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COMPARISON OF SOME SELECTED MULTIAXIAL FATIGUE FAILURE CRITERIA DEDICATED FOR SPECTRAL METHOD

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The paper presents the procedure of estimation of fatigue life in the high-cycle fatigue regime under a multiaxial random loading using the spectral method. The procedure consists of application of the power spectral density function of the equivalent stress in the fatigue life assessment together with the known spectral method developed for the uniaxial stress state. The model proposed by Miles and Dirlik was presented as an example. Two groups of multiaxial fatigue failure criteria have been distinguished: the criteria based on the critical plane approach and the criteria using invariants of the stress state. Two sets of experimental data were used in order to compare the calculated and experimental fatigue lives. It has been shown that under the multiaxial random loading the results of fatigue life calculated according to the considered method are well correlated with the results of experiments if the multiaxial fatigue failure criterion is properly selected for mechanical parameters of the tested material.

Key words: spectral method, multiaxial fatigue, high cycle fatigue, random loading

Notations

a_k	–	coefficients for multiaxial fatigue criteria, $k = 1, \ldots, 6$
f	–	frequency
$G(f)$	–	power spectral density function (PSDF)
$\mathbf{G}_\sigma(f)$	–	matrix of PSDFs
$\mathbf{Q}_M$	–	matrix of coefficients for criterion proposed by Preumont and Piefort
$\mathbf{Q}_{os}, \mathbf{Q}_{on}$	–	coefficients matrices
$\widehat{l}_k, \widehat{m}_k, \widehat{n}_k$	–	for $k = 1, 2, 3$, direction cosines of averaged directions of principal stresses $\sigma_1(t) \geqslant \sigma_2(t) \geqslant \sigma_3(t)$

m_σ, m_τ	–	coefficients of inclination of Wöhler's curves for tension-compression and torsion, respectively
M^+	–	expected rate of peaks
N_f	–	number of cycles to failure
N_σ, N_τ	–	number of cycles read out from Wöhler's curves for tension-compression and torsion, respectively
$N(\sigma_a)$	–	function which returns number of cycle for given stress amplitude σ_a from standard fatigue characteristic
N_0^+	–	expected rate of zero crossing with positive slope
$p(\sigma_a)$	–	probability density function (PDF) of stress amplitudes
I	–	irregularity factor
$r_{\sigma os,\tau os}$	–	correlation coefficient of histories $\tau_{os}(t)$ and $\sigma_{on}(t)$
σ_{eq}	–	equivalent stress
σ_{af}	–	fatigue strength
$\boldsymbol{\sigma}(t)$	–	vector of components of stress tensor
ξ_k	–	k-th moment of PSDF
μ	–	variance
$\Gamma(\cdot)$	–	gamma function
$\mathrm{conj}(\cdot)$	–	complex conjugate function

1. Introduction

The known methods of evaluation of fatigue life of machine elements and structures under random loadings can be divided into two groups. The first one is the cycle counting approach, based on various cycle counting algorithms, which allows obtaining the amplitude histogram of cycles from random loading (Banvillet *et al.*, 2004; Colombi and Doliński, 2001; Dowling, 1972; Łagoda *et al.*, 2001, 2005; Macha *et al.*, 2006). The second group uses specific procedures known form spectral analysis of stochastic processes (Banvillet *et al.*, 2004; Benasciutti and Tovo, 2006; Łagoda *et al.*, 2005; Macha *et al.*, 2006; Niesłony and Macha, 2007; Sherratt *et al.*, 2005), where the probability density function (PDF) of stress amplitudes are approximated with several parameters computed from the power spectral density function (PSDF). In these two groups, final fatigue damage is usually obtained by the damage accumulation hypothesis, where a standard fatigue characteristic of the material, obtained under cyclic loading, is applied. Using the linear Palmgren-Miner hypothesis of damage accumulation, the expected number of cycles to crack initiation N_f

can be derived as follows (Morrow, 1986; Niesłony and Macha, 2007; Sherratt *et al.*, 2005; Szala, 1998)

$$N_f = \Big[\int\limits_0^\infty \frac{p(\sigma_a)}{N(\sigma_a)}\,d\sigma_a \Big]^{-1} \tag{1.1}$$

where $p(\sigma_a)$ is the PDF of stress amplitudes and $N(\sigma_a)$ represents a function which returns the number of cycles for the given stress amplitude σ_a from a standard fatigue characteristic, i.e. Wöhler curve.

In the spectral method, the power spectral density function is usually used (Dirlik, 1985; Sobczyk and Spencer, 1992) to describe the loading. This function represents the power distribution versus frequency, i.e. distribution of the mean square values of particular harmonic components appearing in the random process and in the frequency bandwidth. Observing the shape of the PSDF, it is easily to judge how the signal is distributed in the frequency. However, in practice, parameters which would describe the characteristic of the signal in a quantitative way are required. Usually, the first five moments of the PSDF are used (Sobczyk and Spencer, 1992)

$$\xi_k = \int\limits_0^\infty G(f) f^k\,df \qquad \text{for} \quad k = 0,1,\dots,4 \tag{1.2}$$

Using these moments, it is possible to receive very important, from the point of view of the material fatigue, statistical parameters concerning the time course of the process, like the variance

$$\mu = \xi_0 \tag{1.3}$$

the expected rate of zero crossing with the positive slope

$$N_0^+ = \sqrt{\frac{\xi_2}{\xi_0}} \tag{1.4}$$

the expected rate of peaks

$$M^+ = \sqrt{\frac{\xi_4}{\xi_2}} \tag{1.5}$$

and the irregularity factor

$$I = \frac{N_0^+}{M^+} \tag{1.6}$$

In the case of simple loading states, like uniaxial tension-compression, several proposals for determination of the fatigue life using the spectral method are known (Niesłony and Macha, 2007). In these models, computation is performed in the frequency domain and distribution of stress amplitudes is determined directly from the PSDF. The cycle counting methods, such as the rain flow algorithm, are not necessary.

