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TOPOLOGICAL CLASSES OF STATICALLY DETERMINATE BEAMS WITH ARBITRARY NUMBER OF SUPPORTS

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The paper presents all topologies of statically determinate beams with arbitrary number of pin supports. The geometry of each beam with a fixed topology is optimized by a genetic algorithm, with absolute maximum moment as the objective function. An equality relation between minimum values of this function is defined on the set of all topologies as an equivalence relation. This relation partitions the set of topologies into equivalence classes, called topological classes, for uniform, linear and parabolic gravity loads. An in-depth description of these classes is provided. Exact formulas for optimal locations of supports and hinges are found for the uniform load.

Key words: statically determinate beams, topology optimization, geometry optimization, equivalence classes

Notations

c_E, c_H, c_{EH}	–	number of external, internal and all cantilevers
g	–	number of optimal geometry variants
l, l_E, l_H, L	–	lengths of optimal beam segments and length of beam, see Fig. 3
m	–	number of optimal moment diagrams
M_i, M_i^n	–	optimal moment value of topology $\mathbf{t}_i$ and class $\mathbf{T}_i^n$
n, p, r	–	number of supports, topological classes and no-support bars
q	–	intensity of evenly distributed load
$\{r_n\}$	–	sequence of class moment ratios
R	–	equivalence relation of beam topologies
$\mathbf{t}, \mathbf{t}_i$	–	beam topology, $i = 1, 2, \ldots, \lvert\mathbf{T}^n\rvert$ or $i = 1, 2, \ldots, \lvert\mathbf{T}^{2:n}\rvert$
t_i	–	topological code of support i, $i = 1, 2, \ldots, n$
t_M	–	number of topologies with the same optimal moment diagram

$\mathbf{T}^n, \mathbf{T}^{2:n}$	–	set of all topologies with n supports and with two to n supports
$\mathbf{T}_i^n, \mathbf{T}_i^{2:n}$	–	topological class with n supports and with two to n supports
$\lvert\mathbf{T}^n\rvert, \lvert\mathbf{T}^{2:n}\rvert, \lvert\mathbf{T}_i^n\rvert$	–	number of topologies in set $\mathbf{T}^n$, $\mathbf{T}^{2:n}$ and class $\mathbf{T}_i^n$
$\{\mathbf{T}_k^n\}$	–	sequence of topological classes
x	–	axial coordinate
y_i	–	dimensionless length of cantilever, $i = 1, 2, \ldots, n$
z_i	–	dimensionless length of span, $i = 1, 2, \ldots, n-1$
$(\cdot)^n, (\cdot)^{2:n}, (\cdot)_i^n$	–	quantities in set $\mathbf{T}^n$, $\mathbf{T}^{2:n}$ and class $\mathbf{T}_i^n$

1. Introduction

Structural topology optimization has been identified as one of the most challenging and economically the most rewarding tasks in structural design. It is of substantial practical importance because it can achieve much greater savings than geometry (shape) or sizing (cross-section) optimization (Kirsch, 1989; Rozvany *et al.*, 1995). Topology optimization may not only considerably enhance the design, but also provide the best configuration for further comprehensive shape and sizing optimization (Bojczuk and Szteleblak, 2006).

The idea of topology optimization can be extended to support position design. Design of the optimal support layout is studied in Zhu and Zhang (2006, 2010). Beams are among the most important structural members, particularly statically determinate cases form the basis of solid mechanics (Pedersen and Pedersen, 2009). The optimization of support locations of beams can be found in Mróz and Rozvany (1975), Imam and Al-Shihri (1996), Wang and Chen (1996), Bojczuk and Mróz (1998), Won and Park (1998), Mróz and Bojczuk (2003), Wang (2004, 2006), Friswell (2006), Jang *et al.* (2009). However these papers concern continuous beams which have a single bar with all supports attached to it. Therefore, the topology optimization problem which consists in selecting the pattern of member connections does not concern them. By contrast, statically determinate beams are chains of bars joined by hinges and placed on pin supports. There can be numerous associations between the bars and the supports, but some associations can produce wrong forms. What we need is a constructive rule for generating only the correct topologies. For the case of Gerber beams – in which all supports are moved away from the ends of bars – the topologies were constructed by Golubiewski (1995) in the form of directed graphs. The problem of construction of Gerber and non-Gerber beam topologies was solved by Rychter (Rychter and Kozikowska, 2009) in a much simpler and direct manner.

The optimal design in topological optimization is usually sought in a class of domains, but not in the full domain. Such an approach does not guarantee optimality, because terminal topologies depend on a initial layout, which is adopted arbitrarily. In this paper, the space of all possible candidate topologies is known, exhaustive search in this space is carried out and global optima are found.

Before this exhaustive search is performed, values of the merit function, which ranks beam topologies, are found by geometry optimization of each beam with a fixed topology. In most of the practical beam design problems, reducing the maximal bending moment is of paramount importance (Wang, 2006). Therefore, the objective function in this geometry optimization process has been defined as the absolute maximum moment for the uniform, linear and parabolic load. The function is multi-modal, non-smooth, which means that traditional, gradient-based optimization algorithms fail and much more robust, randomized search techniques must be employed. Among methods of probabilistic optimization, genetic algorithms (Goldberg, 1989) have been widely used because of the simplicity of the search mechanism. Many studies on optimizations of different structures by genetic algorithms have been reported in the literature, including beam structures (Wang and Chen, 1996; Lyu and Saitou, 2005). Therefore, a modified version of the specialized genetic algorithm (Rychter and Kozikowska, 2009) has been applied in this paper to optimize beam geometries for all topologies. For best performance, the algorithm was written by the author in the efficient C programming language (Kernighan and Ritchie, 1988).

Searching for global optima in the space of topologies can be based on a more effective method than exhaustive search. However, the aim of this article is to find not only the best topologies, but also to discover the structure of this space. An equality relation between minimum values of the absolute maximum moment has been defined on the set of all topologies as an equivalence relation. This relation splits that set into a sequence of topological equivalence classes. Typical features of these classes are extensively discussed.

2. Beam topology

The subject of the paper is the set of all statically determinate beams, resting on a fixed number of pin supports (Fig. 1) or a number of pin supports varying within a certain interval. The beams only carry loads perpendicular to their longitudinal axes. Such beams do not undergo horizontal displacements and forces. Therefore, statical determinacy is secured without introducing roller supports.

