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INTEGRAL APPROACH FOR TIME DEPENDENT MATERIALS USING FINITE ELEMENT METHOD

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In this work, we present the development of a mathematical approach for the solution of linear, non-ageing viscoelastic materials undergoing mechanical deformation. We use an integral approach based on a discrete spectrum representation for the creep tensor in order to derive the incremental viscoelastic formulation. Integral operators are discretized using finite difference techniques. The incremental viscoelastic constitutive model contains an internal state variable which represents the influence of the whole past history of stress and strain, thus the difficulty of retaining the stress-strain history in numerical solutions is avoided. A complete general formulation of linear viscoelastic stress-strain analysis is developed in terms of increments of stresses and strains. Numerical simulations are included in order to validate the incremental constitutive equations.

Key words: discrete creep spectrum, integral approach, incremental viscoelastic formulation

1. Introduction

The increasing use of viscoelastic materials for structural applications expected to operate for long periods of time requires understanding of their mechanical behavior. Stress and strain analysis of viscoelastic phenomena can be observed in the behavior of most civil engineering materials, such as concrete, wood and bituminous concrete for road constructions. The assumption of viscoelasticity is one of main characterisation of their behavior and time dependence is a complex problem which is of great importance in the determination of stresses and strains in viscoelastic structures. The main problem in computational mechanics is to know the response of such viscoelastic materials taking into account its complete past history of stress and strain or the hereditary loading. Research focuses on material evaluation, characterisation and development of analytical as well as numerical techniques that are capable of accounting for historical effects of stress and environment.

The objective of the current work is to present a three-dimensional finite element formulation that is suitable for the analysis of non-ageing, linear viscoelastic media. This formulation has been incorporated into a three-dimensional FE program. This code is a general purpose tool capable of predicting the response of a structure to complex loading. The phenomena such as creep, relaxation, and creep-and-recovery can all be predicted using this program. The code also includes automated mesh generators which enable convenient grid generation for problems involving complex geometry.

The criterion used in the selection of papers for discussion herein was that the focus be on incremental differential or integral formulations. Among the several formulations proposed in the literature, Chazal and Moutou Pitti (2010b) and Ghazlan *et al.* (1995) have developed incremental formulation using differential operators. Central to the method was the assumption that the creep compliance could be separated into an "elastic part" and a "creep part" and that the stress could be considered to remain constant across a time step. Chazal and Moutou Pitti (2009a,b, 2010a) have proposed incremental constitutive equations for ageing viscoelastic materials in the finite element context. They discussed conditions under which the requirement of storing all previous solutions could be avoided. Taylor *et al.* (1970) and Christensen (1980a,b) developed integral constitutive models based on the Volterra hereditary integral equations. The current solution in such methods is history dependent. Krishnaswamy *et al.* (1990, 1992) represent modifications of the basic initial strain method which employ variable stiffness. These methods are much more stable but the stiffness matrix changes with time. In fracture viscoelastic mechanics, Chazal and Dubois (2001), Dubois *et al.* (1998, 2002) and recently Moutou Pitti *et al.* (2008) have applied the incremental viscoelastic formulation, initially proposed by Ghazlan *et al.* (1995), in order to evaluate crack growth process in wood. However, the formulation used is based on the spectral decomposition technique using a generalized Kelvin Voigt model. Applications of the finite element method in the field of viscoelastic fracture mechanics have been presented by Krishnaswamy *et al.* (1990), Moran and Knauss (1992) and Chazal and Moutou Pitti (2009a). Brinson and Knauss

(1992) have used the finite element method to conduct an investigation of viscoelastic micromechanics.

Analytical solutions, proposed in the literature, are important to the current discussion. Duenwald *et al.* (2009) developed constitutive equations for ligament using experimental tests. Chazal and Moutou Pitti (2009b), Stouffer and Wineman (1972) developed new constitutive equations for linear ageing viscoelastic materials taking into account environmental dependent viscoelastic behavior. The problem of nonlinear viscoelastic problems has been addressed in the work of Merodio (2006) and Filograna *et al.* (2009). These analytical techniques are obviously limited the method to the solution of relatively simple problems. Most numerical methods cannot deal with complex viscoelastic problems, because these methods require the retaining of the complete past history of stress and strain. In a recent work, Chazal and Moutou Pitti (2010b) have already proposed incremental viscoelastic formulation based on the creep and relaxation differential approach. Differential operators were used to discretize the Volterra integral equations. These methods involve a stepwise integration through time. The key to accomplishing this was the use of a Dirichlet-Prony series (in this case a Kelvin model) to represent the kernel of the Volterra integral equation. These methods made the solution of 'large' viscoelastic problems possible. For a much more in-depth review, the reader is referred to Chazal and Moutou Pitti (2010b).

An efficient incremental formulation, based on discrete creep spectrum, using integral equations in the time domain, is presented. The proposed incremental stress strain constitutive equations are not restricted to isotropic materials and can be used to resolve complex viscoelastic problems. The formulation is developed to deal with three dimensional viscoelastic problems and the solutions of a particular time are found from those at the previous time, this leads to great savings in the amount of computer storage requirements needed to solve real problems involving three dimensional loading.

