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ON THE PROBLEM OF A TRANSVERSELY ISOTROPIC HALF-SPACE WEAKENED BY A PENNY-SHAPED CRACK FILLED WITH A GAS

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An axially-symmetric problem of a penny-shaped crack situated in a position parallel to the boundary of a semi-infinite transversely isotropic solid is formulated with due regard for the presence of an ideal gas in the crack. The method of the Hankel integral transforms is used to solve this problem. The dual integral equations obtained are reduced to a set of integral equations which are solved numerically. The graphs presented illustrate the influence of the gas on the stress intensity factors of Mode I and Mode II.

Key words: transversely isotropic half-space, gas-filled penny-shaped crack, integral equations, stress intensity factors

1. Introduction

The class of problems for solids with cracks and cavities, provided that the defects are filled with some substance, has wide applications in many areas, namely, in geomechanics, the petroleum industry, gas-producing industry, mining geotechnical engineering and others. Such problems are arisen during investigation of hydraulic fracture of rocks, gas-filtration into cavities appearing during coal excavation, spalling of concrete at high temperature, etc. There is a considerable amount of literature on the topic. A great deal of interest is focused on the modelling of liquid or gas-filled cracks (see, for example, Abe *et al.*, 1976; Zazovskii, 1979; Bui and Parnes, 1982; Advani *et al.*, 1997; Feraille-Fresnet and Ehrlacher, 2000; Savitski and Detournay, 2002; Feraille-Fresnet *et* *al.*, 2003). Comprehensive accounts of developments pertaining to fluid-filled crack problems can be found in Chapter 8 of Bui's book (2006). A simplified analysis of these problems, which retains the description of fracture phenomena, can be carried out using the concepts of linear elastic fracture mechanics. This approach was oriented mainly to the construction of appropriate strain-stress solutions for idealized situations with some assumptions involving fluid crack interaction. In this respect, we can mention the research results obtained by Evtushenko and Sulim (1981) for a plane problem involving a crack filled with a compressible barotropic liquid, by Balueva and Dashevskii (1995) and Dashevskii (2007) who studied the growth of gas-filled cracks, by Kit *et al.* (2003) and Machyshyn and Nagórko (2003) for systems with gas-filled contact gaps. Attention should also be paid to an article by Matczyński *et al.* (1999) in which the combined thermal and mechanical influence of the heat-conducting ideal gas filling the crack on stresses is analysed in the case of plane strain.

In the present paper, we continue the investigations originated by the authors (Kaczyński and Monastyrskyy, 2004, 2007), and study an axisymmetric elastostatic problem of determination of disturbances of stresses due to the presence of a gas-filled penny-shaped crack positioned parallel to the boundary of the semi-infinity body treated as a transversely isotropic medium (modelling, for example, a rock layered horizontally (Gil, 1991) or stratified rock mass (Salamon, 1968)). Our goal is to examine the integrated effect of the crack filler on the stress distribution around the crack, essentially on the stress intensity factors.

2. Statement of the problem

2.1. Formulation

Suppose that a penny-shaped crack of radius a is embedded in a transversely isotropic half-space with the axis of elastic symmetry normal to the crack plane as shown in Fig. 1. We refer to a system of cylindrical coordinates (r,θ,z) with the origin placed at the centre of the crack and the z-axis in the transverse direction such that the crack occupies the region $S = \{(r,\theta,z=0) : 0 \leqslant r \leqslant a \quad \wedge \quad 0 < \theta \leqslant 2\pi\}$ and $z = h$ defines the boundary of the solid. Here, h is the distance of the crack away from the half-space surface.

Keeping in mind the presence of a gas in this crack, we touch upon a problem which has two interrelated components regarding the external loading

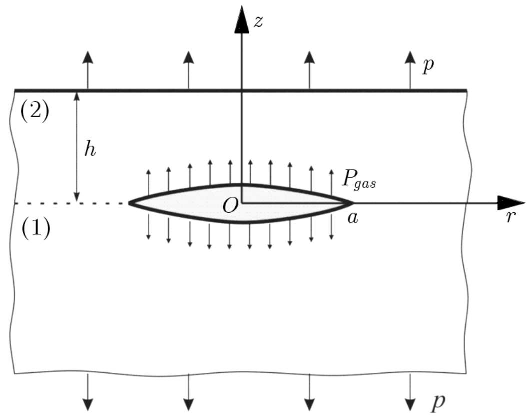

Fig. 1. A half-space with a gas-filled penny-shaped crack

and the behaviour of the filler. For formulation of this problem, some simplified assumptions will be made. It allows effective examination of the stressed-strain state of the half space and determination of pressure of the gas in the crack.

In what follows, we assume that the body is subjected at infinity and the boundary to a constant tensile or compressive load p. The crack is filled with an ideal and compressible gas whose state is described by the well-known Boyle-Clayperon-Mendeleyev equation, written in the simple form

$$P_{gas}V = g_0 = \text{const} \tag{2.1}$$

where P_{gas} and V stand for the pressure and volume of the gas, and g_0 is a constant on the assumption that the mass of the gas and temperature remain constant. The mechanical action of the gas filled the crack is simulated by the internal pressure P_{gas}, so only uniformly distributed normal forces $-P_{gas}$ act on the surfaces of the crack. It is noteworthy that P_{gas} is unknown and according to Eq. (2.1) is a function of gas properties and the volume V which is equal to the volume of the crack depending, in turn, on the external load p. Hence, the gas pressure is an additional unknown parameter of the problem to be determined in the course of its solution.