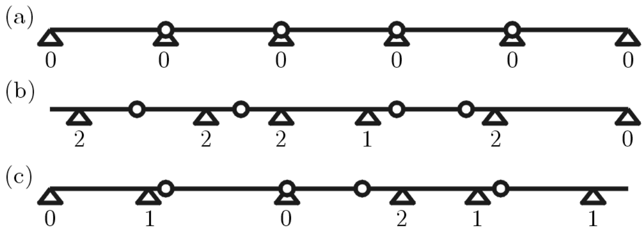

Fig. 1. Beams with topological code: 0 – support at the bar end, 1 – support moved left of the bar end, 2 – support moved right of the bar end

A statically determinate beam with n pin supports has $n-1$ bars. The bars have n endpoints, two external ends and $n-2$ internal, hinged ends. The topology of a statically determinate beam is represented by a vector of topological codes of supports

$$\mathbf{t} = [t_1, \ldots, t_n] \tag{2.1}$$

The topological code t_i describes the location of support i relative to the end of the bar (terminal supports) or two adjacent bars (intermediate supports), Fig. 1. This topology-coding scheme is presented in Rychter and Kozikowska (2009).

The size $|\mathbf{T}^n|$ of the set $\mathbf{T}^n$ of all n-support beam topologies

$$|\mathbf{T}^n| = 2 \cdot \underbrace{3 \cdot \ldots \cdot 3}_{n-2} \cdot 2 = 4 \cdot 3^{n-2} \tag{2.2}$$

and the size $|\mathbf{T}^{2:n}|$ of the set $\mathbf{T}^{2:n}$ of all topologies of beams with two to n supports

$$|\mathbf{T}^{2:n}| = \sum_{i=2}^{n} |\mathbf{T}^n| = \sum_{i=2}^{n} (4 \cdot 3^{i-2}) = 2 \cdot 3^{n-1} - 2 \tag{2.3}$$

grow exponentially with the number of supports.

3. Beam geometry

The geometry of a beam is represented by a set of $2n-1$ parameters divided into two groups. The parameters z_i represent the dimensionless lengths of spans between neighbouring supports. The parameters y_i describe the dimensionless external cantilever lengths and the internal cantilever lengths to span length ratios (Fig. 2).

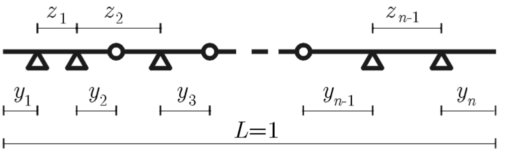

Fig. 2. Beam geometry: span lengths z_i, cantilever lengths y_i

We assume in this work that all the beams have the same length L, normalized to unity

$$L = y_1 + z_1 + z_2 + \ldots + z_{n-1} + y_n = 1 \tag{3.1}$$

A more detailed description of the geometrical parameters is given in Rychter and Kozikowska (2009). The number of nonzero terminal cantilevers (nonzero parameters y_1 and y_n) is equal to c_E. The number of nonzero internal cantilevers (nonzero parameters y_i for $i \in \{2, \ldots, n-1\}$) is equal to c_H.

The minimum number of geometric variables equals the number of supports and hinges, $2n-2$. We use $2n-1$ variables z_i, y_i subjected to constraint (3.1), because this approach is better for geometry optimization by a genetic algorithm.

4. Equivalence relation of beam topologies

4.1. Geometry optimization of the beam with a fixed topology

In this study we concentrate on the topological complexity of beams, which grows exponentially with the number of supports. Therefore, we use simple gravity load distributions: uniform, linear and parabolic.

Let us consider a beam of unit length, with a fixed topology $\mathbf{t}$, under the gravity load. The beam is optimized with respect to geometrical variables. This optimization problem may be stated as follows

$$\begin{aligned} &\text{Minimize} \quad \max_{x\in[0,1]} |M(z_i, y_j, x)| \\ &\text{Subject to} \quad \begin{cases} 0 < z_i < 1 & i = 1, 2, \ldots, n-1 \\ 0 < y_j < 1 & \text{for} \quad t_j \neq 0 \quad j = 1, 2, \ldots, n \\ y_1 + z_1 + z_2 + \ldots + z_{n-1} + y_n = 1 \end{cases} \end{aligned} \tag{4.1}$$

where $\max_{x\in[0,1]}|M(z_i, y_j, x)|$ is the objective function representing the maximum of the absolute bending moment, z_i are the span lengths, y_j denote the nonzero lengths of cantilevers, which are created by movements of supports of nonzero topological codes and x is the axial coordinate. The total number of design variables equals the sum of the number of spans $n-1$ and the number of nonzero external and internal cantilever lengths c_E and c_H, respectively. For a homogeneous beam with a uniform cross-section this optimization process corresponds to design for minimum weight.

A modified version of the genetic algorithm (Rychter and Kozikowska, 2009) is used for optimization of geometrical parameters for all topologies $\mathbf{t}_i$, where $i = 1, 2, \ldots, 4 \cdot 3^{n-2}$ or $i = 1, 2, \ldots, 2 \cdot 3^{n-1} - 2$ in accordance with Eq. (2.2) or Eq. (2.3), respectively. Chromosomes representing n-support beam with fixed topology are vectors of $(n-1)+(c_E+c_H)$ real genes z_i and y_j. Such chromosomes are compact and suitable for genetic operations, particularly crossover and mutation. The minimal value of the absolute maximum bending moment M_i is found as a result of geometry optimization of each beam with topology $\mathbf{t}_i$.

4.2. Definition of equivalence relation of beam topologies

$\mathbf{T}$ is the set of beam topologies: $\mathbf{T}^n$ or $\mathbf{T}^{2:n}$. We define an equivalence relation R on the set $\mathbf{T}$. Any two topologies $\mathbf{t}_i$ and $\mathbf{t}_j$ of the set $\mathbf{T}$ are equivalent with respect to the relation R if the values of the optimal moments M_i and M_j of these topologies are equal

$$\mathbf{t}_i \equiv_R \mathbf{t}_j \qquad \text{if} \;\; M_i = M_j \tag{4.2}$$

The relation R is an equivalence relation because R satisfies the conditions of reflexivity, symmetry and transitivity. The relation R partitions the set $\mathbf{T}^n$ into disjoint subsets $\mathbf{T}_i^n$ called equivalence classes of beam topologies or topological classes. Parameters which concern the class $\mathbf{T}_i^n$ have the superscript n and subscript i. Similarly, the relation R splits the set $\mathbf{T}^{2:n}$ into topological classes $\mathbf{T}_i^{2:n}$.