2. Incrementalization of the viscoelastic equations

The material is assumed as non-ageing which allows a time independence of instantaneous and long term mechanical properties. This is in agreement with constant environmental conditions. In this case, the fourth-order creep tensor $\mathbf{J}(t)$ is written in terms of discrete creep spectrum according to the results of Mandel (1978) and Christensen (1980a,b)

$$\mathbf{J}(t) = \Big[\mathbf{J}^{(0)} + \sum_{m=1}^{M} \Big(1 - \mathrm{e}^{-\lambda_m t}\Big)\mathbf{J}^{(m)}\Big] H(t) \tag{2.1}$$

In the above, the creep tensor is fitted with a Wiechert model. $\mathbf{J}^{(0)}$ and $\mathbf{J}^{(m)}$, $m = 1, \dots, M$, denote for tensors of the fourth rank, $H(t)$ is the Heaviside unit step function and λ_m are positive scalars. The term $\mathbf{J}^{(0)}$ is an elastic compliance while $\mathbf{J}^{(m)}$ is a creep compliance function. $\mathbf{J}^{(0)}$ and $\mathbf{J}^{(m)}$ should be determined in order to represent any particular creep function of interest using experimental data from creep tests.

With the linearity assumption, the constitutive equation can be expressed in the time domain by the hereditary Volterra integral equation. It defines the relationship between strain and stress components, respectively, ε_{ij} and σ_{kl}. According to Boltzmann's principle superposition (Boltzmann, 1878) applied to linear non-ageing viscoelastic material, the constitutive law may be expressed in a tensor notation as

$$\varepsilon_{ij}(t) = \sum_{k=1}^{3}\sum_{l=1}^{3} \int_{-\infty}^{t} J_{ijkl}(t-\tau)\frac{\partial \sigma_{kl}(\tau)}{\partial \tau}\, d\tau \tag{2.2}$$

in which τ is the time variable; and t is the time since loading.

In equation (2.2), it is assumed that the body is in a stress-free state for $t < 0$ and the initial response (elastic) is made implicit. The term J_{ijkl} represents the fourth-order tensor of creep moduli relating stress to mechanical strain and t is referred to as the current time. Equation (2.2) shows that the strain at any given time depends upon the entire "stress history" $\sigma_{kl}(\tau)$, $\tau < t$. The integral in equation (2.2) is called the hereditary integral. The reader will recognize from the form of the constitutive relationship that we have assumed the material to be non-ageing and non-homogeneous. The constitutive relationship given by (2.2) is not suitable for numerical calculus in the context of the finite element method because this leads to the requirement of solving a set of Volterra integrals in order to extract the finite element solution. A different approach will be presented in this work in order to simplify the solution of numerical equations for the simulation of viscoelastic behavior. For this reason, constitutive equation (2.2) is discretized in order to establish incremental constitutive equations and will lead to the requirement of solving a simple set of algebraic equations. Thus the difficulty of retaining the whole past history in computer solution is avoided. A similar approach has been taken by Ghazlan *et al.* (1995) and Zocher *et al.* (1997). In preparation for the developments to follow, we assume that the period of loading is subdivided into discrete intervals Δt_n (time step) such that $t_{n+1} = t_n + \Delta t_n$. Here we will describe the solution process of a step-by-step nature in which the loads are applied stepwise at various time intervals. According to equation (2.2), the strain at time t_n and t_{n+1} may be written as

$$\begin{aligned}
\varepsilon_{ij}(t_n) &= \sum_{k=1}^{3}\sum_{l=1}^{3}\int\limits_{-\infty}^{t_n} J_{ijkl}(t_n-\tau)\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau \\
\varepsilon_{ij}(t_{n+1}) &= \sum_{k=1}^{3}\sum_{l=1}^{3}\int\limits_{-\infty}^{t_{n+1}} J_{ijkl}(t_{n+1}-\tau)\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau
\end{aligned} \tag{2.3}$$

In order to establish the incremental viscoelastic formulation, equation $(2.3)_2$ is separated into a sum as follows

$$\begin{aligned}
\varepsilon_{ij}(t_{n+1}) &= \sum_{k=1}^{3}\sum_{l=1}^{3}\int\limits_{-\infty}^{t_n} J_{ijkl}(t_{n+1}-\tau)\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau \\
&+ \sum_{k=1}^{3}\sum_{l=1}^{3}\int\limits_{t_n}^{t_{n+1}} J_{ijkl}(t_{n+1}-\tau)\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau
\end{aligned} \tag{2.4}$$

The first integral represents the hereditary response, while the second integral deals with the implicit pseudo instantaneous response of the material. Using equations (2.3) and (2.4), the following incremental viscoelastic equation is obtained

$$\begin{aligned}
\varDelta\varepsilon_{ij}(t_n) &= \sum_{k=1}^{3}\sum_{l=1}^{3}\int\limits_{-\infty}^{t_n} [J_{ijkl}(t_{n+1}-\tau)-J_{ijkl}(t_n-\tau)]\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau \\
&+ \sum_{k=1}^{3}\sum_{l=1}^{3}\int\limits_{t_n}^{t_{n+1}} J_{ijkl}(t_{n+1}-\tau)\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau
\end{aligned} \tag{2.5}$$

in which $\varDelta\varepsilon_{ij}(t_n)=\varepsilon_{ij}(t_{n+1})-\varepsilon_{ij}(t_n)$ is the total strain increment during the time step $\varDelta t_n$. In order to simplify the incremental viscoelastic law, let us define the influence of the complete past history of stress by the hereditary integral as follows

$$\widetilde{\varepsilon}_{ij}(t_n) = \sum_{k=1}^{3}\sum_{l=1}^{3}\int\limits_{-\infty}^{t_n} [J_{ijkl}(t_n+\varDelta t_n-\tau)-J_{ijkl}(t_n-\tau)]\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau \tag{2.6}$$

$\widetilde{\varepsilon}_{ij}(t_n)$ is the memory term which involves the whole past solutions. Insertion of equation (2.6) into equation (2.5) results in

$$\Delta\varepsilon_{ij}(t_n) = \sum_{k=1}^{3}\sum_{l=1}^{3}\int\limits_{t_n}^{t_{n+1}} J_{ijkl}(t_{n+1}-\tau)\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau + \widetilde{\varepsilon}_{ij}(t_n) \tag{2.7}$$

In the next Section, the integral in this equation is discretized in order to make it completely incremental.