An example of such solution was presented by Miles in 1954 (Niesłony and Macha, 2007). It was elaborated for the Gaussian narrow-band frequency loading. Using the Palmgren-Miner linear hypothesis of damage accumulation and Rayleigh probability distribution for approximation of stress amplitudes, the following equation for the fatigue life can be derived

$$N_f = \frac{A_\sigma}{(2\mu_\sigma)^{\frac{m_\sigma}{2}} \Gamma\Big(\frac{m_\sigma+2}{2}\Big)} \tag{1.7}$$

where $A_\sigma = \lg(N_f) + m_\sigma \lg(\sigma_a)$ is the coefficient of Wöhler's curves for tension-compression. The theory presented by Miles was developed by other authors (Niesłony, 2008). They proposed much more universal solutions giving stress amplitude distributions similar to those obtained according to the rain flow algorithm. The proposal formulated by Dirlik in 1985 for the wide-band frequency Gaussian loading belongs to the well-known ones (Dirlik, 1985)

$$p(\sigma_a) = \frac{1}{\sqrt{\xi_0}}\Big[\frac{G_1}{Q}\exp\Big(\frac{-Z}{Q}\Big) + \frac{G_2 Z}{R^2}\exp\Big(\frac{-Z^2}{2R^2}\Big) + G_3 Z \exp\Big(\frac{-Z^2}{2}\Big)\Big] \tag{1.8}$$

where

$$Z = \frac{\sigma_a}{\sqrt{\mu_0}} \qquad G_1 = \frac{2(x_m - I^2)}{1 + I^2}$$

$$G_2 = \frac{1 - I - G_1 + G_1^2}{1 - R} \qquad x_m = \frac{\xi_1}{\xi_0}\sqrt{\frac{\xi_2}{\xi_4}}$$

$$G_3 = 1 - G_1 - G_2 \qquad R = \frac{I - x_m - G_1^2}{1 - I - G_1 + G_1^2}$$

$$Q = \frac{5(I - G_3 + G_2 R)}{4G_1} \qquad I = \frac{\xi_2}{\sqrt{\xi_0 \xi_4}}$$

This formula provides the best results of approximation of stress amplitude distribution determined by the rain flow algorithm (Niesłony and Macha, 2007; Sherratt *et al.*, 2005) as compared with other approximations, and can be used for stress histories with narrow- and broad-band frequency spectra. However, in this case, a complicated form of the PDF, Eq. (1.8) does not allow one to simplify the equation for fatigue damage, and the integral occurring in Eq. (1.1) must be solved numerically.

Equation (1.1) is designed for a uniaxial loading and cannot be applied directly under a multiaxial loading. In this case, the spatial stress state should be substituted with the equivalent uniaxial one. To do this, a suitable fatigue failure criterion shall be used. The criterion allows reducing a triaxial random stress state to the equivalent uniaxial one in the sense of material fatigue (Karolczuk and Macha, 2005; Jiang *et al.*, 2007; Niesłony and Sonsino, 2008). Then, the history of the equivalent stress or its suitable probabilistic characteristics may be applied for calculations of the fatigue life, as under the uniaxial random loading.

The stress state in the time domain can be defined using the vector of components of the stress tensor

$$\boldsymbol{\sigma}(t) = [\sigma_{xx}(t), \sigma_{yy}(t), \sigma_{zz}(t), \sigma_{xy}(t), \sigma_{xz}(t), \sigma_{yz}(t)] \tag{1.9}$$

The probabilistic relation between particular components plays an important role in material fatigue. Therefore, in the spectral method, these properties are expressed with a matrix of PSDF $\mathbf{G}_\sigma(f)$ which for the vector, Eq. (1.9), gives a rectangular 6×6 dimensional matrix

$$\mathbf{G}_\sigma(f) = \begin{bmatrix} G_{xx,xx}(f) & \cdots & G_{xx,yz}(f) \\ \vdots & \ddots & \vdots \\ G_{yz,xx}(f) & \cdots & G_{yz,yz}(f) \end{bmatrix} \tag{1.10}$$

On the diagonal of the matrix the autospectral density functions and on the upper and lower triangular part of the matrix the cross-spectral density functions are placed. Since the relationship $G_{i,j}(f) = \mathrm{conj}[G_{j,i}(f)]$ for $i \neq j$ appears, for description of the frequency structure of the random stress state twenty one PSDFs should be known. Using the advantage of multiaxial fatigue failure criteria defined in the frequency domain, the PSDF of the equivalent stress $G_{eq}(f)$ can be computed. Next, the well known uniaxial solutions can be used, for example the models proposed by Miles, Eq. (1.7) or Dirlik, Eq. (1.8), where the PSDF of the equivalent stress state should be applied instead of the uniaxial PSDF (Niesłony and Macha, 2007). The general concept of using the equivalent uniaxial stress state during fatigue life assessment with the spectral method under a multiaxial random loading is shown in Fig. 1.

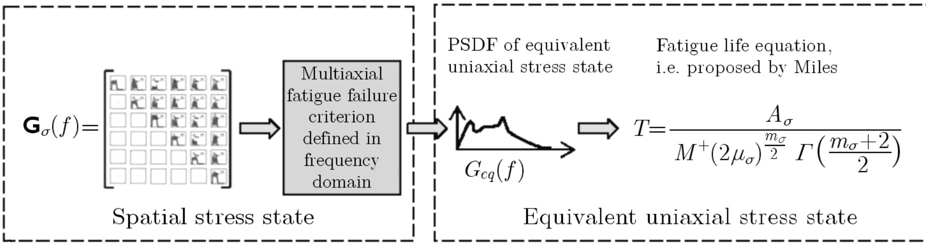

Fig. 1. Flow chart for calculation of fatigue life using spectral method under multiaxial random loading