5. Topological classes for a fixed number of supports under a uniform load

5.1. Optimal bending moment diagram for a fixed topology

A beam of length L from the class $\mathbf{T}_i^n$, with optimal geometry for a fixed topology, is shown in Fig. 3. The beam is found with unique, optimal, uniformly distributed bending moment diagram. All $c^n_{E,i} + c^n_{H,i} + n - 1$ local extreme moment values, at $c^n_{E,i} + c^n_{H,i}$ supports, which were moved away from the ends of bars, and in $n - 1$ spans, are equal to M^n_i.

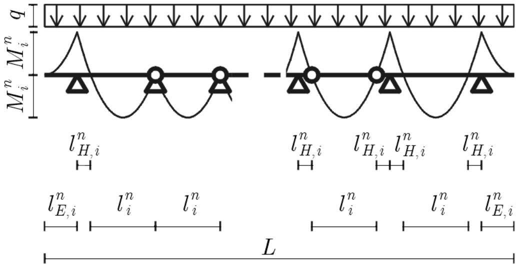

Fig. 3. A beam with optimal geometry for a fixed topology from the class $\mathbf{T}^n_i$ under a uniform load

Geometry optimization of statically determinate beams with a fixed topology, with the absolute maximum bending moment as the objective function, can be found in Imam and Al-Shihri (1996) and Wang (2006). Results of such optimization tasks are given in Siegel (1962), Salvadori and Heller (1975), Kolendowicz (1993) and Allen and Zalewski (2010). All these authors state that to obtain the optimal geometry, it is desirable to equate the moment absolutes at the supports and in spans.

The paper introduces formulas that allow one to calculate the exact optimal geometry for a fixed topology under a uniform load. We can find values of the parameters l^n_i, $l^n_{E,i}$ and $l^n_{H,i}$ (see Fig. 3) solving the system of equations

$$\begin{aligned} &(n-1)l^n_i + c^n_{E,i} l^n_{E,i} + (c^n_{E,i} + 2c^n_{H,i}) l^n_{H,i} = L \\ &\frac{1}{2} l^n_i - l^n_{E,i} = 0 \qquad\qquad (l^n_i)^2 - 4 l^n_i l^n_{H,i} - 4(l^n_{H,i})^2 = 0 \end{aligned} \tag{5.1}$$

where l^n_i is the length of each beam segment with the bottom in tension, $l^n_{E,i}$ denotes the length of each nonzero external cantilever and $l^n_{H,i}$ is the length of each nonzero internal cantilever or the distance between the zero-moment point inside a span and the closest support. The first equation in (5.1) describes the total length of the beam. The second equation represents the comparison between the lengths of a cantilever and a simply supported beam with the same values of the absolute maximum moment. The maximum bending moment value of a simply supported beam of the length $l^n_i + 2l^n_{H,i}$ equals twice this value of a simply supported beam of the length l^n_i in accordance with the third equation.

The solution to system (5.1) is given by

$$l_i^n = \frac{L}{d_i^n} \qquad l_{E,i}^n = \frac{L}{2d_i^n} \qquad l_{H,i}^n = \frac{(\sqrt{2}-1)L}{2d_i^n} \tag{5.2}$$

where $d_i^n = n + 0.5\sqrt{2}c_{H,i}^n + (\sqrt{2}-1)c_{H,i}^n - 1$.

The value of the absolute maximum bending moment M_i^n can be calculated as the moment in the middle of a simply supported, uniformly loaded beam of the length l_i^n

$$M_i^n = \frac{1}{8}q(l_i^n)^2 \tag{5.3}$$

The publications about optimization of statically determinate beams with a uniform load do not contain exact formulas for optimal geometrical parameters, only give approximate values. In Fig. 4, the exact moment values calculated from Eq. (5.3) are compared to results of other authors. The maximum moment values for different optimization tasks are assumed to be all equal to 100%. The comparison reveals the clear advantage of the accurate solutions, found by the author.

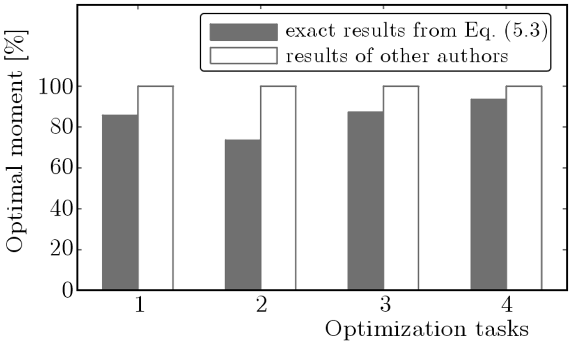

Fig. 4. Comparison of optimization results: 1 – two-support beam from Salvadori and Heller (1975), 2 – six-support beam from Kolendowicz (1993), 3 – two-support beam with one cantilever from Allen and Zalewski (2010), 4 – two-support beam with two cantilevers from Allen and Zalewski (2010)

Figure 3 presents only one variant of topology. An optimal moment diagram can be equivalent to many topologies with unsupported hinges at various points of zero moment, left or right of supports (Fig. 5). The number of different topologies with the same optimal moment diagram $t_{M,i}^n$ equals the number of combinations of $c_{H,i}^n$ unsupported hinges locations

$$t_{M,i}^n = 2^{c_{H,i}^n} \tag{5.4}$$

An optimal moment diagram can correspond to even more numerous optimal geometrical parameter sets, related to the locations of single hinges within spans with two zero-moment points inside (Fig. 5).