3. Incremental constitutive equations

The incremental viscoelastic equations will be derived using a linear approximation of the stress. In fact, we assume that the time derivative during each time increment is constant, a staircase function.

$$\forall\tau\in[t_n, t_n+\Delta t_n] \;\Rightarrow\; \sigma_{kl}(\tau) = \sigma_{kl}(t_n) + \frac{\tau - t_n}{\Delta t_n}\Delta\sigma_{kl}(t_n)H(\tau - t_n) \tag{3.1}$$

in which $\Delta\sigma_{kl}(t_n) = \sigma_{kl}(t_{n+1}) - \sigma_{kl}(t_n)$ is the total stress increment during the time step Δt_n.

Substituting into equation (2.7) the linear approximation of the stress increment given by equation (3.1) and using the exponential approximation for the creep compliance components given by equation (2.1), we obtain

$$\begin{aligned}\Delta\varepsilon_{ij}(t_n) &= \sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\Delta\sigma_{kl}(t_n)}{\Delta t_n}\\ &\cdot\int\limits_{t_n}^{t_n+\Delta t_n}\Big\{J_{ijkl}^{(0)} + \sum_{m=1}^{M} J_{ijkl}^{(m)}\Big[1 - \mathrm{e}^{-\lambda_{ijkl}^{(m)}(t_n+\Delta t_n-\tau)}\Big]\Big\}\,d\tau + \widetilde{\varepsilon}_{ij}(t_n)\end{aligned} \tag{3.2}$$

This equation may be integrated in closed form to produce

$$\Delta\varepsilon_{ij}(t_n) = \sum_{k=1}^{3}\sum_{l=1}^{3}\Pi_{ijkl}(\Delta t_n)\Delta\sigma_{kl}(t_n) + \widetilde{\varepsilon}_{ij}(t_n) \tag{3.3}$$

$\Pi_{ijkl}(\Delta t_n)$ can be interpreted as the viscoelastic compliance tensor and is given by

$$\Pi_{ijkl}(\Delta t_n) = J_{ijkl}^{(0)} + \sum_{m=1}^{M} J_{ijkl}^{(m)}\Big[1 - \frac{1}{\Delta t_n\lambda_{ijkl}^{(m)}}\Big(1 - \mathrm{e}^{-\lambda_{ijkl}^{(m)}\Delta t_n}\Big)\Big] \tag{3.4}$$

The memory term $\widetilde{\varepsilon}_{ij}(t_n)$ in equation (3.3), which involves the whole past solutions, is given by equation (2.6) and defines the influence of the complete past history of stress by a Volterra hereditary integral. Our purpose is now the conversion of this hereditary integral to a more convenient form in order to integrate it in a finite element solution. We now introduce Wiechert model (2.1) onto the formulation by way of substitution into (2.6); doing so yields

$$\widetilde{\varepsilon}_{ij}(t_n) = \sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{M}\Big(1 - \mathrm{e}^{-\lambda_{ijkl}^{(m)}\Delta t_n}\Big)\int\limits_{-\infty}^{t_n} J_{ijkl}^{(m)}\mathrm{e}^{-\lambda_{ijkl}^{(m)}(t_n-\tau)}\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau \tag{3.5}$$

This may also be written as

$$\widetilde{\varepsilon}_{ij}(t_n) = \sum_{k=1}^{3}\sum_{l=1}^{3}\varPsi_{ijkl}(t_n) \tag{3.6}$$

where

$$\varPsi_{ijkl}(t_n) = \sum_{m=1}^{M}\Big(1 - \mathrm{e}^{-\lambda_{ijkl}^{(m)}\Delta t_n}\Big)\varPhi_{ijkl}^{(m)}(t_n) \tag{3.7}$$

In the above, $\varPsi_{ijkl}(t_n)$ is a fourth-order viscoelastic tensor which represents the influence of the whole past history of stress and $\varPhi_{ijkl}^{(m)}(t_n)$ is a pseudo viscoelastic tensor corresponding to one of the spectrum range of the decomposition given by equation (2.1). It is given by

$$\varPhi_{ijkl}^{(m)}(t_n) = \int\limits_{-\infty}^{t_n} J_{ijkl}^{(m)}\mathrm{e}^{-\lambda_{ijkl}^{(m)}(t_n-\tau)}\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau \tag{3.8}$$

Let us divide the domain of integration $-\infty \leqslant \tau \leqslant t_n$ in equation (3.8) into two parts: $-\infty \leqslant \tau \leqslant t_n - \Delta t_n$ and $t_n - \Delta t_n \leqslant \tau \leqslant t_n$. This leads to

$$\begin{aligned}\varPhi_{ijkl}^{(m)}(t_n) = &\int\limits_{-\infty}^{t_n-\Delta t_n} J_{ijkl}^{(m)}\mathrm{e}^{-\lambda_{ijkl}^{(m)}(t_n-\Delta t_n+\Delta t_n-\tau)}\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau \\ &+ \int\limits_{t_n-\Delta t_n}^{t_n} J_{ijkl}^{(m)}\mathrm{e}^{-\lambda_{ijkl}^{(m)}(t_n-\tau)}\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau\end{aligned} \tag{3.9}$$