2. Multiaxial fatigue failure criteria related to the spectral method

2.1. Criteria based on the critical plane approach

In 1988 Macha (Macha, 1988, 1996) introduced the multiaxial fatigue failure criteria based on the critical plane approach defined in the frequency domain to the spectral method. He used his criteria defined in the time domain, being a linear combination of stress or strain tensor components on the critical plane (Karolczuk and Macha, 2005). For these criteria, the general function of material fatigue strength under a multiaxial random loading was defined as follow

$$S(t) = \{\sigma_{ij}(t), P, C\} \tag{2.1}$$

where $\sigma_{ij}(t)$ are components of the stress or strain tensor, P – parameters for determination of critical plane position, and C – parameters characterizing the material. The surface of limit states determining the fatigue life under a random complex state of stress is described by the limit value of the fatigue strength function

$$\max_t\{S(t)\} = \sigma_{af} \tag{2.2}$$

corresponding to the fatigue strength of the material under the alternating cycle of uniaxial tension-compression σ_{af}. Expression (2.2) should also be read as a 100% quantile of the random variable S. Depending on the chosen method for determination of the critical plane position, Macha obtained special forms of the equivalent stress, for example according to the criterion of maximum normal stress on the critical plane

$$\begin{aligned}\sigma_{eq}(t) = \sigma_\eta(t) &= \widehat{l}_1^2\sigma_{xx}(t) + \widehat{m}_1^2\sigma_{yy}(t) + \widehat{n}_1^2\sigma_{zz}(t) + 2\widehat{l}_1\widehat{m}_1\sigma_{xy}(t) \\ &\quad + 2\widehat{l}_1\widehat{n}_1\sigma_{xz}(t) + 2\widehat{m}_1\widehat{n}_1\sigma_{yz}(t)\end{aligned} \tag{2.3}$$

or according to the criterion of maximum shear and normal stresses on the critical plane

$$
\begin{aligned}
\sigma_{eq}(t) = \frac{1}{1+K}\Big\{&[\widehat{l}_1^2 - \widehat{l}_3^2 + K(\widehat{l}_1^2 + \widehat{l}_3^2)^2]\sigma_{xx}(t) + \\
&+[\widehat{m}_1^2 - \widehat{m}_3^2 + K(\widehat{m}_1^2 + \widehat{m}_3^2)^2]\sigma_{yy}(t) + \\
&+[\widehat{n}_1^2 - \widehat{n}_3^2 + K(\widehat{n}_1^2 + \widehat{n}_3^2)^2]\sigma_{zz}(t) + \\
&+2[\widehat{l}_1\widehat{m}_1 - \widehat{l}_3\widehat{m}_3 + K(\widehat{l}_1 + \widehat{l}_3)(\widehat{m}_1 + \widehat{m}_3)]\sigma_{xy}(t) + \\
&+2[\widehat{l}_1\widehat{n}_1 - \widehat{l}_3\widehat{n}_3 + K(\widehat{l}_1 + \widehat{l}_3)(\widehat{n}_1 + \widehat{n}_3)]\sigma_{xz}(t) + \\
&+2[\widehat{m}_1\widehat{n}_1 - \widehat{m}_3\widehat{n}_3 + K(\widehat{m}_1 + \widehat{m}_3)(\widehat{n}_1 + \widehat{n}_3)]\sigma_{yz}(t)\Big\}
\end{aligned}
\tag{2.4}
$$

where $\widehat{l}_k$, $\widehat{m}_k$ and $\widehat{n}_k$ for $k = 1,2,3$ are suitable directional cosines of averaged directions of principal stresses $\sigma_1(t) \geqslant \sigma_2(t) \geqslant \sigma_3(t)$, and K represents the material constant determined in cyclic fatigue tests. As the criteria are based on the notion of the critical plane, three methods of determination of its location are distinguished: weight functions method, variance method and damage accumulation method. From the spectral point of view, a possibility is open for including into the calculation algorithm the two latter ones. Macha reported that the equivalent stress $\sigma_{eq}(t)$ could be understood as the output signal from a physical system with six inputs for which signals representing suitable stress tensor components were delivered, Fig. 2.

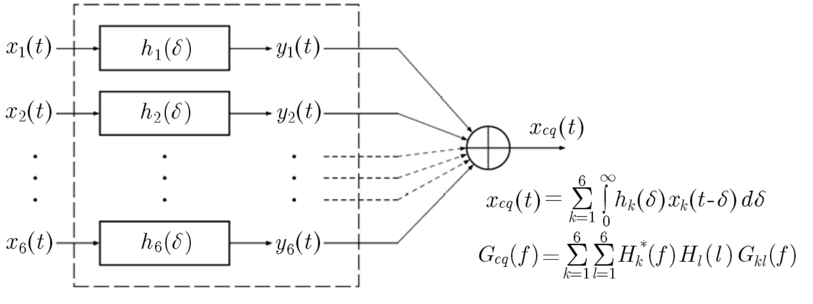

Fig. 2. Interpretation of the damage parameter $x_{eq}(t)$ as an output signal from the physical system with pulse transition functions $h_k(\delta)$, $k = 1,\dots,6$ where signals $x_k(t)$, $k = 1,\dots,6$ were introduced to its input

Thus, the use of transformation for linear systems, known from the spectral analysis of stochastic processes (Bendat and Piersol, 1980), leads to the final equation for the equivalent stress history

$$
\sigma_{eq}(t) = \sum_{k=1}^{6} a_k\sigma_k(t) \tag{2.5}
$$

and the PSDF of the equivalent stress

$$G_{eq}(f) = \sum_{i=1}^{6}\sum_{j=1}^{6} a_i a_j G_{ij}(f) = \sum_{i=1}^{6}\Big[a_i^2 G_{ii}(f) + 2\sum_{j=i+1}^{6} a_i a_j G_{ij}(f)\Big] \tag{2.6}$$

where $\sigma_k(t)$, $k = 1,\dots,6$, are stress tensor components, $G_{ij}(f)$ – components of the matrix of PSDF of the stress tensors, and a_k – coefficients depending on the multiaxial fatigue failure criterion, shown in Table 1 for the criteria (2.3) and (2.4).

The mentioned solution allows one to apply also other linear criteria of multiaxial fatigue, for example the criteria proposed by Łagoda and Ogonowski (2005). The resulting PSDF of the equivalent stress determined according to Eq. (2.6) is equivalent to the PSDF determined by the Fourier transform of the equivalent stress $\sigma_{eq}(t)$ obtained in the time domain according to Eq. (2.5).

