

## NUMERICAL SOLUTION OF NON-HOMOGENOUS FRACTIONAL OSCILLATOR EQUATION IN INTEGRAL FORM

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In this paper, a non-homogenous fractional oscillator equation in finite time interval is considered. The fractional equation with derivatives of order  $\alpha \in (0, 1]$  is transformed into its corresponding integral form. Next, a numerical solution of the integral form of the considered equation is presented. In the final part of this paper, some examples of numerical solutions of the considered equation are shown.

*Keywords:* fractional oscillator equation, fractional integral equation, numerical solution

### 1. Introduction

Fractional differential equations have numerous applications in physics, engineering and biology. There is a number of monographs (Atanackovic *et al.*, 2014; Baleanu *et al.*, 2012; Hilfer, 2000; Kilbas *et al.*, 2006; Klimek, 2009; Leszczynski, 2011; Magin, 2006; Malinowska and Torres, 2012; Podlubny, 1999) and a huge number of papers (Agrawal, 2002; Ciesielski and Leszczynski, 2006; Katsikadelis, 2012; Klimek, 2001; Riewe, 1996; Sumelka, 2014; Sumelka and Blaszczyk, 2014; Zhang *et al.*, 2014) that cover various problems in fractional calculus. The list is large and is growing rapidly. An important issue is that the derivative of fractional order at any point of the domain has a local property only when the order is an integer number. For non-integer cases, the fractional derivative is a nonlocal operator and depends on the past values of the function (left derivative) or future ones (right derivative).

There are two different ways to formulate differential equations containing derivatives of fractional order. The first way is simply to replace integer order derivatives in differential equations by fractional ones. The second one relies on modifying the variational principle by replacing the integer order derivative by a fractional one. This leads to fractional differential equations which are known in the literature as the fractional Euler-Lagrange equations. It involves different types of Lagrangians, e.g., depending on Riemann-Liouville or Caputo fractional derivatives, fractional integrals, and mixed integer-fractional order operators (Agrawal *et al.*, 2011; Almeida and Malinowska, 2012; Baleanu *et al.*, 2014; Klimek *et al.*, 2014; Odziejewicz *et al.*, 2012). This second approach, in recent years, seems to become increasingly important.

The main feature of the fractional Euler-Lagrange equations is that these equations involve simultaneously the left and right derivatives of fractional order. This is also a fundamental problem in finding solutions to equations of a variational type (Baleanu *et al.*, 2012; Klimek, 2009). Consequently, numerous studies have been devoted to numerical schemes for the fractional Euler-Lagrange equations (Blaszczyk *et al.*, 2011; Blaszczyk and Ciesielski, 2014, 2015a,b; Bourdin *et al.*, 2013; Pooseh *et al.*, 2013; Xu and Agrawal, 2014).

In this paper, we present a numerical solution of the non-homogenous fractional oscillator equation in a finite time interval.

## 2. Preliminaries

We consider a variational differential equation containing fractional derivatives of order  $\alpha \in (0, 1]$  in the finite time interval  $t \in [0, b]$

$${}^C D_{b^-}^\alpha D_{0^+}^\alpha x(t) - \omega^{2\alpha} x(t) = f(t) \quad (2.1)$$

where  $\omega > 0$ ,  $x(t)$  is an unknown function and  $f(t)$  is a given function. Equation (2.1) is supplemented by the following boundary conditions

$$x(0) = 0 \quad x(b) = L \quad (2.2)$$

In this type of equation (derived from the Lagrangian with fractional derivative and the properties of the considered variational problem (Klimek, 2009) we assume  $x(0) = 0$  on the left boundary.