We assumed that the partial derivative $\partial\sigma_{kl}(\tau)/\partial\tau$ appearing in equation (3.8) can be approximated as $\Delta\sigma_{kl}/\Delta t_n$. The last equation may be integrated in closed form to produce

$$
\begin{aligned}
\varPhi_{ijkl}^{(m)}(t_n) &= \varPhi_{ijkl}(t_{n-1})\mathrm{e}^{-\lambda_{ijkl}^{(m)}\Delta t_n} \\
&+ \frac{\Delta\sigma_{kl}(t_{n-1})}{\Delta t_{n-1}} \int\limits_{t_n-\Delta t_n}^{t_n} J_{ijkl}^{(m)}\mathrm{e}^{-\lambda_{ijkl}^{(m)}(t_n-\tau)}\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\,d\tau
\end{aligned}
\tag{3.10}
$$

The pseudo viscoelastic tensor at time t_n may now be expressed as

$$
\varPhi_{ijkl}^{(m)}(t_n) = \varPhi_{ijkl}^{(m)}(t_{n-1})\mathrm{e}^{-\lambda_{ijkl}^{(m)}\Delta t_n} + \frac{\Delta\sigma_{kl}(t_{n-1})J_{ijkl}^{(m)}}{\Delta t_{n-1}\lambda_{ijkl}^{(m)}}\Big[1-\mathrm{e}^{-\lambda_{ijkl}^{(m)}\Delta t_n}\Big] \tag{3.11}
$$

Finally, the incremental constitutive law given by equation (3.3) can now be inverted to obtain

$$
\Delta\sigma_{ij}(t_n) = \sum_{k=1}^{3}\sum_{l=1}^{3}\varXi_{ijkl}(\Delta t_n)\Delta\varepsilon_{kl}(t_n) - \widetilde{\sigma}_{ij}(t_n) \tag{3.12}
$$

where $\varXi_{ijkl} = (\varPi_{ijkl})^{-1}$ is the inverse of the compliance tensor and $\widetilde{\sigma}_{ij}(t_n)$ is a pseudo stress tensor which represent the influence of the complete past history of strain. It is given by

$$
\widetilde{\sigma}_{ij}(t_n) = \sum_{k=1}^{3}\sum_{l=1}^{3}[\varPi_{ijkl}(\Delta t_n)]^{-1}\widetilde{\varepsilon}_{ij}(t_n) \tag{3.13}
$$

The incremental constitutive law represented by equation (3.13) is introduced in a finite element discretisation in order to obtain solutions to complex viscoelastic problems.

4. Finite element discretization

The finite element method is used to implement the proposed approach. This allows us to resolve complex viscoelastic problems with real boundary conditions.

4.1. Virtual displacement principle in linear viscoelasticity

A standard finite element procedure is used as explained in Zienkiewicz (1958) and Ghazlan *et al.* (1995). Incremental equilibrium equations for the viscoelastic problem are obtained through the principle of virtual work (Ghazlan *et al.*, 1995). For a three dimensional continuum problem, it yields for any element

$$\begin{aligned}\int\limits_{V^e} \sigma_{ij}(t_{n+1})\delta\varepsilon_{ij}(t_{n+1})\;dV &= \int\limits_{V^e} f_i^v(t_{n+1})\delta u_i(t_{n+1})\;dV\\ &+ \int\limits_{\partial V^e} T_i(t_{n+1})\delta u_i(t_{n+1})\;dS\end{aligned} \tag{4.1}$$

Substituting incremental relations

$$\begin{array}{ll}\sigma_{ij}(t_{n+1}) = \sigma_{ij}(t_n) + \Delta\sigma_{ij} & \qquad \varepsilon_{ij}(t_{n+1}) = \varepsilon_{ij}(t_n) + \Delta\varepsilon_{ij}\\ u_i(t_{n+1}) = u_i(t_n) + \Delta u_i & \end{array} \tag{4.2}$$

into Eq. (4.1) and observing that $\delta\varepsilon_{ij}(t_{n+1}) = \delta(\Delta\varepsilon_{ij})$, $\delta u_i(t_{n+1}) = \delta(\Delta u_i)$ yield

$$\begin{aligned}\int\limits_{V^e} \Delta\sigma_{ij}\delta(\Delta\varepsilon_{ij})\;dV &= \int\limits_{V^e} f_i^v(t_{n+1})\delta(\Delta u_i)\;dV\\ &+ \int\limits_{\partial V^e} T_i(t_{n+1})\delta(\Delta u_i)\;dS - \int\limits_{V^e} \sigma_{ij}(t_n)\delta(\Delta\varepsilon_{ij})\;dV\end{aligned} \tag{4.3}$$

