polegająca na uśrednieniu równań ruchu w przekroju poprzecznym warstwy przy założeniu, że uśrednione wielkości spełniają równania ruchu i warunki brzegowe pozwala na przybliżone rozwiązanie zagadnienia propagacji fali biegnącej w warstwie hipersprężystej zastosowano. Otrzymane w ten sposób równania zastosowano do opisu procesów falowych dla rozpatrywanych w pracy fal biegnących.

Do analizy procesów falowych użyta została technika płaszczyzny fazowej. W oparciu o metodę trajektorii fazowej zinterpretowano warunki propagacji nieliniowej fali oraz ustalono warunki istnienia fizycznie akceptowalnych rozwiązań.

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