We recall some definitions and properties of the fractional operators (Kilbas *et al.*, 2006; Oldham and Spanier, 1974; Podlubny, 1999). The left and right Riemann-Liouville fractional derivatives of order  $\alpha \in (0, 1)$  are defined by

$$\begin{aligned} D_{0^+}^\alpha x(t) &:= D I_{0^+}^{1-\alpha} x(t) \\ D_{b^-}^\alpha x(t) &:= -D I_{b^-}^{1-\alpha} x(t) \end{aligned} \quad (2.3)$$

and the left and right Caputo derivatives are defined as follows

$$\begin{aligned} {}^C D_{0^+}^\alpha x(t) &:= I_{0^+}^{1-\alpha} D x(t) \\ {}^C D_{b^-}^\alpha x(t) &:= -I_{b^-}^{1-\alpha} D x(t) \end{aligned} \quad (2.4)$$

where  $D$  is an operator of the first order derivative and the operators  $I_{0^+}^\alpha$  and  $I_{b^-}^\alpha$  are respectively the left and right fractional integrals of order  $\alpha > 0$  defined by

$$\begin{aligned} I_{0^+}^\alpha x(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} d\tau \quad (t > 0) \\ I_{b^-}^\alpha x(t) &:= \frac{1}{\Gamma(\alpha)} \int_t^b \frac{x(\tau)}{(\tau-t)^{1-\alpha}} d\tau \quad (t < b) \end{aligned} \quad (2.5)$$

If  $\alpha = 1$  then  $D_{0^+}^1 x = {}^C D_{0^+}^1 x = x'$  and  $D_{b^-}^1 x = {}^C D_{b^-}^1 x = -x'$ . The relations between the Caputo and Riemann-Liouville derivatives are of the form

$$\begin{aligned} {}^C D_{0^+}^\alpha x(t) &= D_{0^+}^\alpha x(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x(0) \\ {}^C D_{b^-}^\alpha x(t) &= D_{b^-}^\alpha x(t) - \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)} x(b) \end{aligned} \quad (2.6)$$

The composition rules of the fractional operators (for  $\alpha \in (0, 1]$ ) looks as follows

$$\begin{aligned} I_{0^+}^\alpha {}^C D_{0^+}^\alpha x(t) &= x(t) - x(0) \\ I_{b^-}^\alpha {}^C D_{b^-}^\alpha x(t) &= x(t) - x(b) \end{aligned} \quad (2.7)$$

and the fractional integral of a constant  $C$

$$I_{0+}^{\alpha} C = C \frac{t^{\alpha}}{\Gamma(1 + \alpha)} \tag{2.8}$$

In particular, when  $\alpha = 1$ , then  ${}^C D_{b-}^1 D_{0+}^1 = -D^2$  and Eq. (2.1) becomes

$$-D^2 f(t) - \omega^2 x(t) = f(t) \tag{2.9}$$

and its analytical solution with respect to boundary conditions (2.2) has the form

$$x(t) = \frac{\sin(\omega t)}{\sin(\omega b)} [L + F(b)] - F(t) \tag{2.10}$$

where

$$F(t) = \frac{1}{\omega} \int_0^t f(\tau) \sin(\omega(t - \tau)) d\tau \tag{2.11}$$

This solution we will apply to verification of numerical results obtained by our method for the case  $\alpha = 1$ .

### 3. Equivalent integral form

The first step of our method is to transform a differential equation into an integral equation. We integrate Eq. (2.1) twice: the first time by using right fractional integral operator (2.5)<sub>2</sub> and the second time by using left fractional integral operator (2.5)<sub>1</sub>. Finally, we obtain

$$I_{0+}^{\alpha} I_{b-}^{\alpha} {}^C D_{b-}^{\alpha} D_{0+}^{\alpha} x(t) - \omega^{2\alpha} I_{0+}^{\alpha} I_{b-}^{\alpha} x(t) = I_{0+}^{\alpha} I_{b-}^{\alpha} f(t) \tag{3.1}$$

Next, we use the composition rule of operators (2.7)<sub>2</sub> and get

$$I_{0+}^{\alpha} \left( D_{0+}^{\alpha} x(t) - D_{0+}^{\alpha} x(t) \Big|_{t=b} \right) - \omega^{2\alpha} I_{0+}^{\alpha} I_{b-}^{\alpha} x(t) = I_{0+}^{\alpha} I_{b-}^{\alpha} f(t) \tag{3.2}$$