Table 1. Coefficients a_k in $\sigma_{eq}(t)$ expression according to the criterion of maximum normal stress, Eq. (2.3) and criterion of maximum shear and normal stresses, Eq. (2.4), on the critical plane

a	Eq. (2.3)	Eq. (2.4)
a_1	$\widehat{l}_1^2$	$\dfrac{\widehat{l}_1^2 - \widehat{l}_3^2 + K(\widehat{l}_1^2 + \widehat{l}_3^2)^2}{1+K}$
a_2	$\widehat{m}_1^2$	$\dfrac{\widehat{m}_1^2 - \widehat{m}_3^2 + K(\widehat{m}_1^2 + \widehat{m}_3^2)^2}{1+K}$
a_3	$\widehat{n}_1^2$	$\dfrac{\widehat{n}_1^2 - \widehat{n}_3^2 + K(\widehat{n}_1^2 + \widehat{n}_3^2)^2}{1+K}$
a_4	$2\widehat{l}_1\widehat{m}_1$	$\dfrac{2[\widehat{l}_1\widehat{m}_1 - \widehat{l}_3\widehat{m}_3 + K(\widehat{l}_1 + \widehat{l}_3)(\widehat{m}_1 + \widehat{m}_3)]}{1+K}$
a_5	$2\widehat{l}_1\widehat{n}_1$	$\dfrac{2[\widehat{l}_1\widehat{n}_1 - \widehat{l}_3\widehat{n}_3 + K(\widehat{l}_1 + \widehat{l}_3)(\widehat{n}_1 + \widehat{n}_3)]}{1+K}$
a_6	$2\widehat{m}_1\widehat{n}_1$	$\dfrac{2[\widehat{m}_1\widehat{n}_1 - \widehat{m}_3\widehat{n}_3 + K(\widehat{m}_1 + \widehat{m}_3)(\widehat{n}_1 + \widehat{n}_3)]}{1+K}$

2.2. Criteria using invariants of the stress tensor

Preumont and Piefort (1994) presented a method for determination of the PSDF of the equivalent stress using the Huber-Mises-Hencky hypothesis. To present the method, let us consider the case of the plane stress state. Under this condition, the equivalent stress takes the form

$$\sigma_{eq}^2 = \sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2 \tag{2.7}$$

After defining the vector of stress tensor components $\boldsymbol{\sigma} = [\sigma_{xx}, \sigma_{yy}, \sigma_{xy}]^\top$, Eq. (2.7) can be written according to the rules of matrix calculus

$$\sigma_{eq}^2 = \boldsymbol{\sigma}^\top \mathbf{Q}_M \boldsymbol{\sigma} = \operatorname{tr}\{\mathbf{Q}_M[\boldsymbol{\sigma}\boldsymbol{\sigma}^\top]\} \tag{2.8}$$

where

$$\mathbf{Q}_M = \begin{bmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{2.9}$$

is the matrix of coefficients for the Huber-Mises-Hencky equivalent stress under the plane stress state, $\boldsymbol{\sigma}^\top$ is the vector transposed to $\boldsymbol{\sigma}$, $\operatorname{tr}\{\ldots\}$ is the sum of components of the main diagonal of the square matrix. On the basis of Eq. (2.8), the relation for expected values $E[\cdot]$ could be stated

$$E[\sigma_{eq}^2] = \boldsymbol{\sigma}^\top \mathbf{Q}_M \boldsymbol{\sigma} = \operatorname{tr}\{\mathbf{Q}_M E[\boldsymbol{\sigma}\boldsymbol{\sigma}^\top]\} \tag{2.10}$$

Thus, the obtained formula presents the mean-square value of the reduced stress. The mean-square value can be also determined directly from the matrix of PSDF of the stress tensor

$$E[\sigma_{eq}^2] = \int_0^\infty G_{eq}(f)\, df = \int_0^\infty \operatorname{tr}\{\mathbf{Q}_M \mathbf{G}_\sigma(f)\}\, df \tag{2.11}$$

where $\mathbf{G}_\sigma(f)$ is the matrix of PSDF (autospectral and cross-spectral) of the stress vector $\boldsymbol{\sigma}$. On the basis of the preceding formulae, Preumont and Piefort postulated a method for determination of PSDF of the equivalent stress directly from the matrix of PSDF of the stress vector

$$G_{eq}(f) = \operatorname{tr}\{\mathbf{Q}_M \mathbf{G}_\sigma(f)\} \tag{2.12}$$

This criterion can be related only to materials for which the coefficients of inclination of Wöhler's curves for tension-compression and torsion are equal

$$m_\sigma = m_\tau \tag{2.13}$$

and the fatigue characteristics of the material satisfy the following equality

$$\sigma_{aN} = \sqrt{3}\,\tau_{aN} \tag{2.14}$$

where σ_{aN} and τ_{aN} are amplitudes read out from Wöhler's curves for a constant number of cycles N_f for tension-compression and torsion, respectively.

Pitoiset (2001) carried out many simulations and compared the fatigue lives determined in time and frequency domains. High conformity of the results was reported. This paper also presents an example of calculations of the fatigue life of a nozzle of the Vulcain motor working in the Ariane V rocket. Let us note that while determining the reduced PSDF using Eq. (2.12), the interactions between the stress components σ_{xx} and σ_{xy} as well as σ_{yy} and σ_{xy} have not been taken into account because of the zeros in the coefficient matrix $\mathbf{Q}_M$, Eq. (2.9). It is an unfavourable property of this method. The above approach was applied by Pitoiset *et al.* (2001) for formulation of the criteria proposed by Matake and Crossland in the frequency domain with the same weakness of the method.

Bonte *et al.* (2007) criticized the solution proposed by Preumont and Piefort due to the fact that not all phase shifts between the stress tensor components had been taken into account while determining the PSDF of the equivalent stress. They proposed their own solution and obtained the equivalent stress including phase displacements. Let us note, however, that their solution leads to a lower value of the determined equivalent stress which does not correlate with typical results obtained from fatigue tests. In such cases, the phase displacement of stress tensor components with constant amplitudes and frequencies usually influences reduction of the fatigue life, so the determined equivalent stress should be higher. The limitation expressed by Eqs. (2.13) and (2.14) is presented in both Preumont and Piefort's proposals and the solution developed by Bonte *et al.* Thus, possibilities of their application are rather small.

The restriction expressed by Eq. (2.14) seems to be the most important because a wide range of materials does not satisfy it. However, it is possible to formulate a criterion in the frequency domain without the mentioned restriction, which allows one to obtain correct values under simple loading states, i.e. tension-compression and torsion (Niesłony, 2008). It is defined directly in the frequency domain, and its equivalent form in the time domain for random processes does not exist. It is assumed that PSDF of shear and normal stresses acting on the octahedral plane, Fig. 3, strongly influences the material fatigue.