Note that the virtual work made by load increments during the time step Δt_n is given by the difference between the two first terms and the third term on the right-hand side of equation (4.3). Introducing incremental viscoelastic equation (3.12) into the first hand side of equation (4.3), the following expression can be obtained

$$\begin{aligned}\int\limits_{V^e} \Xi_{ijkl}(\Delta t_n)\Delta\varepsilon_{kl}\delta(\Delta\varepsilon_{ij})\;dV &= \int\limits_{V^e} f_i^v(t_{n+1})\delta(\Delta u_i)\;dV\\ + \int\limits_{\partial V^e} T_i(t_{n+1})\delta(\Delta u_i)\;dS &- \int\limits_{V^e} \sigma_{ij}(t_n)\delta(\Delta\varepsilon_{ij})\;dV + \int\limits_{V^e} \widetilde{\sigma}_{ij}(t_n)\delta(\Delta\varepsilon_{ij})\;dV\end{aligned} \tag{4.4}$$

where Δu_i is the incremental displacement field between t_n and t_{n+1}, $f_i^v(t_{n+1})$ are the body forces per unit volume, $T_i(t_{n+1})$ is the surface tensions per unit surface, δ is the variation symbol and V^e is the volume of the element. Assuming small displacements, the strains are derived from shape functions using a standard manner in the context of the finite element method. Using matrix notation, the strain increment can be written as

$$\{\Delta\varepsilon_{ij}\} = [D]\{\Delta U^e\} \tag{4.5}$$

where ΔU^e is the local element displacement increment and $[D]$ is the strain-displacement transformation matrix.

4.2. Viscoelastic stiffness matrix computation

The equilibrium equations, in matrix notation, can be simplified by introducing the approximation of the strain components given by equation (4.5) into equation (4.4). This leads to

$$\begin{aligned}\int_{V^e} [D][\delta(\Delta U^e)]^\top [\Theta(\Delta t_n)][D][\Delta U^e]\, dV &= \int_{V^e} [\delta(\Delta[U^e)]^\top \{f_i^v(t_{n+1})\}\, dV \\ &+ \int_{\partial V^e} [\delta(\Delta U^e)]^\top \{T_i(t_{n+1})\}\, dS - \int_{V^e} [D][\delta(\Delta U^e)]^\top \sigma_{ij}\, dV \\ &+ \int_{V^e} [D][\delta(\Delta U^e)]^\top \widetilde{\sigma}_{ij}(t_n)\, dV\end{aligned} \tag{4.6}$$

Interpolation functions must be used in order to obtain equilibrium equations in discrete form. The displacement field ΔU^e is approximated by

$$\{\Delta U^e\} = [N^e]\{\Delta q^e\} \tag{4.7}$$

where $[N^e]$ is the matrix of shape functions and $\{\Delta q^e\}$ are the nodal element displacement increment. Substituting equation (4.7) into equation (4.6), we get

$$\begin{aligned}\int_{V^e} [B_L]^\top [\Theta(\Delta t_n)][B_L]\{\Delta q^e\}\, dV &= \int_{V^e} [N^e]^\top \{f^v(t_{n+1})\}\, dV \\ &+ \int_{\partial V^e} [N^e]^\top \{T(t_{n+1})\}\, dS - \int_{V^e} [B_L]^\top \{\sigma(t_n)\}\, dV + \int_{V^e} [B_L]^\top \{\widetilde{\sigma}(t_n)\}\, dV\end{aligned} \tag{4.8}$$

where $[B_L] = [D][N^e]$ is the total strain-displacement transformation matrix. These equilibrium equations for linear viscoelastic behavior can be rewritten as

$$[K_T(\Delta t_n)]\{\Delta q^e\} = \{F^v(t_{n+1})\} + \{F^s(t_{n+1})\} - \{F^\sigma(t_n)\} + \{F^{vis}(t_n)\} \tag{4.9}$$

Difference between $\{F^v(t_{n+1})\} + \{F^s(t_{n+1})\}$ and $\{F^\sigma(t_n)\}$ is the external load increment during time step Δt_n while $\{F^{vis}(t_n)\}$ is the viscous load vector corresponding to the complete past history. Equation (4.9) can then be simplified to obtain

$$[K_T(\Delta t_n)]\{\Delta q^e\} = \{\Delta F^{ext}(t_n)\} + \{F^{vis}(t_n)\} \tag{4.10}$$

where the stiffness matrix $[K_T(\Delta t_n)]$ is given by

$$[K_T\Delta(t_n)] = \int\limits_{V^e} [B_L]^\top [\Theta(\Delta t_n)][B_L]\, dV \tag{4.11}$$

and the nodal force vector is denoted as the sum of two vectors: $\{\Delta F^{ext}(t_n)\}$ is the external load increment vector, $\{F^{vis}(t_n)\}$ is the memory load vector and $[\Theta(\Delta(t_n)]$ is the viscoelastic constitutive matrix. The viscous load vector increment is given by

$$\{F^{vis}(t_n)\} = \int\limits_{V^e} [B_L]^\top \{\widetilde{\sigma}(t_n)\}\, dV \tag{4.12}$$

4.3. Incremental viscoelastic algorithm

The formulation is introduced in the software Cast3m used by the French Energy Atomic Agency. The software can be employed for plane linear viscoelasticity structures. The global incremental procedure for the creep integral approach is described as below:

1. The instantaneous response (elastic solution) to the applied load is first obtained. Thus at the time $t = t_0$, the stress σ_0, the displacements q_0 and the strains ε_0 are all known.
2. Given a solution at the time t_n. That is, given the stresses $\sigma(t_n)$ and displacements $q(t_n)$ that satisfy the equilibrium equations, our objective is to determine the solution at the time $t_{n+1} = t_n + \Delta t_n$. Loop over the time interval of study
   (a) compute the compliance moduli $\Pi_{ijkl}(\Delta t_n)$ from equation (3.4) and the tangent moduli $\Xi_{ijkl}(\Delta t_n)$ from the relation $\Xi_{ijkl}(\Delta t_n) = [\Pi_{ijkl}(\Delta t_n)]^{-1}$ and then get the viscoelastic constitutive matrix $[\Theta(\Delta(t_n)] = [\Xi_{ijkl}(\Delta t_n)]$
   (b) determine the pseudo stress tensor $\widetilde{\sigma}_{ij}(t_n)$ from equation (3.13) and then compute the viscous load vector $\{F^{vis}(t_n)\}$ from equation (4.12)
   (c) update the linear viscoelastic stiffness matrix $[K_T(\Delta t_n)]$ from equation (4.11)
   (d) assemble and solve the system of equilibrium equations (4.10) to obtain the displacement increment $\Delta q(\Delta t_n)$
   (e) compute the strain increment $\Delta\varepsilon^e(\Delta t_n)$ from equations (4.10)
   (f) use the result of step (b) to compute the stress increment $\Delta\sigma^e(\Delta t_n)$ from equation (3.12)

