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THE INVERSE DETERMINATION OF VOLUME FRACTION OF FIBRES IN REINFORCED COMPOSITE WITH IMPERFECT THERMAL CONTACT BETWEEN CONSTITUENTS

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The paper considers the problem of determination of the volume fraction of fibres in an unidirectionally reinforced composite in order to provide the appropriate effective thermal conductivity. The problem formulated in such a way should be treated as an inverse heat transfer problem. The thermal conductivities of constituents (fibres and matrix) and fibres arrangement are known. The calculations are carried out for an imperfect thermal contact between the fibres and matrix.

Key words: effective thermal conductivity, inverse heat transfer problem, boundary collocation method, Newton method

1. Introduction

In the literature, the following problems are considered to be the classical inverse heat conduction problems:

i) determination of heat sources (i.e. Yan *et al.*, 2008),
ii) determination of heat transfer coefficient (i.e. Hon and Wei, 2004),
iii) Cauchy's problem (i.e. Marin, 2005),
iv) determination of temperature-dependent thermal conductivity (i.e. Chantasiriwan, 2002),
v) determination of the unknown initial temperature (i.e. Hon, 2009).

The above mentioned problems usually apply to homogeneous media. In the case of the composite materials (nonhomogeneous media) other practically important issues might have to be considered. One of them is the inverse problem of determination of the volume fraction of constituents in order to obtain the appropriate effective thermal conductivity.

Let us consider a unidirectional fibrous composite with a regular arrangement of fibres (Fig. 1a). If the thermal conductivity coefficients of constituents and their volume fractions are known then the composite can be treated as a homogeneous region for which the effective thermal conductivity can be determined as a function of known parameters. Currently, there are many papers in which the effective thermal conductivity coefficient is determined for a regular arrangement of fibres for the given thermal conductivity of constituents and volume fraction of fibres (direct problem). The method of determination is usually based on the solution to the heat transfer equation at the microstructure level in repeated elements of the array (i.e. Han and Cosner, 1981). In almost all these papers, the thermal analysis of fibrous composites is based on the assumption that thermal contact between fibres and matrix is perfect. This assumption implies continuity of temperature and heat flux on the boundary of fibres and matrix, which simplifies the theoretical analysis of the problem. However, in real composites, there is always a thin layer between the pure fibre and pure matrix material that participates in thermal interaction between the composite constituents. To the best knowledge of the authors, the problem of the effective thermal conductivity of the composite with imperfect thermal contact was considered for the first time in the paper by Benveniste and Miloh (1986), in which the Kapitza thermal resistance boundary condition was used (direct problem). Benveniste and Miloh (1986) developed the Hashim method and applied it to the analysis of composites with different types of inclusions, but not fibres. For fibrous composites with imperfect thermal contact between constituents the determination of the effective thermal conductivity (direct problem) was considered in the paper by Goncalves and Kołodziej (1993).

The purpose of this paper is to propose an analytic-numerical algorithm for determination of the volume fraction of fibres in order to obtain a given value of the transverse effective thermal conductivity λ_z. Such a problem can be treated as one kind of the inverse heat conduction problem for a composite material. Because the algorithm for the inverse problem is, in some sense, based on the solution to the direct problem, a brief outline of this paper is as follows: in Section 2 the direct problem is considered in a similar way as *First Model* in the paper by Goncalves and Kołodziej (1993). The inverse problem of determination of the volume fraction of fibres in the composite for a given effective thermal conductivity is considered in Section 3. In Section 4, we present numerical results for both direct and inverse problems. Finally, concluding remarks are given in Section 5.

2. Direct problem: determination of the effective thermal conductivity for the composite with an imperfect thermal contact between the components

In real composite materials, the contact surface between the components is a transitional layer of a certain thickness with mechanical, thermal and chemical interactions between the components. The transition layer of a finite thickness is taken into account in the formulation of contact condition of the fibre-matrix in the form of so-called Kapitza condition

$$\lambda_f \frac{\partial \widetilde{T}_f}{\partial r} = \beta(\widetilde{T}_m - \widetilde{T}_f) \tag{2.1}$$

where $\widetilde{T}_f$, $\widetilde{T}_m$ are the temperature field of the fibre and matrix respectively, λ_f is the thermal conductivity of fibres, β is the factor determining the thermal contact resistance components of the composite. When $\beta \to 0$, we have a case of ideal insulation, and when $\beta \to \infty$ there is perfect contact between the components.