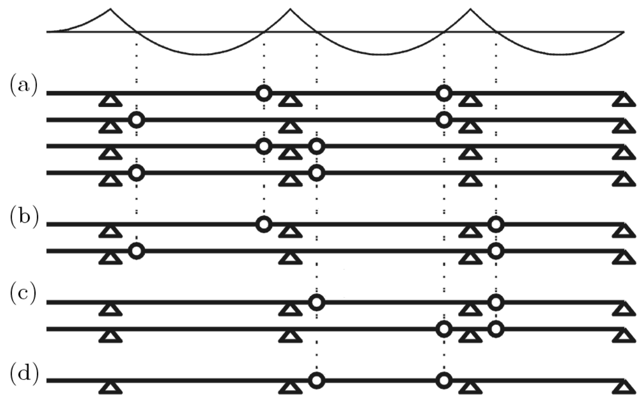

Fig. 5. All topologies and geometries corresponding to the same moment diagram: (a) topology $[2,2,2,0]$, (b) topology $[2,2,1,0]$, (c) topology $[2,1,1,0]$, (d) topology $[2,1,2,0]$

5.2. Features of beam topologies and geometries in a topological class

All optimal bending moment diagrams from a topological class, under a uniform gravity load, are shown in Fig. 6.

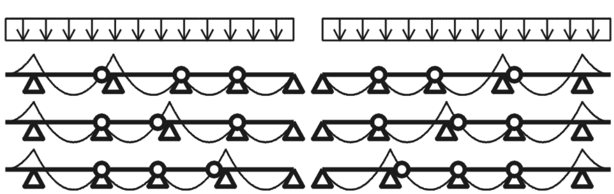

Fig. 6. All optimal moment diagrams in the class $\mathbf{T}_8^5$ under a uniform load

The quality measure of the class $\mathbf{T}_i^n$ is the value of moment M_i^n, which is dependent on the length l_i^n, in accordance with Eq. (5.3). The length l_i^n depends on parameters $c_{E,i}^n$ and $c_{H,i}^n$ (see Eq. (5.2)). Thus for two topologies $\mathbf{t}_i$ and $\mathbf{t}_j$ of the set $\mathbf{T}^n$ under a uniform load the equivalent condition from Eq. (4.2) can be expressed as

$$\mathbf{t}_i \equiv_R \mathbf{t}_j \qquad \text{if} \quad c_{E,i} = c_{E,j} \quad \text{and} \quad c_{H,i} = c_{H,j} \tag{5.5}$$

where $c_{E,i}$, $c_{H,i}$, $c_{E,j}$, $c_{H,j}$ are the numbers of external and internal cantilevers for the topology $\mathbf{t}_i$ and $\mathbf{t}_j$, respectively. All topologies of uniformly loaded beams in the class $\mathbf{T}_i^n$ have the same values of parameters $c_{E,i}^n$ and $c_{H,i}^n$. It does not make any difference which supports are moved away from the beam end and hinges. Thus, the total number of different bending moment diagrams in the class $\mathbf{T}_i^n$, m_i^n equals the product of binomial coefficients

$$m_i^n = \binom{2}{c_{H,i}^n}\binom{n-2}{c_{H,i}^n} \tag{5.6}$$

The total number of different topologies in the class $\mathbf{T}_i^n$, $|\mathbf{T}_i^n|$ equals the product of the number of diverse moment diagrams m_i^n and the number of moves of $c_{H,i}^n$ unsupported hinges $2^{c_{H,i}^n}$

$$|\mathbf{T}_i^n| = m_i^n \cdot 2^{c_{H,i}^n} = \binom{2}{c_{H,i}^n}\binom{n-2}{c_{H,i}^n}2^{c_{H,i}^n} \tag{5.7}$$

The minimal value of the moment M_i^n, has the first class $\mathbf{T}_1^n$ with all supports moved away from the ends of bars. An algebraic formula from Eq. (5.8), found by the author, determines the number of geometry variants in this class. In each of $n-1$ spans of each beam, there are two points of zero bending moment. Depending on the beam topology, in a span there can be none or one or two hinges placed at the zero-moment points. To discover this formula, we must solve the problem of finding all proper placements of $n-2$ hinges in $2(n-1)$ zero crossings of the moment diagram. A beam from this class consists of three types of bars: with two, one or no supports. A beam with $n \geqslant 2$ supports has $n-1$ bars of which $r = 0, 1, \ldots, \mathrm{Floor}[n/2]-1$ can be no-support bars, where the function $\mathrm{Floor}[y]$ gives the greatest integer less than or equal to y. Each pair of two-support bars must be separated by exactly one unsupported bar, thus giving a $(2r+1)$-element chain of $r+1$ two-support bars and r no-support bars between them. Any number of one-support bars can be placed anywhere before, inside and after the $(2r+1)$-element chain. The chain of $2r+1$ elements can be placed arbitrarily in $n-1$ locations of bars, preserving their order in the chain, thus giving $\binom{n-1}{2r+1}$ combinations of possible placements (topologies). The remaining $(n-1)-(2r+1) = n-2-2r$ locations of the total $n-1$ places are occupied by one-support bars. Each such bar creates a span with one hinge inside and each such hinge has two possible locations, at two points of zero moment. This yields 2^{n-2-2r} combinations of single hinge locations (variants of geometry) for each beam topology with

n supports and r no-support bars. Multiplying the number of topologies $\binom{n-1}{2r+1}$ by the number of variants of geometry 2^{n-2-2r} for fixed r, summing over all possible values of r and using *Mathematica* software package to simplify the result, we get the number of optimal geometry variants in the class $\mathbf{T}_1^n$

$$g_1^n = \sum_{r=0}^{\mathrm{Floor}[n/2]-1} \binom{n-1}{2r+1} 2^{n-2-2r} = \frac{3^n - 3}{6} \tag{5.8}$$

5.3. Comparison of topological classes

The whole set of four-support topological classes, under a uniform load, with all optimal moment diagrams is presented in Fig. 7.