Thus the problem under study lies in the determination of the stress-and-strain state of the body, paying much attention to the distribution of stresses in the neighbourhood of the crack. In particular, the stress intensity factors as the local important parameters controlling the fracture instability are of prime interest.

2.2. Governing equations

The above-mentioned problem may be treated as axially-symmetric (independent of the angle θ) with the only non-vanishing displacements in the radial and axial directions $u_r(r,z)$, $u_z(r,z)$ and components of the stress tensor σ_{zz}, σ_{rz}, σ_{rr}, $\sigma_{\theta\theta}$. In this case, the linear constitutive relations of a transversely isotropic medium characterised by the five elastic moduli c_{33}, c_{13}, c_{44}, c_{11}, c_{12} are (Lekhnitskii, 1963)

$$
\begin{aligned}
\sigma_{zz}(r,z) &= c_{33}\frac{\partial u_z(r,z)}{\partial z} + c_{13}\frac{1}{r}\frac{\partial[ru_r(r,z)]}{\partial r}\\
\sigma_{rz}(r,z) &= c_{44}\Big(\frac{\partial u_r(r,z)}{\partial z} + \frac{\partial u_z(r,z)}{\partial r}\Big)\\
\sigma_{rr}(r,z) &= c_{11}\frac{\partial u_r(r,z)}{\partial r} + c_{12}\frac{u_r(r,z)}{r} + c_{13}\frac{\partial u_z(r,z)}{\partial z}\\
\sigma_{\theta\theta}(r,z) &= c_{12}\frac{\partial u_r(r,z)}{\partial r} + c_{11}\frac{u_r(r,z)}{r} + c_{13}\frac{\partial u_z(r,z)}{\partial z}
\end{aligned}
\tag{2.2}
$$

The equilibrium equations for the unknown displacements $u_r(r,z)$ and $u_z(r,z)$, in the absence of body forces, are given by

$$
\begin{aligned}
&c_{11}\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial[ru_r(r,z)]}{\partial r}\Big) + (c_{13}+c_{44})\frac{\partial^2 u_z(r,z)}{\partial r\partial z} + c_{44}\frac{\partial^2 u_r(r,z)}{\partial z^2} = 0\\
&c_{44}\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u_z(r,z)}{\partial r}\Big) + (c_{13}+c_{44})\frac{1}{r}\frac{\partial^2[ru_r(r,z)]}{\partial r\partial z} + c_{33}\frac{\partial^2 u_z(r,z)}{\partial z^2} = 0
\end{aligned}
\tag{2.3}
$$

3. The boundary-value problem and its solution

Following the classical approach in crack problems based on the superposition principle, we separate the problem under study into two parts: the first relating to the body with no crack subjected to the given exterior load p and the second, corrective part involving local perturbations caused by the gas-filling penny-shaped crack. Since the first part is trivial, we will draw attention to finding the corrective solution. To formulate the perturbed problem with the crack located on the plane $z=0$, it is convenient to treat the considered solid as a layer of thickness h, described by $0 \leqslant z \leqslant h$ joined to the half-space $z \leqslant 0$ of the same transversely isotropic material (see Fig. 1). Bearing in mind all aforementioned assumptions, we arrive at the following boundary conditions:

– on the crack

$$\begin{aligned} &\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0) = -P_{gas} - p && 0 \leqslant r \leqslant a \\ &\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0) = 0 && 0 \leqslant r \leqslant a \end{aligned} \tag{3.1}$$

– outside the crack (continuity of stresses and displacements)

$$\begin{aligned} &\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0) && \sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0) && a < r < \infty \\ &u_z^{(1)}(r,0) = u_z^{(2)}(r,0) && u_r^{(1)}(r,0) = u_r^{(2)}(r,0) && a \leqslant r < \infty \end{aligned} \tag{3.2}$$

– on the boundary

$$\sigma_{zz}^{(2)}(r,h) = \sigma_{rz}^{(2)}(r,h) = 0 \tag{3.3}$$

– at infinity (regularity conditions)

$$\sigma_{zz}^{(1)}(r,-\infty) = \sigma_{rz}^{(1)}(r,-\infty) = 0 \tag{3.4}$$

where superscripts (1) and (2) refer to quantities associated with the region $z \leqslant 0$ and $0 \leqslant z \leqslant h$, respectively.

Besides, recall that the unknown gas pressure P_{gas} is found during solving the problem by using Eq. (2.1), in which

$$V = \iint\limits_S (u_z^{(2)} - u_z^{(1)})\, dS = 2\pi \int\limits_0^a r[u_z^{(2)}(r,0) - u_z^{(1)}(r,0)]\, dr \tag{3.5}$$

The solution to the non-trivial perturbed problem is grounded on the jump displacement method. It lies in reducing this problem to a set of simultaneous integral equations the solution to which can be obtained only in a numerical fashion. The use of the Hankel integral transforms (see Sneddon and Lowengrub, 1969) is a mathematical tool.