The criterion can be written in the following general form

$$G_{eq}(f) = g[G_{\tau\, os}(f), G_{\sigma\, on}(f), \boldsymbol{P}] \tag{2.15}$$

where $\boldsymbol{P}$ is the vector of material constants, $G_{\tau\, os}(f)$ and $G_{\sigma\, os}(f)$ are the PSDFs of shear and normal stresses on the octahedral plane, respectively.

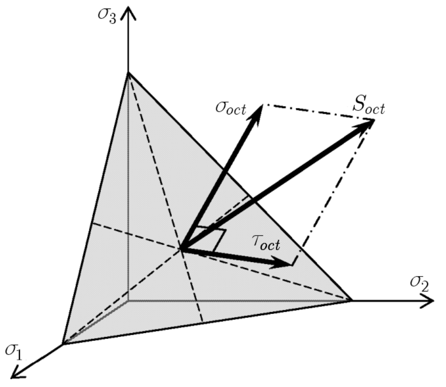

Fig. 3. Normal and shear stresses acting on the octahedral plane

In order to compute the particular PSDFs, the same procedure proposed by Pitoised and Premount was used

$$G_{\tau\,os}(f) = \operatorname{tr}\{\mathbf{Q}_{os}\mathbf{G}_{\sigma}(f)\} \qquad G_{\sigma\,on}(f) = \operatorname{tr}\{\mathbf{Q}_{on}\mathbf{G}_{\sigma}(f)\} \tag{2.16}$$

with the suitably defined coefficient matrix

$$\mathbf{Q}_{os} = \frac{1}{9}\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \qquad \mathbf{Q}_{on} = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.17}$$

PSDF of the equivalent stress is defined as the sum of these functions with suitable weights A and B. According to the summation theory for two one-dimensional stochastic processes, it can be written

$$G_{eq}(f) = A^2 G_{\tau\,os}(f) + B^2 G_{\sigma\,on}(f) + 2A^2B^2 \operatorname{Re}\{G_{\tau\,os,\sigma\,on}(f)\} \tag{2.18}$$

where $\operatorname{Re}\{G_{\tau\,os,\sigma\,on}(f)\}$ is the real part of cross PSDF of $\tau_{os}(t)$ and $\sigma_{on}(t)$. Depending on the material, the ratio of stresses amplitude from the Wöhler curves for tension-compression and torsion varies, and in a general case depends on the number of cycles. The advanced criteria include that fact in order to determine the correct equivalent values in simple loading cases, i.e. tension-compression and torsion. Such solutions can be found in the criteria proposed by Sonsino, Dang Van (Niesłony and Sonsino, 2008; Karolczuk and Macha, 2005), or Łagoda and Ogonowski (2005). Also in this case, the coefficients A

and B are especially determined so as to obtain correct results for uniaxial tension-compression and torsion

$$
\begin{aligned}
A &= \frac{\sqrt{6}}{2}\frac{\sigma_{af}}{\tau_{af}} \\
B &= -\sqrt{3}r_{\tau\,os,\sigma\,os}\frac{\sigma_{af}}{\tau_{af}} + \frac{1}{\tau_{af}}\sqrt{3r^2_{\tau\,os,\sigma\,os}\sigma^2_{af} + 9\tau^2_{af} - 3\sigma^2_{af}}
\end{aligned}
\tag{2.19}
$$

Assuming that the histories $\tau_{os}(t)$ and $\sigma_{on}(t)$ are correlated, the maximum value of PSDF of the equivalent stress leads to the following form of the coefficient B

$$
B = -\sqrt{2}A + 3 \tag{2.20}
$$

3. Experimental verification

For the verification purposes, experimental data which can be found in the literature were used. While searching for the suitable data, special attention was paid to the kind of experimental tests and their complexity. For these reasons, two sets of experimental results presented by Simbürger (1975) and by Niesłony and Macha (2007) were selected. For both data sets, the computation of PSDF of the equivalent stress were done according to four criteria:

- C1 – criterion of maximum normal stress on the critical plane, Eq. (2.3),
- C2 – criterion of maximum normal and shear stresses on the critical plane, Eq. (2.4),
- C3 – criterion proposed by Preumont and Piefort, Eq. (2.12),
- C4 – criterion of shear and normal stresses acting on the octahedral plane, Eq. (2.18).

3.1. Fatigue data obtained by cyclic bending with torsion

The testing program by Simbürger (1975) comprised plain and notched specimens made of Ck 45 steel (SAE 1045). The tests were load controlled. A constant amplitude loading was used with equal frequency of stress tensor components to perform several tests under bending and/or torsion. The criterion for fatigue failure was the crack initiation defined as crack depth of 0.25 mm and it was monitored and detected during the tests by the ultrasonic method. While uniaxial tests for determining only material data, unnotched plain and hollow specimens were tested, Table 2. In the case of bending, torsion and combined loading, notched specimens, with the notch radius 5 mm, having the theoretical stress concentration factor $K_{tt} = 1.24$ under torsion and $K_{tb} = 1.49$ under bending, were tested. Six tests series were carried out for the notched specimens by bending (S01), torsion (S02), and four combinations of bending with torsion (S03-S06). The tests under combined bending with torsion were carried out for the constant ratio of amplitudes of nominal stresses $\tau_{a,xy}/\sigma_{a,xx} = 0.575$. In this case, the series S03-S06 differed in the value of phase shift between bending and torsion with values $\varphi_{\sigma\tau} = 0$, 30, 60 and 90 degree. Figure 4 presents geometry of the specimen with scheme of the loading.

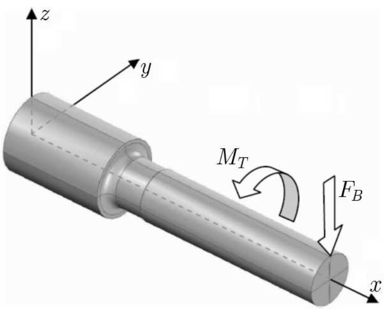

Fig. 4. Geometry of the specimen with scheme of the loading used by Simbürger

Application of the spectral method presented in the previous section for fatigue life calculation in the case of constant amplitude loading is incorrect. They assume that the loading has a random character with normal distribution, i.e. the Gaussian process, what is not appearing in this case. However, it is obvious that the components of the vector of the loading are changing sinusoidally in time and they are characterized by the same frequency. It is expected that the equivalent stress to be also a constant amplitude course with the frequency equal to the frequency of components of the loading vector. Therefore, by computing the amplitude of the equivalent stress the variance of this process can be used

$$\sigma_{a,eq} = \sqrt{2\mu_{eq}} = \sqrt{2\int\limits_0^\infty G_{eq}(f)\,df} \tag{3.1}$$

Like during calculations in the time domain, also in the spectral method the number of cycles to crack initiation is determined directly from the stress-life characteristic

$$N_{cal} = A_\sigma(\sigma_{a,eq})^{-m_\sigma} \tag{3.2}$$

It leads to simplification of the calculations, since the estimate of the PDF of amplitudes is unnecessary. The result is not burdened with uncertainness of the damage accumulation hypothesis and of approximation of the PDF of amplitudes. In order to obtain calculated fatigue lives for each considered case of loading and criterion of multiaxial fatigue, the fatigue limit was not taken into account during the calculations, Eq. (3.2). The calculation results N_{cal} were compared with the test results N_{exp} in Fig. 5.