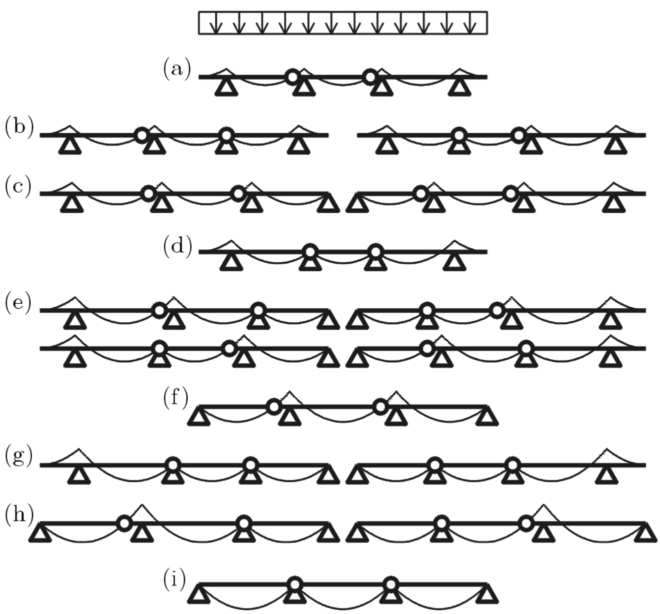

Fig. 7. All four-support topological classes with their optimal moment diagrams under a uniform load: (a) $\mathbf{T}_1^4$, (b) $\mathbf{T}_2^4$, (c) $\mathbf{T}_3^4$, (d) $\mathbf{T}_4^4$, (e) $\mathbf{T}_5^4$, (f) $\mathbf{T}_6^4$, (g) $\mathbf{T}_7^4$, (h) $\mathbf{T}_8^4$, (i) $\mathbf{T}_9^4$

The set of all n-support classes is described by the set of all possible ordered pairs $(c_{E,i}^n, c_{H,i}^n)$ where $c_{E,i}^n \in \{0,1,2\}$ and $c_{H,i}^n \in \{0,1,\ldots,n-2\}$. Thus the set of all classes is characterized by the Cartesian product of the three-element and $(n-1)$-element sets. The total number of classes p^n is the product of the numbers of set members

$$p^n = 3(n-1) \tag{5.9}$$

The set of n-support topological classes can be arranged by their optimal moment values in the monotonically increasing sequence

$$\{\mathbf{T}_k^n\}_{k=1}^{3(n-1)} = \{\mathbf{T}_1^n, \mathbf{T}_2^n, \dots, \mathbf{T}_{3(n-1)}^n\} \tag{5.10}$$

The class $\mathbf{T}_i^n$ precedes the class $\mathbf{T}_j^n$ in the sequence $\{\mathbf{T}_k^n\}$, if M_i^n is smaller than M_j^n. For beams with fewer than five supports, the class $\mathbf{T}_i^n$ precedes the class $\mathbf{T}_j^n$ if the class $\mathbf{T}_i^n$ has more cantilevers

$$c_{E,i}^n + c_{H,i}^n > c_{E,j}^n + c_{H,j}^n \tag{5.11}$$

or the class $\mathbf{T}_i^n$ has more external cantilevers with the same total number of cantilevers

$$c_{E,i}^n > c_{E,j}^n \qquad \text{and} \qquad c_{E,i}^n + c_{H,i}^n = c_{E,j}^n + c_{H,j}^n \tag{5.12}$$

For beams with five or more supports, the topological class $\mathbf{T}_i^n$ precedes the class $\mathbf{T}_j^n$ if condition (5.11) or (5.12) is fulfilled unless the following condition is satisfied

$$\begin{gathered} c_{E,i}^n = 2 \qquad \text{and} \qquad c_{E,j}^n = 0 \qquad \text{and} \\ c_{E,i}^n + c_{H,i}^n = c_{E,j}^n + c_{H,j}^n - 1 = c_{H,j}^n - 1 \end{gathered} \tag{5.13}$$

If the compared classes meet condition (5.13), then $\mathbf{T}_i^n$ immediately precedes $\mathbf{T}_j^n$ although the total number of cantilevers in $\mathbf{T}_j^n$ is greater by one than in $\mathbf{T}_i^n$ (see Fig. 8).

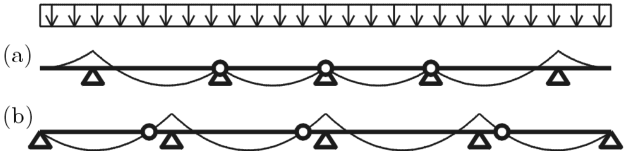

Fig. 8. Two successive topological classes under a uniform load: the class $\mathbf{T}_6^5$ (a) precedes the class $\mathbf{T}_7^5$ (b) with the total number of cantilevers greater by one

The number of cantilevers in the two-support class $\mathbf{T}_i^2$ can be computed from

$$c_{E,i}^2 = 3 - i \qquad\qquad c_{H,i}^2 = 0 \qquad \text{for} \quad i = 1, 2, 3 \tag{5.14}$$

The number of cantilevers in the three-support class $\mathbf{T}_i^3$ is given by the following

$$c_{E,i}^3 = \mathrm{Floor}[(6-i)/2] \qquad c_{H,i}^3 = \mathrm{Mod}[i,2] \qquad \text{for } i = 1,2,\dots,6 \tag{5.15}$$

where the function $\mathrm{Mod}[a,b]$ returns the remainder on division of a by b. The formulas for the number of cantilevers in classes with at least four supports are shown in Table 1.

Table 1. Number of cantilevers in n-support topological classes for $n \geqslant 4$ under a uniform load

$\mathbf{T}_i^n$	$\mathbf{T}_1^n$	$\mathbf{T}_2^n$	$\mathbf{T}_3^n$	$\mathbf{T}_4^n$	$\mathbf{T}_i^n,\ i \in [5, 3n-7]$	$\mathbf{T}_{3n-6}^n$	$\mathbf{T}_{3n-5}^n$	$\mathbf{T}_{3n-4}^n$	$\mathbf{T}_{3n-3}^n$
$c_{E,i}^n$	2	2	1	2	$\mathrm{Mod}[i+2,3]$	0	1	0	0
$c_{H,i}^n$	$n-2$	$n-3$	$n-2$	$n-4$	A	2	0	1	0

$A = (3n-i+1-5\mathrm{Mod}[i+2,3])/3$

Let us consider the sequence of real numbers, $\{r_n\}_{n=2}^{\infty}$, whose members are ratios of moment values of extreme classes

$$r_n = \frac{M_{3(n-1)}^n}{M_1^n} = \frac{1}{2}\Big(\frac{2n+\sqrt{2}-2}{n-1}\Big)^2 \tag{5.16}$$

The sequence $\{r_n\}$ decreases monotonically and converges to the limit 2. This means that the values of class moments become closer to each other with a growing number of supports, but the moment value of the worst class $\mathbf{T}_{3(n-1)}^n$ is always more than twice the value of the best class $\mathbf{T}_1^n$. A growing rapprochement between class moment values is also seen in Fig. 9, which compares the moment values M_i^n of all topological classes for beams with 2, 3, 4 and 5 supports.