At the first stage, we solve an auxiliary problem described by Eqs. (3.3) and (3.4), and the following boundary conditions on the whole plane $z = 0$

$$\begin{aligned} &\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0) && \sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0) \\ &u_z^{(2)}(r,0) - u_z^{(1)}(r,0) = \Delta u_z(r) && u_r^{(2)}(r,0) - u_r^{(1)}(r,0) = \Delta u_r(r) \end{aligned} \tag{3.6}$$

where $\Delta u_z(r)$ and $\Delta u_r(r)$ stand for the jumps of normal and tangential displacements, respectively, unknown beforehand.

A commonly employed method of obtaining the solution is based on the theory of Hankel's transforms. Only some relevant results will be given. We apply Hankel's transforms of the first order for Eq. $(2.3)_1$ and the zero order for Eq. $(2.3)_2$, with the aid of the definitions

$$\widetilde{f}_n(\xi,z) \equiv H_n[f(r,z);r\rightarrow\xi] \equiv \int\limits_0^\infty rf(r,z)J_n(r\xi)\,d\xi \qquad n=0,1 \tag{3.7}$$

where J_n stands for the Bessel function of the first kind of the order n, and ξ is the transform parameter. Using some properties of Hankel's transform, we arrive at two-coupled ordinary differential equations for the Hankel transforms of displacements $(\widetilde{u}_z)_0$ and $(\widetilde{u}_r)_1$. Solving the set of these equations separately for the half-space $z \leqslant 0$ (bearing Eq. (3.4) in mind) and the layer $0 \leqslant z \leqslant h$, we get the following expressions:
— for $z \leqslant 0$

$$\begin{aligned}
(\widetilde{u}_z^{(1)})_0 &= S_1(\xi)\exp(k_1\xi z) + S_2(\xi)\exp(k_2\xi z)\\
(\widetilde{u}_r^{(1)})_1 &= \frac{c_{44}-k_1^2c_{33}}{(c_{13}+c_{44})k_1}S_1(\xi)\exp(k_1\xi z) + \frac{c_{44}-k_2^2c_{33}}{(c_{13}+c_{44})k_2}S_2(\xi)\exp(k_2\xi z)
\end{aligned} \tag{3.8}$$

— for $0 \leqslant z \leqslant h$

$$\begin{aligned}
(\widetilde{u}_z^{(1)})_0 &= X_1(\xi)\cosh(k_1\xi z) + X_2(\xi)\sinh(k_1\xi z) + X_3(\xi)\cosh(k_2\xi z) +\\
&\quad + X_4(\xi)\sinh(k_2\xi z)\\
(\widetilde{u}_r^{(1)})_1 &= \frac{c_{44}-k_1^2c_{33}}{(c_{13}+c_{44})k_1}[X_2(\xi)\cosh(k_1\xi z) + X_1(\xi)\sinh(k_1\xi z)] +\\
&\quad + \frac{c_{44}-k_2^2c_{33}}{(c_{13}+c_{44})k_1}[X_4(\xi)\cosh(k_2\xi z) + X_3(\xi)\sinh(k_2\xi z)]
\end{aligned} \tag{3.9}$$

in which the functions S_1, S_2 and X_1, X_2, X_3, X_4 are unknown and remain to be found. Moreover, k_i^2 $(i=1,2)$ are the roots of the equation

$$c_{33}c_{44}k^4 + (c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33})k^2 + c_{44}c_{11} = 0 \tag{3.10}$$

Here we have confined to the case of distinct positive roots given explicitly by Ding *et al.* (2006)

$$\begin{aligned}
k_1 &= \sqrt{\frac{(\sqrt{c_{11}c_{33}} - c_{13})(\sqrt{c_{11}c_{33}} + c_{13} + 2c_{44})}{4c_{33}c_{44}}} +\\
&\quad - \sqrt{\frac{(\sqrt{c_{11}c_{33}} + c_{13})(\sqrt{c_{11}c_{33}} - c_{13} - 2c_{44})}{4c_{33}c_{44}}}
\end{aligned} \tag{3.11}$$

$$k_2 = \sqrt{\frac{(\sqrt{c_{11}c_{33}} - c_{13})(\sqrt{c_{11}c_{33}} + c_{13} + 2c_{44})}{4c_{33}c_{44}}} + \\ + \sqrt{\frac{(\sqrt{c_{11}c_{33}} + c_{13})(\sqrt{c_{11}c_{33}} - c_{13} - 2c_{44})}{4c_{33}c_{44}}}$$

Note that there are no principal difficulties to solve the problem in the special case of equal roots, but in the present paper it is omitted.