The quantities on the axes in the figures are presented in a logarithmic order. The dashed lines mark the scatter band with coefficient 3 which indicates that the points within the band results from the comparison of two quantities, x and y, for which the expression $1/3 < y/x < 3$ is satisfied. The diagonal (full line) defines an ideal case for which the two compared quantities are equal $y/x = 1$.

For the case of loading under pure bending (S01), the criteria based on invariants of the stress state (C3 and C4) are giving the best results in relationship to the critical plane concept criteria (C1 and C2). Probably, these criteria are taking the special condition of the stresses arising at the bottom of the notch into account appropriately. In Fig. 5b, only the results of calculations received according to C2 and C3 criteria are included into the established scatter band. This is a result of using the relationship σ_{af}/τ_{af} which was applied in these criteria. In Fig. 5c-f, the results of calculations are presented for the combined loading with phase shifts $\varphi_{\sigma\tau} = 0$, 30, 60 and 90 degrees. It is easy to notice that the C1 and C2 criteria taking into account the non-proportionality of the loading are not correct and tend to overestimate the calculated fatigue life. Under those criteria, worse results of the calculation for greater values of the phase shifts were received. It can be assumed that the criteria based on the invariants of the stress state should be used for fatigue life assessment of machines made of Ck45 steel.

3.2. Fatigue data obtained under random bending with torsion

The data obtained while fatigue tests under combined random bending with torsion were presented by Niesłony and Macha (2007). Round smooth specimens made of 18G2A steel were tested. A typical product, i.e. a drawn bar was tested. Since it was assumed that a commercially available material could be subjected to fatigue tests, the purchased material had not been subjected to any preliminary selection. The main material constants given in Table 2 were obtained from the basic cyclic fatigue tests.

In the tests, histories of Gaussian distributions and narrow frequency bands were used. The dominating frequency was 20 Hz. In Fig. 6, geometry of the

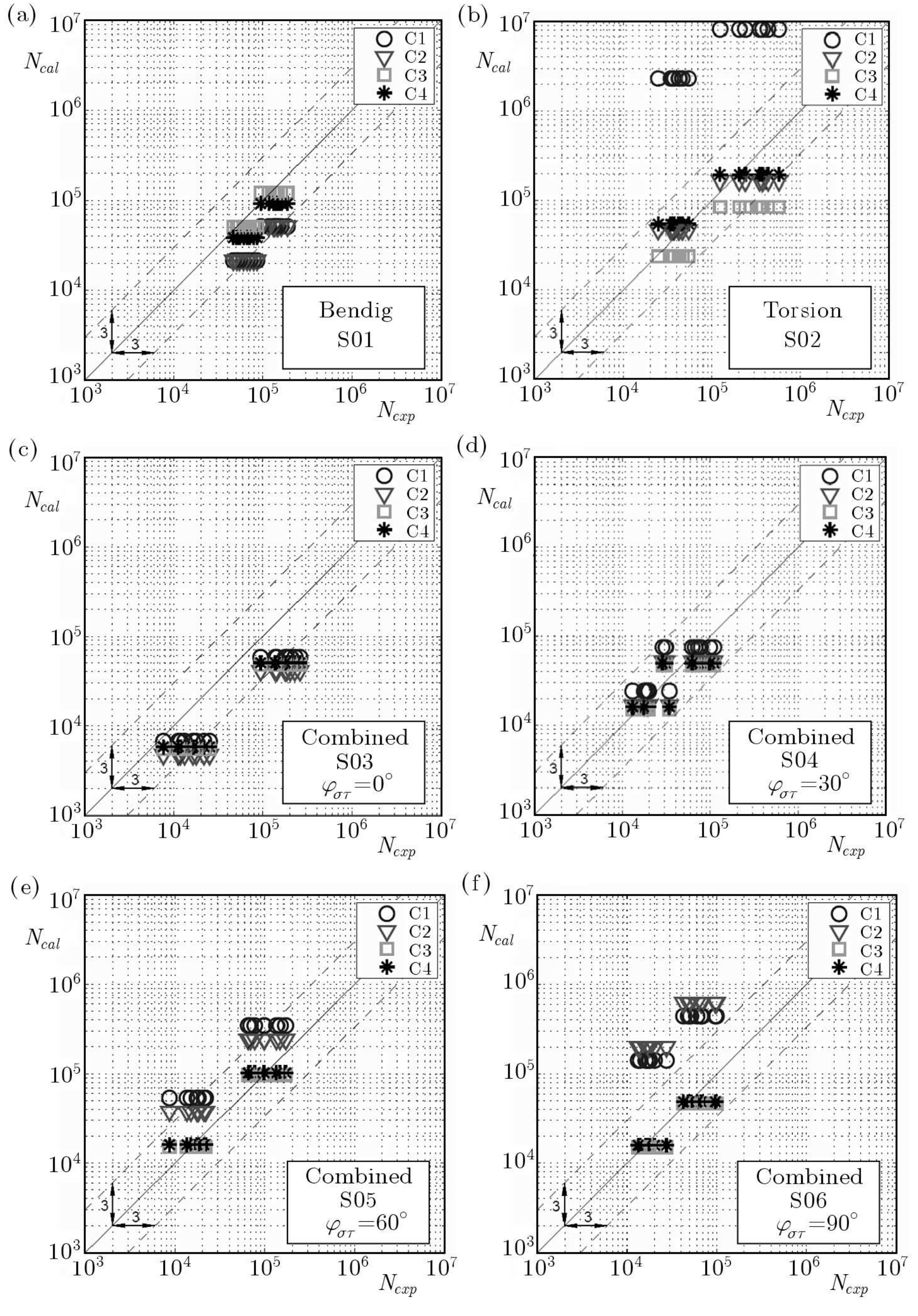

Fig. 5. Comparison of the calculated N_{cal} and experimental N_{exp} cycles to failure