The total number of different optimal moment diagrams in all n-support topological classes m^n is the sum of the number of diverse diagrams in each class $\mathbf{T}_i^n$, m_i^n over all possible values of parameters $c_{E,i}^n$ and $c_{H,i}^n$. Substituting m_i^n from Eq. (5.6) and simplifying by software *Mathematica*, we obtain

$$m^n = \sum_{c_{E,i}^n=0}^{2} \sum_{c_{H,i}^n=0}^{n-2} m_i^n = \sum_{c_{E,i}^n=0}^{2} \sum_{c_{H,i}^n=0}^{n-2} \binom{2}{c_{E,i}^n}\binom{n-2}{c_{H,i}^n} = 2^n \tag{5.17}$$

An algorithm which calculates the total number of optimal geometry variants in all n-support topological classes was written by the author. The algorithm

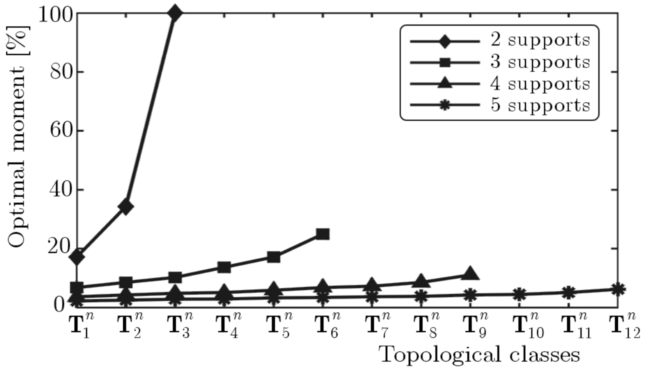

Fig. 9. Optimal moments in topological classes for a fixed number of supports under a uniform load

generates all topological codes of n-support beams and adds up the numbers of geometry variants associated with them. The sequence of numbers of optimal geometries was generated by this algorithm. The recurrence formula for the total number of optimal geometries g^n was found using the website search engine for The On-Line Encyclopedia of Integer Sequences (http://oeis.org/).

$$g^n = \begin{cases} 4 & \text{for} \quad n = 2 \\ 16 & \text{fo} \quad n = 3 \\ 4g^{n-1} - g^{n-2} + 1 & \text{for} \quad n > 3 \end{cases} \tag{5.18}$$

The numbers of topologies and optimal geometry variants are shown in Fig. 10.

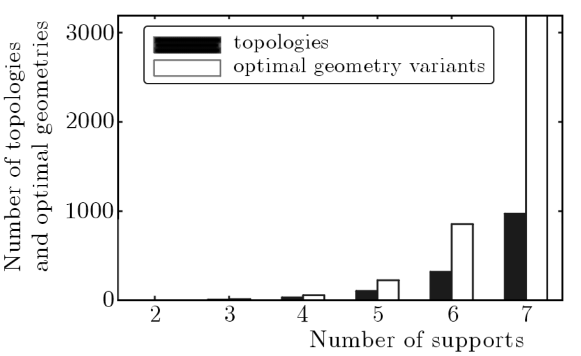

Fig. 10. Number of supports versus number of topologies and optimal geometry variants

6. Topological classes for a fixed number of supports under a non-symmetric linear or parabolic load

Beams with optimal geometry under a non-symmetric load have moment diagrams whose local extreme values are the same, like in the case of beams under a uniform load. Optimal beams for a non-symmetric load, with almost equal absolute values of the support and span moments, can be found in Siegel (1962) and Mróz and Rozvany (1975).

Under a non-symmetric load, there is only one moment diagram in each class. The diverse topologies in the class arise only from different locations of hinges in zero points of this single diagram. The number of topologies is equal to $2^{c^n_{H,i}}$, according to Eq. (5.4). The number of classes is equal to the total number of moment diagrams 2^n, according to Eq. (5.17). Figure 11 shows consecutive topological classes under a linear load. The topologies from Fig. 11 form one class under a uniform load (Fig. 6). For a non-symmetric load, each topology with cantilevers in different places belongs to another class. The class $\mathbf{T}^n_i$ with the same number of external and internal cantilevers as in the class $\mathbf{T}^n_j$, $c^n_{E,i} = c^n_{E,j}$ and $c^n_{H,i} = c^n_{H,j}$, precedes the class $\mathbf{T}^n_j$ if the cantilevers of the class $\mathbf{T}^n_i$ are created under a lower load. If the compared classes $\mathbf{T}^n_i$ and $\mathbf{T}^n_j$ have a different number of external and/or internal cantilevers, then $\mathbf{T}^n_i$ precedes $\mathbf{T}^n_j$ if condition (5.11) or (5,12) is satisfied for $n < 5$ or if condition (5.11) or (5.12) is fulfilled unless condition (5.13) is true for $n \geqslant 5$.

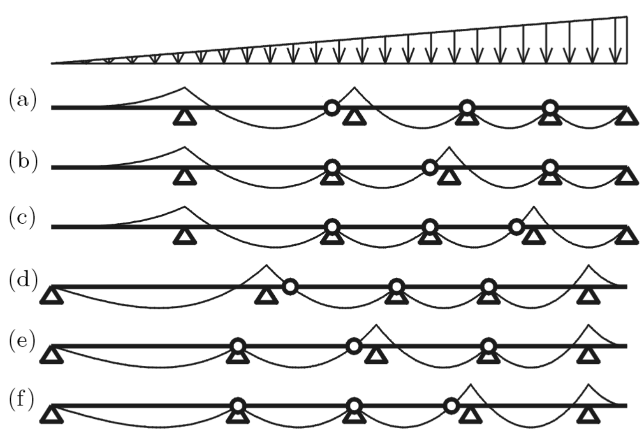

Fig. 11. Successive five-support topological classes with one internal and one external cantilever under a linear load: (a) $\mathbf{T}^5_{18}$, (b) $\mathbf{T}^5_{19}$, (c) $\mathbf{T}^5_{20}$, (d) $\mathbf{T}^5_{21}$, (e) $\mathbf{T}^5_{22}$, (f) $\mathbf{T}^5_{23}$

7. Topological classes for a fixed number of supports under a symmetric parabolic load

Figure 12 shows successive topological classes under a symmetric quadratic load. All the topologies from these classes are members of one class under a uniform load (see Fig. 6) and belong to six classes under a linear load (see Fig. 11). For a symmetric quadratic load, there is a single moment diagram or there are two moment diagrams in the class. The number of different topologies in the class $\mathbf{T}_i^n$ is equal to $2^{c_{H,i}^n}$ or $2^{c_{H,i}^n+1}$.