With the help of Eqs. (3.8) and (3.9), the Hankel transforms of stresses σ_{zz} and σ_{rz} are found from Eqs. $(2.2)_{1,2}$ to be:

— for $z \leqslant 0$

$$\begin{aligned}
(\widetilde{\sigma}_{zz}^{(1)})_0 &= \frac{c_{44}}{(c_{13}+c_{44})}\xi \cdot \\
&\cdot\Big[\frac{c_{13}+c_{33}k_1^2}{k_1}S_1(\xi)\exp(k_1\xi z) + \frac{c_{13}+c_{33}k_2^2}{k_2}S_2(\xi)\exp(k_2\xi z)\Big] \\
(\widetilde{\sigma}_{rz}^{(1)})_1 &= -\frac{c_{44}}{c_{13}+c_{44}}\xi \cdot \\
&\cdot\Big[(c_{13}+c_{33}k_1^2)S_1(\xi)\exp(k_1\xi z) + (c_{13}+c_{33}k_2^2)S_2(\xi)\exp(k_2\xi z)\Big]
\end{aligned} \tag{3.12}$$

— for $0 \leqslant z \leqslant h$

$$\begin{aligned}
(\widetilde{\sigma}_{zz}^{2})_0 &= \frac{c_{44}}{(c_{13}+c_{44})}\xi\Big(\frac{c_{13}+c_{33}k_1^2}{k_1}[X_2(\xi)\cosh(k_1\xi z) + X_1(\xi)\sinh(k_1\xi z)] + \\
&+ \frac{c_{13}+c_{33}k_2^2}{k_2}[X_4(\xi)\cosh(k_2\xi z) + X_3(\xi)\sinh(k_2\xi z)]\Big) \\
(\widetilde{\sigma}_{rz}^{(1)})_1 &= -\frac{c_{44}}{c_{13}+c_{44}}\xi\Big((c_{13}+c_{33}k_1^2)[X_1(\xi)\cosh(k_1\xi z) + X_2(\xi)\sinh(k_1\xi z)] + \\
&+(c_{13}+c_{33}k_2^2)[X_3(\xi)\cosh(k_2\xi z) + X_4(\xi)\sinh(k_2\xi z)]\Big)
\end{aligned} \tag{3.13}$$

The next step in the solution is to express the above transforms in terms of the Hankel transforms of two functions given in the region of the crack, namely, the jumps of normal and tangential displacements: $\Delta u_z(r)$ and $\Delta u_r(r)$. To this end, we apply the Hankel transformation to boundary conditions (3.3) and (3.6), and with the aid of Eqs. (3.8), (3.9), (3.12), (3.13), we see that these conditions are satisfied if the six unknown functions S_1, S_2 and X_1, X_2, X_3, X_4 fulfil the following set of linear equations

$$\begin{aligned}
&\frac{c_{13}+c_{33}k_1^2}{k_1}[X_2(\xi)\cosh(k_1\xi h)+X_1(\xi)\sinh(k_1\xi h)]+\\
&\qquad+\frac{c_{13}+c_{33}k_2^2}{k_2}[X_4(\xi)\cosh(k_2\xi h)+X_3(\xi)\sinh(k_2\xi h)]=0\\
&(c_{13}+c_{33}k_1^2)[X_1(\xi)\cosh(k_1\xi h)+X_2(\xi)\sinh(k_1\xi h)]+\\
&\qquad+(c_{13}+c_{33}k_2^2)[X_3(\xi)\cosh(k_2\xi h)+X_4\sinh(k_2\xi h)]=0\\
&\frac{(c_{13}+c_{33}k_1^2)}{k_1}[X_2(\xi)-S_1(\xi)]+\frac{(c_{13}+c_{33}k_2^2)}{k_2}[X_4(\xi)-S_2(\xi)]=0\\
&(c_{13}+c_{33}k_1^2)[X_1(\xi)-S_1(\xi)]+(c_{13}+c_{33}k_2^2)[X_3(\xi)-S_2(\xi)]=0\\
&X_1(\xi)-S_1(\xi)+X_3(\xi)-S_2(\xi)=\big(\widetilde{\Delta u_z}(\xi)\big)_0\\
&\frac{c_{44}-c_{33}k_1^2}{(c_{13}+c_{44})k_1}[X_2(\xi)-S_1(\xi)]+\frac{c_{44}-c_{33}k_2^2}{(c_{13}+c_{44})k_2}[X_4(\xi)-S_2(\xi)]=\big(\widetilde{\Delta u_r}(\xi)\big)_1
\end{aligned}\tag{3.14}$$

Its solution is written in the form

$$\begin{aligned}
X_1(\xi)&=S_1(\xi)-\frac{c_{13}+c_{33}k_2^2}{c_{33}(k_1^2-k_2^2)}\big(\widetilde{\Delta u_z}(\xi)\big)_0\\
X_2(\xi)&=S_1(\xi)-\frac{(c_{13}+c_{33}k_2^2)k_1}{c_{33}(k_1^2-k_2^2)}\big(\widetilde{\Delta u_r}(\xi)\big)_1\\
X_3(\xi)&=S_2(\xi)+\frac{c_{13}+c_{33}k_1^2}{c_{33}(k_1^2-k_2^2)}\big(\widetilde{\Delta u_z}(\xi)\big)_0\\
X_4(\xi)&=S_2(\xi)+\frac{(c_{13}+c_{33}k_1^2)k_2}{c_{33}(k_1^2-k_2^2)}\big(\widetilde{\Delta u_r}(\xi)\big)_1
\end{aligned}\tag{3.15}$$