Table 2. Parameters of Wöhler's curves used in the fatigue life calculation

Material	Tension			Torsion		
	σ_{af} [MPa]	m_σ	N_σ	τ_{af} [MPa]	m_τ	N_τ
Ck45	390	8.5	200000	275	29	300000
18G2A	204	7.9	1120000	142	12.3	1982900

specimen with scheme of the loading is presented. The tests were performed for different correlation coefficients $r_{\sigma\tau}$ between the histories of bending $\sigma(t)$ and torsion $\tau(t)$ and different variances μ_σ and μ_τ. Eight fatigue tests were performed and marked with symbols as follows:

- N01 – pure bending,
- N02 – pure torsion,
- N03 – non-proportional bending with torsion, $r_{\sigma\tau} \approx 0$, $\sqrt{\mu_\sigma/\mu_\tau} \approx 2$
- N04 – non-proportional bending with torsion, $r_{\sigma\tau} \approx 0$, $\sqrt{\mu_\sigma/\mu_\tau} \approx 1$
- N05 – proportional bending with torsion, $r_{\sigma\tau} = 1$, $\sqrt{\mu_\sigma/\mu_\tau} \approx 2$
- N06 – proportional bending with torsion, $r_{\sigma\tau} = 1$, $\sqrt{\mu_\sigma/\mu_\tau} \approx 1$
- N07 – non-proportional bending with torsion, $r_{\sigma\tau} \approx 0.5$, $\sqrt{\mu_\sigma/\mu_\tau} \approx 1$
- N08 – non-proportional bending with torsion, $r_{\sigma\tau} \approx 0.5$, $\sqrt{\mu_\sigma/\mu_\tau} \approx 2$

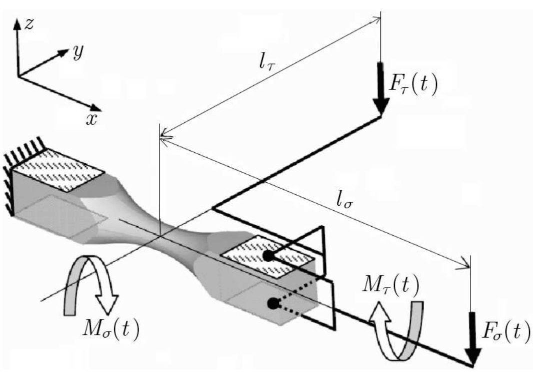

Fig. 6. Geometry of the specimen with scheme of the loading presented by Niesłony and Macha (2007)

In order to determine the fatigue life from PSDF of the equivalent stress, a suitable spectral formula was derived, assuming that the loading had a normal probability distribution with the narrow-band frequency spectrum and damages was accumulated according to the Serensen-Kogayev hypothesis. The following final formula was obtained

$$T_{cal} = \frac{b_{SK}}{M^+}\Big[\int\limits_{a_{SK}}^{\infty} \frac{p(\sigma_a)}{N(\sigma_a)}\;d\sigma_a\Big]^{-1} \tag{3.3}$$

where

$$b_{SK} = \frac{\sqrt{2\mu_\sigma}\varGamma\Big(\frac{3}{2},\frac{a_{SK}^2\sigma_{af}^2}{2\mu_\sigma}\Big)e^{\frac{a_{SK}^2\sigma_{af}^2}{2\mu_\sigma}} - a_{SK}\sigma_{af}}{\sigma_{a\,max} - a_{SK}\sigma_{af}} \tag{3.4}$$

is the Serensen-Kogayev coefficient formulated for the spectral method, a_{SK} is the coefficient enabling one to include amplitudes below the fatigue limit and PDF $p(\sigma_a)$ is computed using the Dirlik model, Eq. (1.8). Details according to the spectral formulation of the coefficient b_{SK} can be found in literature (Łagoda *et al.*, 2005). The calculation results were compared with the fatigue test results in Figs. 7 and 8.

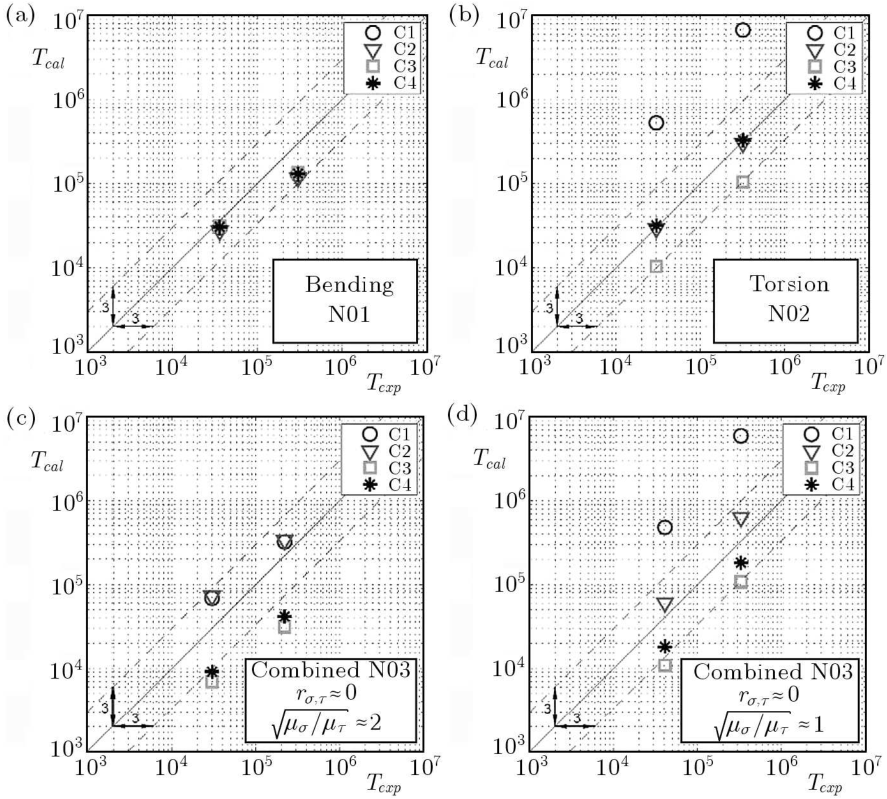

Fig. 7. Comparison of the calculated T_{cal} and experimental T_{exp} time to failure for loading cases N01-N04