In the above equation there occurs the value  $D_{0+}^{\alpha} x(t) \Big|_{t=b}$ , and it is unknown at this stage. Here, we treat it as a constant. By using property (2.7)<sub>1</sub> and the assumption  $x(0) = 0$  in the boundary condition, and the fractional integral of constant (2.8) we obtain the following form of the considered equation

$$x(t) - D_{0+}^{\alpha} x(t) \Big|_{t=b} \frac{t^{\alpha}}{\Gamma(\alpha + 1)} - \omega^{2\alpha} I_{0+}^{\alpha} I_{b-}^{\alpha} x(t) = I_{0+}^{\alpha} I_{b-}^{\alpha} f(t) \tag{3.3}$$

The unknown value  $D_{0+}^{\alpha} x(t) \Big|_{t=b}$  in the above equation can be determined on the basis of the boundary condition. Substituting the value  $t = b$  into Eq. (3.3)

$$x(b) - D_{0+}^{\alpha} x(t) \Big|_{t=b} \frac{b^{\alpha}}{\Gamma(\alpha + 1)} - \omega^{2\alpha} I_{0+}^{\alpha} I_{b-}^{\alpha} x(t) \Big|_{t=b} = I_{0+}^{\alpha} I_{b-}^{\alpha} f(t) \Big|_{t=b} \tag{3.4}$$

we obtain

$$D_{0+}^{\alpha} x(t) \Big|_{t=b} = \frac{\Gamma(\alpha + 1)}{b^{\alpha}} \left[ x(b) - \omega^{2\alpha} I_{0+}^{\alpha} I_{b-}^{\alpha} x(t) \Big|_{t=b} - I_{0+}^{\alpha} I_{b-}^{\alpha} f(t) \Big|_{t=b} \right] \tag{3.5}$$

Next, we put the right-hand side of formula (3.5) into Eq. (3.3) and get the integral form of Eq. (2.1)

$$x(t) - \omega^{2\alpha} \left[ I_{0+}^{\alpha} I_{b-}^{\alpha} x(t) - \left( \frac{t}{b} \right)^{\alpha} I_{0+}^{\alpha} I_{b-}^{\alpha} x(t) \Big|_{t=b} \right] = \left( \frac{t}{b} \right)^{\alpha} \left[ L - I_{0+}^{\alpha} I_{b-}^{\alpha} f(t) \Big|_{t=b} \right] + I_{0+}^{\alpha} I_{b-}^{\alpha} f(t) \tag{3.6}$$

### 4. Numerical solution

In this Section, we present a numerical scheme for Eq. (3.6). We introduce a homogeneous grid of  $n + 1$  nodes (with the constant time step  $\Delta t = b/n$ ):  $0 = t_0 < t_1 < \dots < t_i < t_{i+1} < \dots < t_n = b$ , and  $t_i = i\Delta t$ ,  $i = 0, 1, \dots, n$ . For every grid node  $t_i$ , we write the following equation

$$\begin{aligned} x(t_i) - \omega^{2\alpha} \left[ I_{0+}^\alpha I_{b-}^\alpha x(t) \Big|_{t=t_i} - \left( \frac{t_i}{t_n} \right)^\alpha I_{0+}^\alpha I_{b-}^\alpha x(t) \Big|_{t=t_n} \right] \\ = \left( \frac{t_i}{t_n} \right)^\alpha \left[ L - I_{0+}^\alpha I_{b-}^\alpha f(t) \Big|_{t=t_n} \right] + I_{0+}^\alpha I_{b-}^\alpha f(t) \Big|_{t=t_i} \end{aligned} \tag{4.1}$$