provided that

$$\begin{aligned}
S_1(\xi)&=\frac{c_{13}+c_{33}k_2^2}{c_{33}(k_1^2-k_2^2)}\Big[D_{1z}(\xi)\big(\widetilde{\Delta u_z}(\xi)\big)_0+D_{1r}(\xi)\big(\widetilde{\Delta u_r}(\xi)\big)_1\Big]\\
S_2(\xi)&=\frac{c_{13}+c_{33}k_1^2}{c_{33}(k_1^2-k_2^2)}\Big[D_{2z}(\xi)\big(\widetilde{\Delta u_z}(\xi)\big)_0+D_{2r}(\xi)\big(\widetilde{\Delta u_r}(\xi)\big)_1\Big]
\end{aligned}\tag{3.16}$$

where $(i=1,2)$

$$\begin{aligned}
D_{iz}(\xi)&=\frac{1}{2}\Big((-1)^{3-i}+\frac{k_1+k_2}{k_1-k_2}\exp(-2k_i\xi h)-\frac{2k_i}{k_1-k_2}\exp[-(k_1+k_2)\xi h]\Big)\\
D_{ir}(\xi)&=\frac{k_i}{2}\Big((-1)^{3-i}-\frac{k_1+k_2}{k_1-k_2}\exp(-2k_i\xi h)+\frac{2k_{3-i}}{k_1-k_2}\exp[-(k_1+k_2)\xi h]\Big)
\end{aligned}\tag{3.17}$$

Thus, we can establish the representations of the Hankel transforms of displacements (3.8), (3.9) and stresses (3.12), (3.13) through the transforms $\big(\widetilde{\Delta u_z}(\xi)\big)_0$ and $\big(\widetilde{\Delta u_r}(\xi)\big)_1$ with the help of Eqs. (3.15)-(3.17). It can be observed that these representations satisfy all the boundary conditions of the posed principal problem, except conditions (3.1) and $(3.2)_2$. Applying the Hankel inversion to Eqs. (3.8), (3.9) and (3.12), (3.13), we find that conditions (3.1) and $(3.2)_2$ yield the system of simultaneous dual integral equations for the functions $\big(\widetilde{\Delta u_z}(\xi)\big)_0$ and $\big(\widetilde{\Delta u_r}(\xi)\big)_1$:

— for $0 \leqslant r \leqslant a$

$$\begin{aligned}
&\int\limits_0^\infty \Big[\Big(\frac{1}{k_1k_2(k_1+k_2)} - U_{1z}(\xi h)\Big)\big(\widetilde{\Delta u_z}(\xi)\big)_0 - U_{1r}(\xi h)\big(\widetilde{\Delta u_r}(\xi)\big)_1\Big]\xi^2 J_0(\xi r)\, d\xi = \\
&\qquad = \frac{2c_{33}(c_{13}+c_{44})(P_{gas}+p)}{(c_{13}+c_{33}k_1^2)(c_{13}+c_{33}k_2^2)c_{44}} \\
&\int\limits_0^\infty \Big[\Big(\frac{1}{k_1+k_2} - U_{2r}(\xi h)\Big)\big(\widetilde{\Delta u_r}(\xi)\big)_1 - U_{2z}(\xi h)\big(\widetilde{\Delta u_z}(\xi)\big)_0\Big]\xi^2 J_1(\xi r)\, d\xi = 0
\end{aligned} \tag{3.18}$$

— for $r > a$

$$\begin{aligned}
&\int\limits_0^\infty \xi\big(\widetilde{\Delta u_z}(\xi)\big)_0 J_0(\xi r)\, d\xi = 0 \\
&\int\limits_0^\infty \xi\big(\widetilde{\Delta u_r}(\xi)\big)_1 J_1(\xi r)\, d\xi = 0
\end{aligned} \tag{3.19}$$

In the above, the quantities $U_{iz}(\xi h)$, $U_{ir}(\xi h)$, $i = 1,2$ stand for

$$\begin{aligned}
U_{1z}(\xi h) &= \frac{1}{(k_1-k_2)^2}\sum_{i,j=1}^{2}(-1)^{i+j}\frac{2\exp[-(k_i+k_j)\xi h]}{k_i+k_j} \\
U_{1r}(\xi h) &= U_{2r}(\xi h) = -\frac{1}{(k_1-k_2)^2}[\exp(-k_2\xi h) - \exp(-k_1\xi h)]^2 \\
U_{2z}(\xi h) &= \frac{1}{(k_1-k_2)^2}\sum_{i,j=1}^{2}(-1)^{i+j}\frac{2k_ik_j\exp[-(k_i+k_j)\xi h]}{k_i+k_j}
\end{aligned} \tag{3.20}$$