(g) evaluate the pseudo viscoelastic tensor $\Phi_{ijkl}^{(m)}(t_n)$ at time t_{n+1} from equation (3.11)

(h) compute the memory term $\widetilde{\varepsilon}_{ij}(t_{n+1})$ from equations (3.6) and (3.7)

(i) update the state (displacement, stress and strain) at the end of the time increment Δt_n:

$$\begin{aligned}\{q(t_{n+1})\} &= \{q(t_n)\} + \{\Delta q(\Delta t_n)\}\\ \{\sigma_{ij}(t_{n+1})\} &= \{\sigma_{ij}(t_n)\} + \{\Delta\sigma_{ij}(\Delta t_n)\}\\ \{\varepsilon_{ij}(t_{n+1})\} &= \{\varepsilon_{ij}(t_n)\} + \{\Delta\varepsilon_{ij}(\Delta t_n)\}\end{aligned}$$

(j) go to step (a)

5. Numerical applications

5.1. Viscoelastic plane stress panel

This example is used to check the validity of the creep incremental constitutive formulation proposed in this work. The structure analyzed is a plane stress panel subjected to compressive load in the x direction and transverse load in the y direction distributed along the length L. The panel is made of a

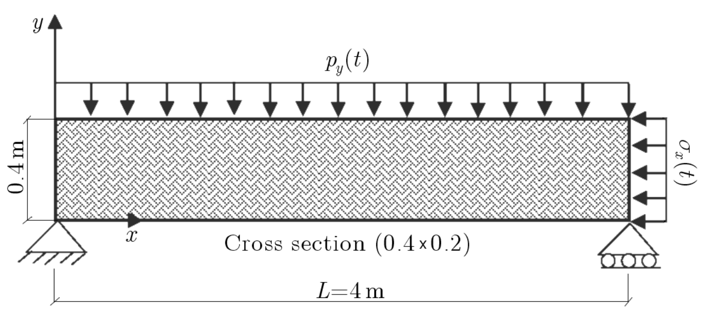

Fig. 1. Viscoelastic panel submitted to axial and transverse load

homogeneous and isotropic viscoelastic material with dimensions specified in Fig. 1 and is pinned at the ends. Horizontal and vertical loads are applied:

- Transverse load in the y direction: $p_y(t) = 32\,\mathrm{kN/m}\ \ \forall t \geqslant 0$
- Compressive load in the x direction (t in hours):

$$\sigma_x(t) = \begin{cases} \sigma_0 H(t) & t \leqslant 400 \ \text{ and } \ t \geqslant 600 \\ 0 & 400 < t < 600 \end{cases}$$

The initial stress applied at time $t = 0$ is given by $\sigma_x(t = 0) = 3\,\text{MPa}$. The creep function of the constitutive material is represented by equation (2.1) with $M = 10$. The constants of the creep function are shown in Table 1, and the analytical solution of this constant stress plate is given by $u(y,t) = 5.96[1 + 0.0135t^{0.3358}]$.

Table 1. Constants used in creep function

J_i	λ_i
$4.65 \cdot 10^{-7}$	1520
$7.25 \cdot 10^{-10}$	133
$9.17 \cdot 10^{-10}$	11.7
$9.43 \cdot 10^{-7}$	1.02
$4.31 \cdot 10^{-7}$	$8.95 \cdot 10^{-2}$
$5.21 \cdot 10^{-6}$	$7.68 \cdot 10^{-3}$
$2.44 \cdot 10^{-7}$	$6.90 \cdot 10^{-4}$
$2.28 \cdot 10^{-5}$	$6.05 \cdot 10^{-5}$
$1.40 \cdot 10^{-5}$	$5.30 \cdot 10^{-6}$
$8.77 \cdot 10^{-5}$	$4.64 \cdot 10^{-7}$

Figure 2 shows a comparison between the analytical and numerical results for the deflection at the center of the panel and its variation with time. It can be observed that the numerical results coincide with the analytical values and a good result is achieved.

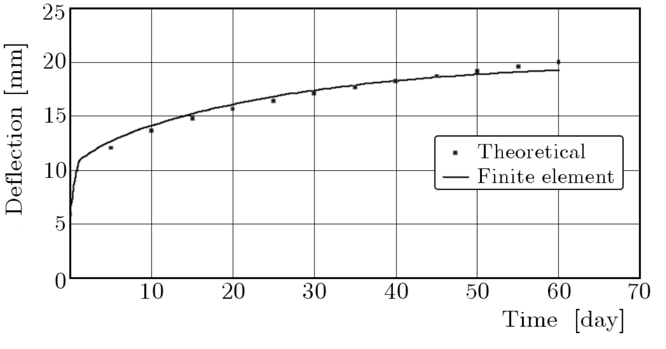

Fig. 2. Deflection in viscoelastic panel versus time

In Fig. 3, we present the percentage change in creep deflection

$$\Big[\frac{u(y,t)-u(y,t=0)}{u(y,t=0)}\Big]\times 100$$

After 4 days, the creep displacement is equal to the instantaneous displacement. It is equal to 200% after 42 days.