In order to simplify notation, we denote the values of functions  $x(t)$  and  $f(t)$  at the node  $t_i$  by  $x_i = x(t_i)$  and  $f_i = f(t_i)$ , respectively. The numerical problem is to approximate the values of  $I_{0+}^\alpha I_{b-}^\alpha x(t) \Big|_{t=t_i}$  and  $I_{0+}^\alpha I_{b-}^\alpha f(t) \Big|_{t=t_i}$ , for  $i = 0, 1, \dots, n$ , properly. In our previous works (Błaszczyk and Ciesielski, 2015a,b) we have determined the discrete form of the composition of operators. On the basis of these results, we present the final discrete forms

$$\begin{aligned} I_{0+}^\alpha I_{b-}^\alpha x(t) \Big|_{t=t_i} &\approx \sum_{j=0}^i u_{i,j}^{(\alpha)} \sum_{k=j}^n v_{j,k}^{(\alpha)} x_k \\ I_{0+}^\alpha I_{b-}^\alpha f(t) \Big|_{t=t_i} &\approx \sum_{j=0}^i u_{i,j}^{(\alpha)} \sum_{k=j}^n v_{j,k}^{(\alpha)} f_k \end{aligned} \tag{4.2}$$

where the coefficients  $u_{i,j}^{(\alpha)}$  and  $v_{j,k}^{(\alpha)}$  are as follows

$$\begin{aligned} u_{i,j}^{(\alpha)} = \frac{(\Delta t)^\alpha}{\Gamma(\alpha + 2)} \begin{cases} 0 & \text{for } i = 0 \text{ and } j = 0 \\ (i-1)^{\alpha+1} - i^{\alpha+1} + i^\alpha(\alpha+1) & \text{for } i > 0 \text{ and } j = 0 \\ (i-j+1)^{\alpha+1} - 2(i-j)^{\alpha+1} + (i-j-1)^{\alpha+1} & \text{for } i > 0 \text{ and } 0 < j < i \\ 1 & \text{for } i > 0 \text{ and } j = i \end{cases} \\ v_{j,k}^{(\alpha)} = \frac{(\Delta t)^\alpha}{\Gamma(\alpha + 2)} \begin{cases} 0 & \text{for } j = n \text{ and } k = n \\ (n-j-1)^{\alpha+1} - (n-j)^{\alpha+1} + (n-j)^\alpha(\alpha + 1) & \text{for } j < n \text{ and } k = n \\ (k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1} & \text{for } j < n \text{ and } j < k < n \\ 1 & \text{for } j < n \text{ and } k = j \end{cases} \end{aligned} \tag{4.3}$$

One can observe that in order to compute the values of operators (4.2) at every node  $t_i$ , we need to use the values of functions at all nodes of the domain.

Now we present the numerical scheme of integral equation (3.6). If we substitute (4.2) into (4.1), then the solution can be written as a system of  $n + 1$  linear equations

$$\begin{aligned} x_i - \omega^{2\alpha} \left[ \sum_{j=0}^i u_{i,j}^{(\alpha)} \sum_{k=j}^n v_{j,k}^{(\alpha)} x_k - \left( \frac{i}{n} \right)^\alpha \sum_{j=0}^n u_{n,j}^{(\alpha)} \sum_{k=j}^n v_{j,k}^{(\alpha)} x_k \right] \\ = \left( \frac{i}{n} \right)^\alpha \left[ L - \sum_{j=0}^n u_{n,j}^{(\alpha)} \sum_{k=j}^n v_{j,k}^{(\alpha)} f_k \right] + \sum_{j=0}^i u_{i,j}^{(\alpha)} \sum_{k=j}^n v_{j,k}^{(\alpha)} f_k \quad i = 0, \dots, n \end{aligned} \tag{4.4}$$

The above system can also be written in the matrix form

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b} \tag{4.5}$$

with the coefficients of matrices  $\mathbf{A}$  and  $\mathbf{b}$

$$A_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} - \omega^{2\alpha} \left[ \sum_{k=0}^{\min(i,j)} u_{i,k}^{(\alpha)} v_{k,j}^{(\alpha)} - \left(\frac{i}{n}\right)^\alpha \sum_{k=0}^j u_{n,k}^{(\alpha)} v_{k,j}^{(\alpha)} \right]$$

$$i = 0, \dots, n \quad j = 0, \dots, n \tag{4.6}$$

$$b_i = \left(\frac{i}{n}\right)^\alpha \left[ L - \sum_{j=0}^n u_{n,j}^{(\alpha)} \sum_{k=j}^n v_{j,k}^{(\alpha)} f_k \right] + \sum_{j=0}^i u_{i,j}^{(\alpha)} \sum_{k=j}^n v_{j,k}^{(\alpha)} f_k \quad i = 0, \dots, n$$

respectively, and  $\mathbf{x} = [x_0, x_1, \dots, x_n]^T$ .