Following Sneddon (1996), if the unknown transforms are taken to be of the form

$$
\begin{aligned}
&\big(\widetilde{\varDelta u_z}(\xi)\big)_0 = \xi^{-1} \int\limits_0^a \varphi_z(t)\sin(\xi t)\;dt \\
&\big(\widetilde{\varDelta u_r}(\xi)\big)_1 = \xi^{-1} \int\limits_0^a \varphi_r(t)\Big(\frac{\sin(\xi t)}{\xi} - \cos(\xi t)\Big)\;dt
\end{aligned}
\tag{3.21}
$$

then Eqs. (3.19) are identically satisfied, and the inserting of Eqs. (3.21) into (3.19) yields two equations for the new auxiliary functions φ_z and φ_r defined in $[0, a]$

$$
\begin{aligned}
&\frac{1}{k_1k_2(k_1+k_2)r}\frac{\partial}{\partial r}\int\limits_0^r \frac{t\varphi_z(t)\,dt}{\sqrt{r^2-t^2}} - \int\limits_0^a \varphi_z(t)\;dt\int\limits_0^\infty \xi U_{1z}(\xi h)\sin(\xi t)J_0(\xi r)\;d\xi + \\
&\quad - \int\limits_0^a \varphi_r(t)\;dt\int\limits_0^\infty \xi U_{1r}(\xi h)\Big(\frac{\sin(\xi t)}{\xi} - \cos(\xi t)\Big)J_0(\xi r)\;d\xi = \\
&\quad = \frac{2c_{33}(c_{13}+c_{44})(P_{gas}+p)}{(c_{13}+c_{33}k_1^2)(c_{13}+c_{33}k_2^2)c_{44}} \\
&\frac{1}{k_1+k_2}\Big(\frac{\partial}{\partial r}+\frac{1}{r}\Big)\int\limits_0^r \frac{\varphi_r(t)\,dt}{\sqrt{r^2-t^2}} - \int\limits_0^a \varphi_z(t)\;dt\int\limits_0^\infty \xi U_{2z}(\xi h)\sin(\xi t)J_1(\xi r)\;d\xi + \\
&\quad - \int\limits_0^a \varphi_r(t)\;dt\int\limits_0^\infty \xi U_{2r}(\xi h)\Big(\frac{\sin(\xi t)}{\xi} - \cos(\xi t)\Big)J_1(\xi r)\;d\xi = 0
\end{aligned}
\tag{3.22}
$$

Finally, Eqs. (3.22) may be inverted to give

$$
\begin{aligned}
&\frac{\pi}{2}\frac{1}{k_1k_2(k_1+k_2)}\varphi_z(r) - \int\limits_0^a \varphi_z(t)K_{1z}(r,t)\;dt - \int\limits_0^a \varphi_r(t)K_{1r}(r,t)\;dt = \\
&\quad = \frac{2c_{33}(c_{13}+c_{44})(P_{gas}+p)}{(c_{13}+c_{33}k_1^2)(c_{13}+c_{33}k_2^2)c_{44}} \\
&\frac{\pi}{2}\frac{1}{k_1+k_2}\Big(\varphi_r(r) + \int\limits_0^r \frac{\varphi_r(t)\,dt}{t}\Big) - \int\limits_0^a \varphi_z(t)K_{2z}(r,t)\;dt + \\
&\quad - \int\limits_0^a \varphi_r(t)K_{2r}(r,t)\;dt = 0
\end{aligned}
\tag{3.23}
$$

Here, the following notations for the kernels have been employed

$$
\begin{aligned}
K_{1z}(r,t) &= \frac{1}{(k_1-k_2)^2}\sum_{i,j=1}^{2}(-1)^{i+j}\frac{2}{k_i+k_j}I_1(h,k_i+k_j,r,t)\\
K_{1r}(r,t) &= -\frac{1}{(k_1-k_2)^2}\sum_{i,j=1}^{2}(-1)^{i+j}[I_5(h,k_i+k_j,r,t)-I_2(h,k_i+k_j,r,t)]\\
K_{2z}(r,t) &= -\frac{1}{(k_1-k_2)^2}\sum_{i,j=1}^{2}(-1)^{i+j}I_3(h,k_i+k_j,r,t)\\
K_{2r}(r,t) &= \frac{1}{(k_1-k_2)^2}\sum_{i,j=1}^{2}(-1)^{i+j}\frac{2k_ik_j}{k_i+k_j}\cdot\\
&\quad\cdot[I_6(h,k_i+k_j,r,t)-I_4((h,k_i+k_j,r,t)]
\end{aligned}
\tag{3.24}
$$

in which

$$
\begin{aligned}
I_1(h,k,r,t) &= \int\limits_0^\infty \exp(-h\xi k)\sin(\xi t)\sin(\xi r)\,d\xi =\\
&= \frac{2hkrt}{[h^2k^2+(r-t)^2][h^2k^2+(r+t)^2]}\\
I_2(h,k,r,t) &= \int\limits_0^\infty \exp(-h\xi k)\cos(\xi t)\sin(\xi r)\,d\xi =\\
&= \frac{r(h^2k^2+r^2-t^2)}{[h^2k^2+(r-t)^2][h^2k^2+(r+t)^2]}\\
I_3(h,k,r,t) &= \int\limits_0^\infty \exp(-h\xi k)\sin(\xi t)[1-\cos(\xi r)]\,d\xi =\\
&= \frac{r^2t(3h^2k^2+r^2-t^2)}{[h^2k^2+(r-t)^2](h^2k^2+t^2)[h^2k^2+(r+t)^2]}\\
I_4(h,k,r,t) &= \int\limits_0^\infty \exp(-h\xi k)\cos(\xi t)[1-\cos(\xi r)]\,d\xi =\\
&= \frac{r^2hk(h^2k^2+r^2-3t^2)}{[h^2k^2+(r-t)^2](h^2k^2+t^2)[h^2k^2+(r+t)^2]}\\
I_5(h,k,r,t) &= \int\limits_0^\infty \exp(-h\xi k)\frac{\sin(\xi t)}{\xi t}\sin(\xi r)\,d\xi = \frac{1}{4t}\ln\frac{h^2k^2+(r+t)^2}{h^2k^2+(r-t)^2}
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
I_6(h,k,r,t) &= \int\limits_0^\infty \exp(-h\xi k)\frac{\sin(\xi t)}{\xi t}[1-\cos(\xi r)]\,d\xi = \\
&= \frac{1}{2t}\Big(\arctan\frac{r-t}{hk} + 2\arctan\frac{t}{hk} - \arctan\frac{r+t}{hk}\Big)
\end{aligned}
$$