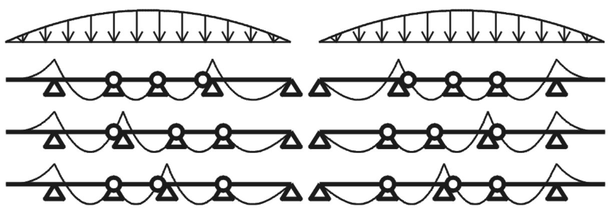

Fig. 12. Successive five-support topological classes with one internal and one external cantilever under a symmetric parabolic load: (a) $\mathbf{T}_{12}^5$, (b) $\mathbf{T}_{13}^5$, (c) $\mathbf{T}_{14}^5$

We need to find the number of topological classes for a symmetric load. The problem, solved by the author, is easiest to analyse for an odd and even number of supports separately. For beams with n supports and c_{EH} cantilevers $(c_{EH} = c_E + c_H)$, the number of all distinct optimal moment diagrams is equal to $\binom{n}{c_{EH}}$. For odd n, among these diagrams $\binom{\text{Floor}[n/2]}{\text{Floor}[c_{EH}/2]}$ are symmetric and $\binom{n}{c_{EH}} - \binom{\text{Floor}[n/2]}{\text{Floor}[c_{EH}/2]}$ are non-symmetric. Symmetric diagrams form classes independently, while each class with non-symmetric diagrams has two such diagrams. We sum over all possible values of c_{EH} and simplify using *Mathematica*. Thus, the number of topological classes for an odd number of supports is equal to

$$
\begin{aligned}
p_{odd}^n = & \underbrace{\sum_{c_{EH}=0}^{n} \frac{1}{2}\left[\binom{n}{c_{EH}} - \binom{\text{Floor}[n/2]}{\text{Floor}[c_{EH}/2]}\right]}_{\substack{\text{number of classes} \\ \text{with two moment diagrams}}} \\
& + \underbrace{\sum_{c_{EH}=0}^{n} \binom{\text{Floor}[n/2]}{\text{Floor}[c_{EH}/2]}}_{\substack{\text{number of classes} \\ \text{with one moment diagram}}} = 2^{n-1} + 2^{\text{Floor}[n/2]}
\end{aligned}
\tag{7.1}
$$

For an even number of supports n and an odd number of cantilevers c_{EH}, all $\binom{n}{c_{EH}}$ optimal moment diagrams are non-symmetric. If n and c_{EH} are even, there are $\binom{n/2}{c_{EH}/2}$ symmetric and $\binom{n}{c_{EH}} - \binom{n/2}{c_{EH}/2}$ non-symmetric diagrams. Replacing an odd c_{EH} with $2k+1$ and an even c_{EH} with $2k$, where k is an integer, summing over all possible values of k and using *Mathematica* to simplify, we finally get the number of classes for an even number of supports

$$\begin{aligned} p^n_{even} = & \underbrace{\sum_{k=0}^{n/2-1} \frac{1}{2}\binom{n}{2k+1}}_{\substack{\text{number of classes}\\ \text{with two moment diagrams}\\ \text{for odd number of cantilevers}}} \\ & \underbrace{+\underbrace{\sum_{k=0}^{n/2} \frac{1}{2}\left[\binom{n}{2k} - \binom{n/2}{k}\right]}_{\substack{\text{number of classes}\\ \text{with two moment diagrams}}} + \underbrace{\sum_{k=0}^{n/2}\binom{n/2}{k}}_{\substack{\text{number of classes}\\ \text{with one moment diagram}}}}_{\text{even number of cantilevers}} = 2^{n-1} + 2^{n/2-1} \end{aligned} \tag{7.2}$$

The number of topological classes under a symmetric parabolic load can be computed from one formula for any number of supports

$$p^n = 2^{n-1} + 2^{\mathrm{Floor}[(n+1)/2]-1} \tag{7.3}$$

8. Comparison of topological classes for a fixed number of supports under different loads

The numbers of classes for three loading types are given in Fig. 13. The number of classes grows linearly with the number of supports for a uniform load and exponentially – for the other two types of load.

The optimal moment values in all classes of three-support beams are presented in Fig. 14. The resultants of all the loading types are the same. The moment values are normalized relative to the largest value in the figure. The order of topological classes for a uniform load remains the same for a non-uniform load. If the topology $\mathbf{t}_i$ belongs to a better class than the topology $\mathbf{t}_j$ for a uniform load, then the topology $\mathbf{t}_i$ also belongs to a better class for the other two loading types. If the topologies $\mathbf{t}_i$ and $\mathbf{t}_j$ are elements of the same class for a uniform load and their topological differences only concern movements of the same supports (one or more) in opposite directions, then

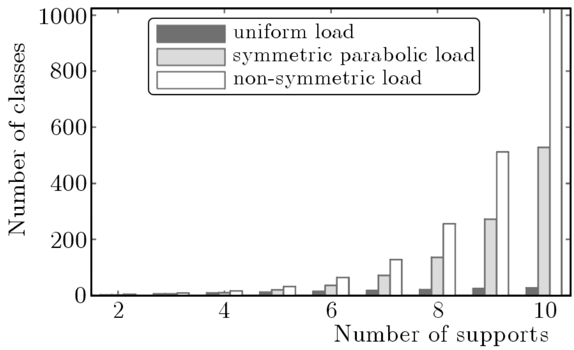

Fig. 13. Number of supports versus number of topological classes for different loads