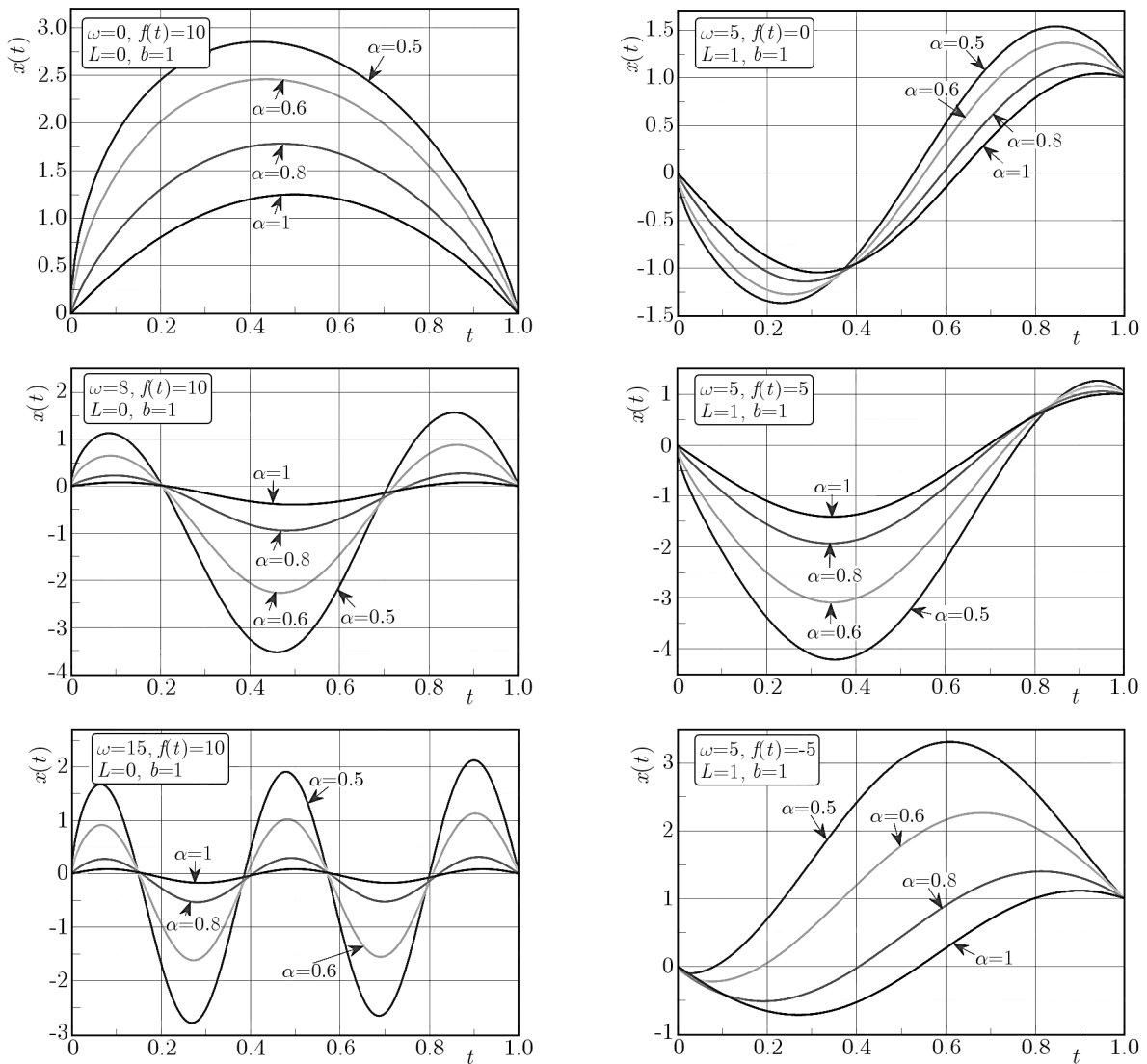


Fig. 1. Numerical solution of Eq. (2.1) for  $\alpha \in \{0.5, 0.6, 0.8, 1\}$ ,  $b = 1$ , and different values of parameters  $\omega$ ,  $L$  and functions  $f(t)$ ; left-side: different  $\omega$  and  $f(t) = 10$ ; right-side: constant  $\omega = 5$  and different  $f(t)$