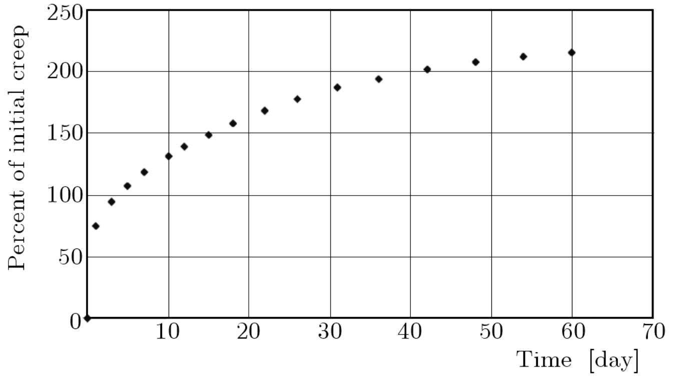

Fig. 3. Creep deflection as a percentage of elastic deflection

The results of the numerical process for the compressive load are shown in Fig. 4. The displacement at the right end of the panel in the x direction is plotted versus time.

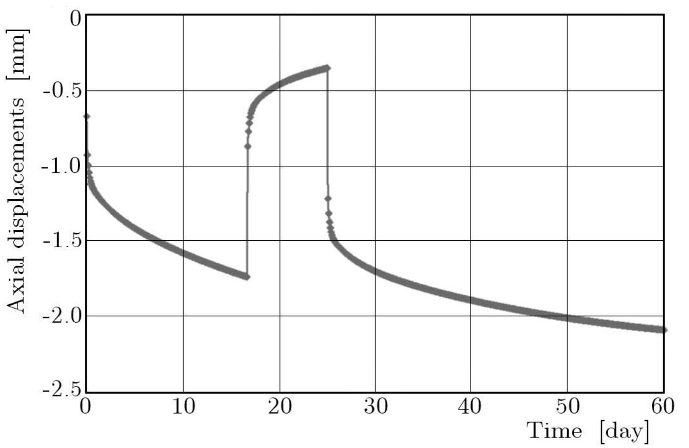

Fig. 4. Axial displacement in viscoelastic panel versus time

5.2. Viscoelastic cylinder bonded to thin elastic case

In this example, a viscoelastic hollow cylinder bonded to a thin elastic case is analyzed when subjected to an internal pressure p. The cylinder is representative of a solid propellant rocket motor. The fuel is represented by the viscoelastic cylinder and the thin shell represents the rocket motor casing. The cylinder is made of a homogeneous and isotropic viscoelastic material and shown in Fig. 5

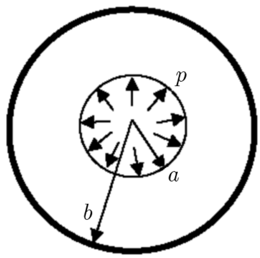

Fig. 5. Viscoelastic cylinder submitted to internal pressure

For this problem, the exact solution has been provided by Lee *et al.* (1959) and a step by step solution by Zienkiewich *et al.* (1968). A finite element model as well as a theoretical solution is also provided by Zocher *et al.* (1997). The viscoelastic material is represented by the uniaxial relaxation modulus

$$E(t) = E_\infty + E_1 \mathrm{e}^{-t/\mu} \tag{5.1}$$

The values of these constants are given in Table 2.

Table 2. Constants used in the relaxation function

E_∞ [MPa]	E_1 [MPa]	μ [s]
0.1	0.4	1.0

The outer steel casing had the properties $E = 68910^2$ MPa and Poisson's ratio is taken to be 0.3015. The internal pressure is $p(t) = p_0 H(t)$, where $p_0 = 100$ Pa is the initial pressure applied at time $t = 0$.

When we apply the viscoelastic correspondence principle to the elastic solution, it is easy to derive an analytical expression for the radial displacement u_r. One finds

$$u_r(r,t) = \frac{p_0 a^2 b (1+\upsilon)(1-2\upsilon)}{a^2 + (1-2\upsilon)b^2} \Big(\frac{b}{r} - \frac{r}{b}\Big) J(t) \tag{5.2}$$

in which $a = 2\,\mathrm{m}$ the inner radius, $b = 4\,\mathrm{m}$ the outer radius of the cylinder and $D(t)$ is the creep compliance, and it is easily determined from relaxation modulus (5.1) and is given by

$$J(t) = J_0 + J_1\Big(1 - \mathrm{e}^{-\lambda_1 t}\Big) \tag{5.3}$$

where

$$J_0 = \frac{1}{E_0} \qquad\qquad E_0 = E_\infty + E_1$$
$$J_1 = \Big(\frac{1}{E_\infty} - \frac{1}{E_0}\Big) \qquad\qquad \lambda_1 = \frac{E_\infty}{E_0\mu}$$

Figure 6 shows the mesh used in the analysis of this viscoelastic cylinder in which symmetry conditions were used so that the mesh of the entire cylinder is not necessary to be modeled.