The same type of graph was used for the data set by Simburger. Because of the random loading, the experimental T_{exp} and calculated T_{cal} time to failure

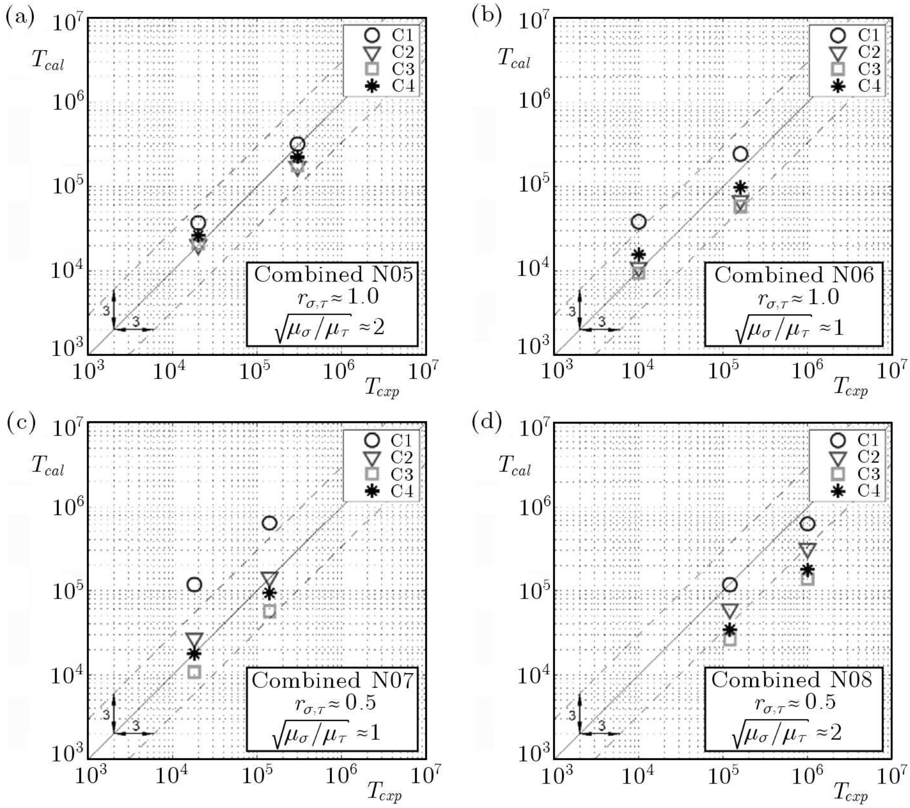

Fig. 8. Comparison of the fatigue life calculated according to the four criteria C1-C4 with the experimental life, loading cases N05-N08

was used in figures instead of the number of cycles. Like previously, only the criteria C2 and C3 give acceptable results under pure torsion. Worse results were obtained by application of the criterion C1 because of the big scatter range. The criteria based on the invariants of the stress state (C3 and C4) tends to underestimate the computed life what is acceptable from the engineering point of view. Generally, good results of calculations were obtained owing to the C2 criterion which can be recommended for the fatigue life assessment of 18G2A steel.

4. Conclusions

The spectral method of fatigue life determination has been successfully applied under uniaxial and very rarely under multiaxial random loadings so far. Some fatigue failure criteria, such as the criterion of maximum principal stress or the criterion based on the Huber-Mises-Hencky strength hypothesis, give satisfactory results only for some constructional materials. From theoretical considerations it appears that those criteria do not include the influence of changes of the principal stress directions under a multiaxial loading on the material fatigue. Therefore, it is necessary to apply these criteria for materials which are not sensitive to non-proportionality of the loading, i.e. tension-compression with phase shifts or non-correlated random loadings. The criteria being a linear combination of the stress state components based on the critical plane approach take into account the non-proportionality of the loading. All of the presented criteria do not include the loading mean value, and the range of their application should be more verified with experimental data.

The presented multiaxial fatigue failure criteria reduce the three-dimensional stress state to the equivalent uniaxial one directly in the frequency domain working on PSDFs of stresses. This fact give us possibility to use the known uniaxial models during calculations, see Fig. 1, what is a strong advantage of this kind of criteria. Because of the calculation rate and good correlation with the experimental results (Łagoda *et al.*, 2005), the introduced method is useful for the fatigue life assessment of real structures under random or variable amplitude loadings using FEM (Niesłony, 2008).

Development of numerical methods, especially virtual test methods using FEM joined with multi-body dynamic analysis allows one to state that the criteria of multiaxial fatigue defined in the frequency domain could be very interesting in the nearest future.

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Porównanie kilku wybranych kryteriów wieloosiowego zniszczenia zmęczeniowego dedykowanych metodzie spektralnej

Streszczenie

W pracy przedstawiono metodologię wyznaczania trwałości zmęczeniowej w zakresie dużej liczby cykli przy obciążeniu losowym z wykorzystaniem metody spektralnej. Metodologia ta wykorzystuje funkcję gęstości widmowej mocy naprężenia ekwiwalentnego przy wyznaczaniu trwałości zmęczeniowej wraz ze znanymi jednoosiowymi modelami spektralnymi. Jako przykład przedstawiono modele proponowane przez Milesa i Dirlika. Wyróżniono dwie grupy kryteriów wieloosiowego zmęczenia: kryteria opierające się na pojęciu płaszczyzny krytycznej i kryteria wykorzystujące niezmienniki stanu naprężenia. W celu porównania obliczeniowej i eksperymentalnej trwałości zmęczeniowej wykorzystano dwa zestawy danych otrzymanych przy kombinacji zginania ze skręcaniem. Wykazano, że przy wieloosiowym obciążeniu losowym trwałości obliczeniowe korelują dobrze z otrzymanymi podczas eksperymentu, jeżeli zostanie zastosowane odpowiednie do właściwości mechanicznych materiału kryterium wieloosiowego zniszczenia zmęczeniowego.

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