Note that Eqs. (3.23) contain the unknown pressure of the gas P_{gas}. To determine this quantity, we use governing state equation (2.1) and, in view of Eqs. (3.5) and $(3.21)_1$, we can obtain the following formula

$$
P_{gas} = \frac{g_0}{2\pi\int\limits_0^a r\varphi_z(r)\,dr} \tag{3.26}
$$

Thus, the original problem is reduced to the solution of the system of simultaneous integral equations (3.23) supplemented with Eq. (3.26). Once the functions $\varphi_z(r)$ and $\varphi_r(r)$ are known, the displacements and stresses in the half-space can be find by applying Hankel's inversion theorem to Eqs. (3.8), (3.9) and (3.12), (3.13) with the known functions given by Eqs. (3.15), (3.16) and (3.21).

From the viewpoint of linear fracture mechanics (see Kassir and Sih, 1975), it is of great importance to investigate highly intensified normal and tangential stresses around the crack edge resulting in fracture initiation under the environment of a given external load and the presence of the gas in the crack. The most widely used fracture criterions in Mode I and II of crack propagation are based on the knowledge of the stress intensity factors (SIFs) K_{I}, K_{II} describing the asymptotic behaviour of stresses in the immediate vicinity of the crack border $r > a$, i.e.

$$
\sigma_{zz}(r,\theta,0) = \frac{K_{\mathrm{I}}}{\sqrt{2\pi(r-a)}} + O(1) \qquad \sigma_{rz}(r,\theta,0) = \frac{K_{\mathrm{II}}}{\sqrt{2\pi(r-a)}} + O(1) \tag{3.27}
$$

It turns out that in our case these factors are given in terms of the solution to governing integral equations (3.23) as

$$
\begin{aligned}
K_{\mathrm{I}} &= \sqrt{\frac{\pi}{a}}\frac{c_{44}(c_{13}+c_{33}k_1^2)(c_{13}+c_{33}k_2^2)}{2(c_{13}+c_{44})c_{33}k_1k_2(k_1+k_2)}\varphi_z(a) \\
K_{\mathrm{II}} &= \sqrt{\frac{\pi}{a}}\frac{c_{44}(c_{13}+c_{33}k_1^2)(c_{13}+c_{33}k_2^2)}{2(c_{13}+c_{44})c_{33}(k_1+k_2)}\varphi_r(a)
\end{aligned} \tag{3.28}
$$

4. Numerical analysis

4.1. Numerical procedure

The complicated system of coupled integral equations (3.23) can be solved only by recourse to numerical techniques. A certain numerical procedure, briefly outlined below, was used.

We shall proceed on the well-known fact that any continuous function in a bounded domain can be uniformly approximated up to any accuracy by a polynomial. By some arguments resulting from the structure of kernels and geometry of the problem in hand, we represent the approximated solutions to governing equations (3.23) as

$$\begin{aligned} \varphi_z(r) &\approx \varphi_{zN}(r) = c_{z1}r + c_{z2}r^3 + \ldots + c_{zN}r^{2N-1} \\ \varphi_r(r) &\approx \varphi_{rM}(r) = c_{r1}r^2 + c_{r2}r^4 + \ldots + c_{rM}r^{2M} \end{aligned} \tag{4.1}$$

where c_{zn} $(n = 1, 2, \ldots, N)$ and c_{rm} $(m = 1, 2, \ldots, M)$ stand for the unknown coefficients to be determined. Substituting the above-assumed expressions into Eqs. (3.23) and (3.26), and then satisfying them in the set of chosen collocation points (Eq. $(4.1)_1$ at the points $r_n = na/N$ and Eq. $(4.1)_2$ at the points $r_m = ma/M$, we arrive at a set of non-linear algebraic equations (the discrete analogue of Equations (3.23), (3.26)) for the unknown coefficients c_{zn}, c_{rm} and parameter P_{gas}. Its solution is found by Newton's method. The desired accuracy is achieved by increasing the power of approximating polynomials in Eqs. (4.1).