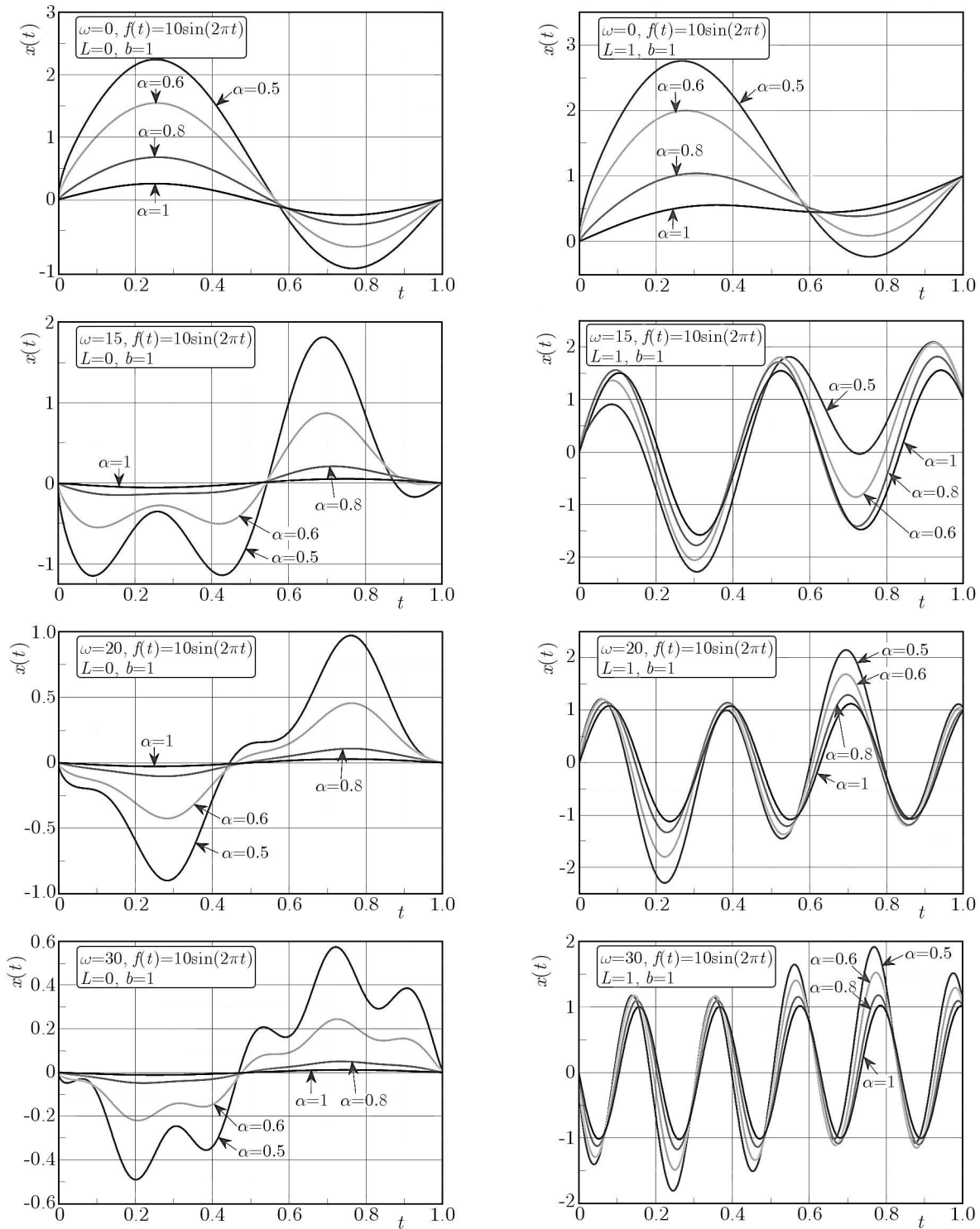


Fig. 2. Numerical solution of Eq. (2.1) for  $\alpha \in \{0.5, 0.6, 0.8, 1\}$ ,  $f(t) = 10 \sin(2\pi t)$ ,  $b = 1$ , different values of parameter  $\omega$ , and  $L = 0$  (left-side),  $L = 1$  (right-side)

### 5. Example of computations

We present the results of computations obtained by our numerical scheme to the forced fractional oscillator equation. The system of linear equations (4.5) has been solved by the Gaussian elimination algorithm (the LUP decomposition (Press *et al.*, 2007)). We present several exam-

ples of calculations for different values of the parameters  $\alpha$ ,  $\omega$  and different functions  $f$ . In the examples we assume:  $b = 1$  and  $L \in \{0, 1\}$  in the right boundary condition. In Figs. 1 and 2 we present results of the numerical solution of Eq. (2.1). The values of the parameters used in the solution of equation (2.1) are given in plot legends. The time domain  $t \in [0, 1]$  has been divided into  $n = 1000$  subintervals in all examples.

**5.1. Numerical error analysis**

The discretization of differential or integral equations always entails certain errors. The knowledge of an analytical solution  $x_{anal}(t)$  allows us to evaluate the errors of our numerical solution  $x_{num}(t)$ . We define the average error as