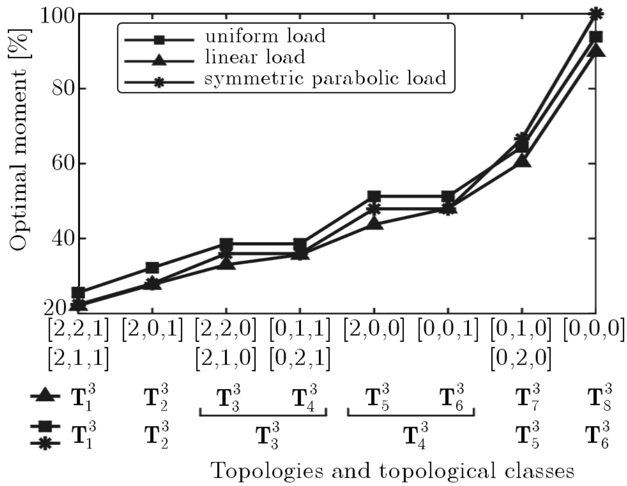

Fig. 14. Comparison of three-support topological classes for different loads

they belong to the same class for the non-uniform load too. If the topologies $\mathbf{t}_i$ and $\mathbf{t}_j$ are members of the same class for a uniform load and have different supports moved away from the ends of bars, then the topologies are members of different classes for a non-symmetric load but for a symmetric load they belong to either the same class or different classes.

9. Topological classes for a different number of supports under a uniform load

Let us consider the set $\mathbf{T}^{2:n}$ consisting of beam topologies with two to n supports and topological classes $\mathbf{T}_i^{2:n}$. The plot in Fig. 15 shows the moment values $M_i^{2:4}$ of all classes for beams with two to four supports under a uniform load. The classes contain topologies with two successive numbers of supports ($\mathbf{T}_6^{2:4}$, $\mathbf{T}_8^{2:4}$ and $\mathbf{T}_{12}^{2:4}$) or topologies with only one number of supports (the remaining classes). The class $\mathbf{T}_6^{2:4}$ with its two optimal moment diagrams is presented in Fig. 16.

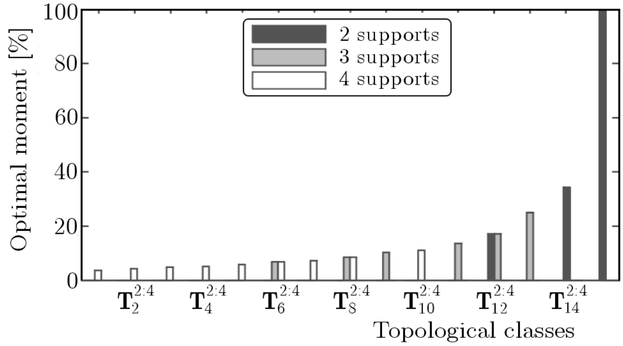

Fig. 15. Optimal moments in topological classes for a different number of supports under a uniform load

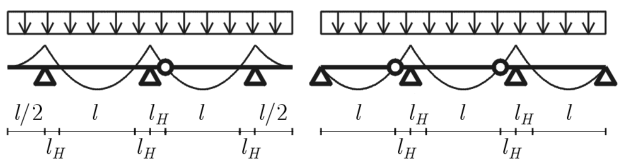

Fig. 16. Class $\mathbf{T}_6^{2:4}$ with its optimal moment diagrams

The two topologies $\mathbf{t}_i^k$ and $\mathbf{t}_j^{k+1}$ with the number of supports k and $k+1$, where $k \in \{2, \ldots, n-1\}$, are in the same class if they meet the condition

$$\mathbf{t}_i^k \equiv_R \mathbf{t}_j^{k+1} \qquad \text{if} \;\; c_{E,i} = 2 \;\wedge\; c_{E,j} = 0 \;\wedge\; c_{H,i} = d \;\wedge\; c_{H,j} = d+1 \tag{9.1}$$

where $d \in \{0, 1, \ldots, k-2\}$ and $c_{E,i}$, $c_{H,i}$, $c_{E,j}$, $c_{H,j}$ are the numbers of external and internal cantilevers for the topology $\mathbf{t}_i^k$ and $\mathbf{t}_j^{k+1}$, respectively. Substituting the numbers of supports and cantilevers into Eq. (5.2), respectively for both topologies, we get the same length of the beam segment with the bottom in tension and the same moment value from Eq. (5.3).

10. Conclusions

Topological optimization, which belongs to the complex area of discrete, combinatorial optimization, usually refers to finding the optimal layout of the structure within a specified design domain. The goal of the present paper is to study the whole space of statically determinate beam topologies and to present not only the best topologies, but also to give the full description of the whole space.

Because all topologies of statically determinate beams are known, an exhaustive search of the space of beam topologies is carried out. The merit function in this search is found as a result of geometry optimization of each beam with a fixed topology. This optimization process is performed by a genetic algorithm, with the absolute maximum bending moment as the objective function, for a uniform, linear and parabolic gravity load. The beams with optimal geometry have uniformly distributed moment diagrams for each topology and load. Under a uniform load, the exact formulas for the locations of supports and hinges of optimal beams have been found for all topologies.

An equality relation between minimum values of the absolute maximum moment has been defined as the equivalence criterion for the classification of beam topologies. This criterion partitions the whole space of topologies into topological classes. Typical features of the classes have been found.

The results of the present work can be used as a guide to the beam structure design. Topological classes found here are worthy of further research with additional design variables, such as cross-sectional and material properties, with other equivalence relations including constraints on strength, stiffness, stability, with more complex load distributions, with multiple load cases and with multiple objective functions.

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Klasy topologiczne statycznie wyznaczalnych belek o dowolnej liczbie podpór

Streszczenie

W pracy omówiono wszystkie topologie statycznie wyznaczalnych belek o dowolnej liczbie przegubowych podpór. Geometrię każdej belki o ustalonej topologii zoptymalizowano za pomocą algorytmu genetycznego z bezwzględnie maksymalnym momentem jako funkcją celu. Relację równości minimalnych wartości tej funkcji zdefiniowano na zbiorze wszystkich topologii jako relację równoważności. Na podstawie tej relacji dokonano podziału zbioru topologii na klasy równoważności, zwane klasami topologicznymi, pod równomiernym, linowym i kwadratowym grawitacyjnym obciążeniem. Przedstawiono szczegółową charakterystykę tych klas. Znaleziono ścisłe wzory na optymalne położenie podpór i przegubów belek obciążonych równomiernie.

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