4.2. Numerical results

The numerical analysis was carried out for the following dimensional parameters

$$\begin{array}{lll} \overline{P}_{gas} = 10^3 \dfrac{P_{gas}}{c_{44}} & \overline{p} = 10^3 \dfrac{p}{c_{44}} & \\ \overline{c}_{11} = \dfrac{c_{11}}{c_{44}} = 3.88 & \overline{c}_{33} = \dfrac{c_{33}}{c_{44}} = 3.15 & \overline{c}_{13} = \dfrac{c_{13}}{c_{44}} = 1.31 \\ \overline{h} = \dfrac{h}{a} & \overline{g}_0 = \dfrac{g_0}{c_{44}a^3} = 10^{-5} & \\ \overline{K}_{\rm I} = 10^3 \dfrac{K_{\rm I}\sqrt{a}}{c_{44}} & \overline{K}_{\rm II} = 10^4 \dfrac{K_{\rm II}\sqrt{a}}{c_{44}} & \end{array} \tag{4.2}$$

In Fig. 2 a graph of the internal gas pressure is plotted against the external load. As the load increases, the pressure of the gas decreases. The high slope of the curve is observed for the compressive load. While the external load becomes tensile, the slope decreases and $\overline{P}_{gas}$ tends to zero.

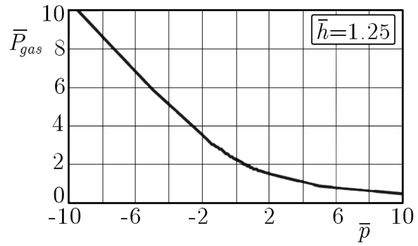

Fig. 2. Dependence of the gas pressure on the external load

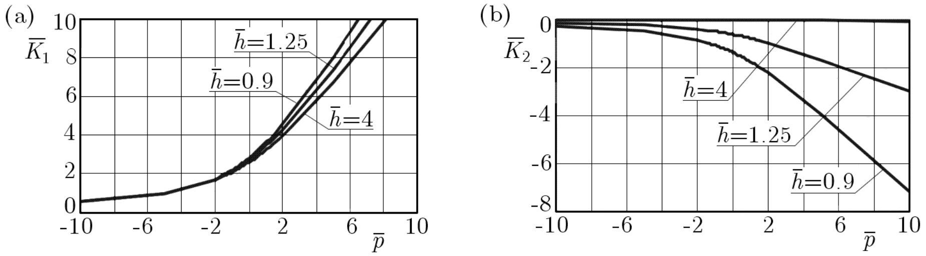

Fig. 3. Variations of SIF of Mode I (a) and Mode II (b) with the external load

A graphical representation of the SIF of Mode I and II is given in Fig. 3a and Fig. 3b, respectively. As in the case presented in Fig. 2, the dependences of the SIFs on the external load are nonlinear. It should be remarked here that we obtain physically reasonable values of the SIFs within the range of compressive external load. This directly indicates the effect of the crack filler. Moreover, it can be seen that $\overline{K}_{\mathrm{I}}$ and $|\overline{K}_{\mathrm{II}}|$ increase in magnitude as the boundary is approached, i.e. for decreasing values of $\overline{h} = h/a$. This tendency has also been observed in Fig. 4 for $p = 0$. Unlike that, Fig. 5 shows that the pressure of the gas in this case slowly decreases as the crack surface approaches the half-space boundary.

5. Conclusions

The presented research was carried out with the aim of demonstrating the role of the gas filling a penny-shaped crack situated parallel to the boundary of a semi-infinite transversely isotropic space on the limiting equilibrium stress

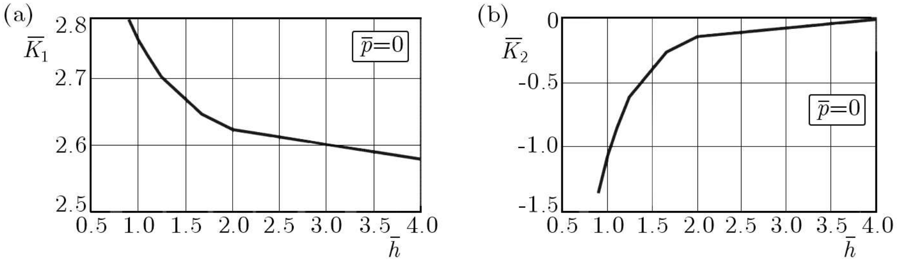

Fig. 4. SIF of Mode I (a) and Mode II (b) versus $\overline{h} = h/a$

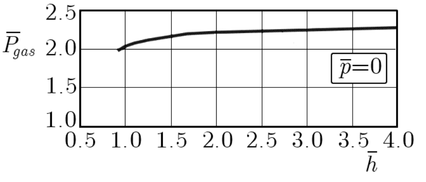

Fig. 5. Boundary effect on the gas pressure

state. As can be seen from the numerical solution to complex integral equations (3.23), a change in the mechanical behaviour has been noted in comparison to the case of the non-filled crack. The main result of the paper is the non-linearity of the relations between the external load and the internal pressure of the gas as well as the stress intensity factors. This phenomenon is due to the non-linear response of the filler on the change of its volume, governed by the state equation.

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O zagadnieniu poprzecznie izotropowej półprzestrzeni osłabionej szczeliną kołową wypełnioną gazem

Streszczenie

W pracy rozważono osiowo-symetryczne zagadnienie kołowej szczeliny wypełnionej idealnym gazem i położonej równolegle do brzegu półprzestrzeni sprężystej poprzecznie izotropowej. Używając techniki transformacji całkowej Hankela, rozpatrywany problem został sprowadzony do złożonego układu równań całkowych. Na podstawie procedury numerycznej zbadano i zilustrowano wpływ gazu na współczynniki intensywności naprężeń.

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