$$Err_{avg} = \frac{1}{m + 1} \sum_{i=0}^m |x_{num}(t_i) - x_{anal}(t_i)| \quad \text{for } i = 0, \dots, m \tag{5.1}$$

where  $m$  is the number of measurement points.

Let us consider Eq. (2.1) with the parameters  $\omega = 0$ ,  $L = 0$  and the function  $f(t) = (b - t)^{2-\alpha}$ . In this case

$$I_{0+}^\alpha I_{b-}^\alpha f(t) = I_{0+}^\alpha I_{b-}^\alpha (b - t)^{2-\alpha} = t^\alpha b^2 \Gamma(3 - \alpha) \sum_{j=0}^2 \frac{(-1)^j}{\Gamma(3 - j)\Gamma(\alpha + 1 + j)} \left(\frac{t}{b}\right)^j \tag{5.2}$$

and the analytical solution (by using Eq. (3.6) and further simplifications) has form

$$x_{anal}(t) = t^\alpha b^2 \Gamma(3 - \alpha) \sum_{j=0}^2 \frac{(-1)^j [(t/b)^j - 1]}{\Gamma(3 - j)\Gamma(j + \alpha + 1)} \tag{5.3}$$

In Tables 1 and 2, we present the values of the analytical (exact) solution  $x_{anal}(t)$  for  $\alpha \in \{0.5, 0.75\}$  at the selected values of nodes  $t_i$ ,  $i = 0, \dots, 10$  and the corresponding numerical values  $x_{num}(t)$  calculated for different numbers of grid nodes ( $n \in \{100, 200, 400, 800\}$ ). The errors generated by numerical schema are also included in Tables 1 and 2. One can note that the error decreases with a decrease in the time step of the grid.