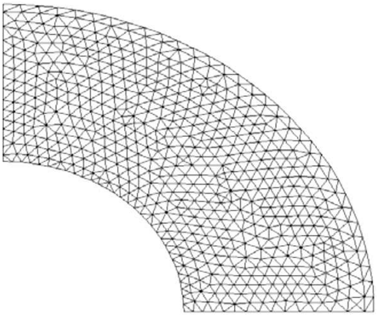

Fig. 6. Viscoelastic cylinder under internal pressure: finite element mesh

The analysis was performed in the axisymmetric mode with the viscoelastic properties extracted from Zocher *et al.* (1997) and presented in Table 2. The numerical responses were compared with the analytical solution to Eq. (5.2), in which $J(t)$ is the creep compliance interconverted from the relaxation modulus $E(t)$ also extracted from Zocher *et al.* (1997). The results of the viscoelastic numerical process are shown in Figs. 7-9. The variation of radial and circumferential stress with time is displayed in Figs. 7 and 8, while the radial displacement of the mid-thickness versus time is presented in Fig. 9 and compared to theoretical solution.

The results of Fig. 9 indicate the time function of the radial displacement $u_r(r,t)$ at the radial position $r = 3\,\mathrm{m}$. For FE numerical analysis, it was evaluated two time steps $\Delta t = 1\,\mathrm{s}$ and $\Delta t = 0.5\,\mathrm{s}$ which are very small according to the response of the viscoelastic material analyzed at the constant load condition. It can be observed that our finite element prediction is in good agreement with the analytical solution.

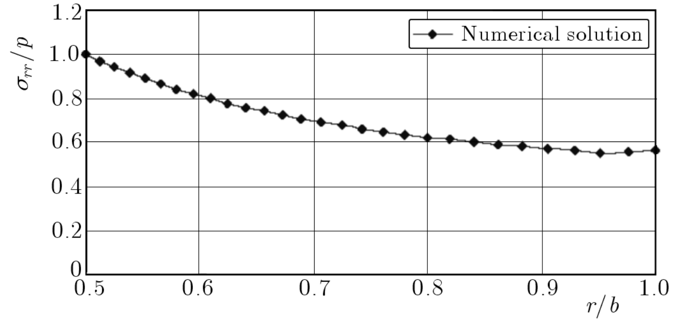

Fig. 7. Variation of the distribution of radial compressive stress

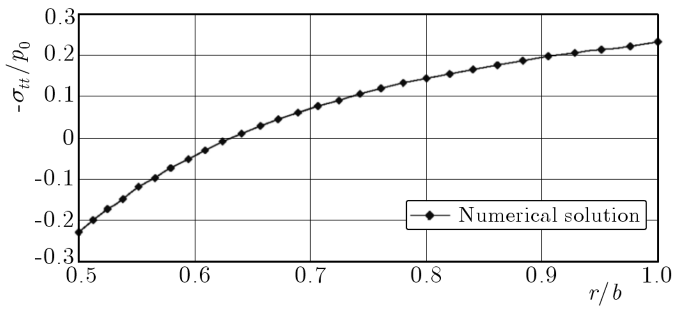

Fig. 8. Variation of the distribution of circumferential compressive stress

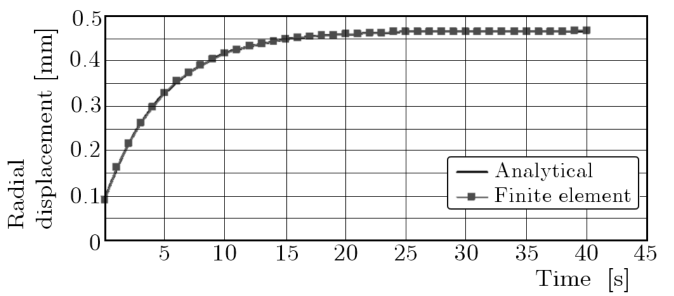

Fig. 9. Viscoelastic cylinder bonded to the thin elastic case

6. Conclusions

A three dimensional finite element formulation in the time domain is presented. The incremental formulation is adapted to linear non-ageing viscoelastic materials submitted to mechanical deformation. The method is based on an integral approach using a discrete spectrum representation for the creep tensor. The governing equations are then obtained using a discretized form of Boltzmann's principle. The analytical solution to constitutive integral equations is then obtained using a finite difference discretization in the time domain. In this way, the incremental constitutive equations for the linear viscoelastic material using a pseudo fourth order rigidity tensor have been proposed. The influence of the whole past history on the behavior of the material at the current time is given by a pseudo second-order tensor, and the final incremental law is explained in terms of strain and stress increment. Finally, the formulation is introduced in a finite element discretization in order to resolve complex boundary viscoelastic problems, and the numerical results obtained from the proposed method show good accuracy.

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Całkowy opis właściwości materiałów zmiennych w czasie z użyciem metody elementów skończonych

Streszczenie

W pracy przedstawiono analityczną metodę badania zachowania się liniowych, lepko-sprężystych i niestarzejących się materiałów poddanych mechanicznej deformacji. Użyto całkowej metody opartej na dyskretnej widmowej reprezentacji tensora pełzania w celu wyznaczenia inkrementalnego opisu właściwości lepko-sprężystych. Operatory całkowe zdyskretyzowano metodą różnic skończonych. Inkrementalny, lepko-sprężysty model konstytutywny materiału określono zmienną stanu wewnętrznego, odzwierciedlającą historię naprężeń i odkształceń, co pozwoliło na uniknięcie konieczności zachowywania jej podczas symulacji numerycznych. Zupełne i ogólne sformułowanie zagadnienia lepko-sprężystości otrzymano w funkcji przyrostów naprężeń i odkształceń. Dodano przykłady wyników symulacji do weryfikacji inkrementalnych równań konstytutywnych modelu.

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