**Table 1.** Values of analytical and numerical solutions of Eq. (2.1) for  $\alpha = 0.5$ ,  $\omega = 0$ ,  $f(t) = (b - t)^{2-\alpha}$ ,  $b = 1$ ,  $L = 0$

$t_i$	$x_{anal}(t_i)$	$x_{num}(t_i)$ $n = 100$	$x_{num}(t_i)$ $n = 200$	$x_{num}(t_i)$ $n = 400$	$x_{num}(t_i)$ $n = 800$
0	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.1	0.159378794072	0.159379032826	0.159378842522	0.159378803745	0.159378795979
0.2	0.186040855728	0.186041292916	0.186040941860	0.186040872522	0.186040858974
0.3	0.184034779322	0.184035363697	0.184034893461	0.184034801419	0.184034783568
0.4	0.166968260457	0.166968963299	0.166968397209	0.166968286848	0.166968265515
0.5	0.141421356237	0.141422156701	0.141421511713	0.141421386199	0.141421361972
0.6	0.111541920371	0.111542800274	0.111542091205	0.111541953283	0.111541926669
0.7	0.080319362547	0.080320302735	0.080319545231	0.080319397768	0.080319369292
0.8	0.050087922696	0.050088897461	0.050088112561	0.050087959382	0.050087929735
0.9	0.022768399153	0.022769357797	0.022768587015	0.022768435655	0.022768406192
1	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
$Err_{avg}$	–	$5.92466 \cdot 10^{-7}$	$1.15654 \cdot 10^{-7}$	$2.2385 \cdot 10^{-8}$	$4.30118 \cdot 10^{-9}$



**Table 2.** Values of analytical and numerical solutions of Eq. (2.1) for  $\alpha = 0.75$ ,  $\omega = 0$ ,  $f(t) = (b - t)^{2-\alpha}$ ,  $b = 1$ ,  $L = 0$

$t_i$	$x_{anal}(t_i)$	$x_{num}(t_i)$ $n = 100$	$x_{num}(t_i)$ $n = 200$	$x_{num}(t_i)$ $n = 400$	$x_{num}(t_i)$ $n = 800$
0	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.1	0.067645942013	0.067646050586	0.067645965523	0.067645947053	0.067645943086
0.2	0.094996909603	0.094997100277	0.094996949929	0.094996918109	0.094996911394
0.3	0.105395550994	0.105395806812	0.105395604628	0.105395562237	0.105395553351
0.4	0.104362700328	0.104363008356	0.104362764589	0.104362713751	0.104362703135
0.5	0.095196902130	0.095197250210	0.095196974490	0.095196917206	0.095196905277
0.6	0.080331627323	0.080332002341	0.080331705051	0.080331643482	0.080331630691
0.7	0.061751765560	0.061752151496	0.061751845314	0.061751782104	0.061751769002
0.8	0.041170523241	0.041170897406	0.041170600282	0.041170539178	0.041170526550
0.9	0.020119449943	0.020119771392	0.020119515708	0.020119463484	0.020119452744
1	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
$Err_{avg}$	–	$2.42522 \cdot 10^{-7}$	$5.03981 \cdot 10^{-8}$	$1.04972 \cdot 10^{-8}$	$2.19045 \cdot 10^{-9}$

## 6. Conclusions

In this work, the method of solving of the non-homogenous fractional oscillator equation with derivatives of order  $\alpha \in (0, 1]$  has been presented. First, the equation has been transformed into integral form. Next, the numerical scheme for the integral form of the equation has been proposed. Analysing the solutions of the equation for different values of parameters  $\alpha$ ,  $\omega$ , various boundary conditions and different functions  $f(t)$  one can understand the influence of these parameters on the solution. One can note that if the value of  $\alpha$  decreases then the amplitude of oscillations increases, and if the value of  $\omega$  increases then the oscillation frequency increases as well. The analytical solution of this type of fractional differential equation (except for the case  $\alpha = 1$ ) is not yet known. Our proposed numerical method of solutions for  $\alpha \rightarrow 1$  is consistent with the analytical solution for  $\alpha = 1$ . This type of considered equation can be used in mathematical modelling of the oscillatory behavior of systems where the use of the classical oscillator equation is insufficient.

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