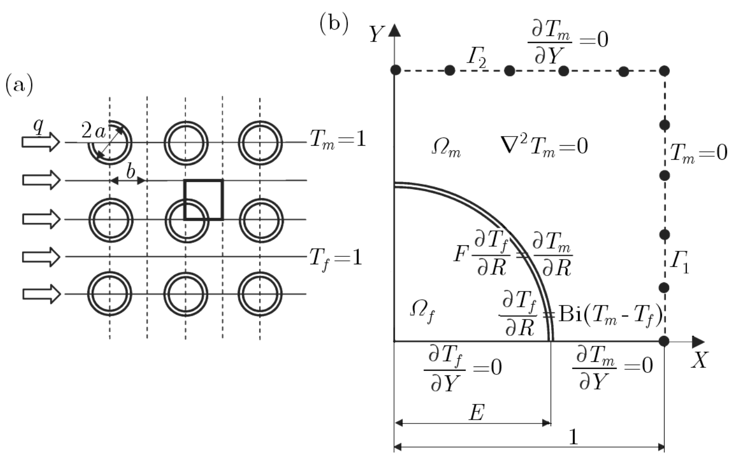

Fig. 1. A unidirectional reinforced fibrous composite with fibre arrangement according to a square array for the imperfect thermal contact between the fibre and matrix: (a) general view, (b) formulation of the nondimensional boundary value problem in a repeated element

Let us consider a unidirectional composite with fibres arranged in a matrix in a regular, square array with an imperfect thermal contact between the fibre and matrix (Fig. 1a), where a is the radius of the fibres, $2b$ is the distance between neighbouring fibres, $E = a/b$, and $\varphi = \pi E^2/4$ is the volume fraction of fibres. The ratio of thermal conductivity of fibres λ_f to matrix λ_m is denoted as $F = \lambda_f/\lambda_m$, $R = r/b$ is the dimensionless radius, $(X = x/b, Y = y/b)$ are dimensionless Cartesian coordinates, $T_{f/m} = (\widetilde{T}_{f/m} - \widetilde{T}_R)/(\widetilde{T}_L - \widetilde{T}_R)$ is the dimensionless temperature field, $\widetilde{T}_L$, $\widetilde{T}_R$ are the temperature on the left and on the right boundary in a repeated element, respectively.

In order to solve the nondimensional boundary value problem in the repeated element of the composite (Fig. 1b), the boundary collocation method is used (Kołodziej and Zieliński, 2009). The general solution to the Laplace equation $\nabla^2 T = 0$ in the polar coordinate system (R, θ) has the form

$$\begin{aligned} T = {} & A_0 + A_1\theta + A_2\theta\ln R + A_3\ln R \\ & + \sum_{k=1}^{\infty}[(B_kR^k + C_kR^{-k})\cos(k\theta) + (D_kR^k + E_kR^{-k})\sin(k\theta)] \end{aligned} \tag{2.2}$$

where $A_0, A_1, A_2, A_3, B_k, C_k, D_k, E_k$ are integral constants.

Given the repeated element of the square array $\Omega = \Omega_f \cup \Omega_m$ in the region of the fibre Ω_f and the matrix Ω_m, the solution is predicted to has form (2.2). Some of the constants must be determined strictly by the conditions at the bottom and on the left side of the repeated element

$$\begin{aligned} & \frac{\partial T_f}{\partial\theta} = \frac{\partial T_m}{\partial\theta} = 0 && \text{for} \quad \theta = 0 \\ & T_f = T_m = 1 && \text{for} \quad \theta = \frac{\pi}{2} \end{aligned} \tag{2.3}$$

and the contact conditions of fibre-matrix

$$\begin{aligned} & F\frac{\partial T_f}{\partial R} = \frac{\partial T_m}{\partial R} && \text{for} \quad R = E \\ & \frac{\partial T_f}{\partial R} = \mathrm{Bi}(T_m - T_f) && \text{for} \quad R = E \end{aligned} \tag{2.4}$$

where $\mathrm{Bi} = b\beta/\lambda_f$ is the dimensionless resistance number.

After determining the constants from boundary conditions (2.3) and from contact conditions of fibre-matrix (2.4), marking the remaining constants as w_k, $(w_1 = B_1, w_2 = B_3, \ldots, w_k = B_{2k-1}, \ldots)$ and cutting off an infinite number of the test functions to N expressions, we obtain a solution for the temperature field of the fibre and matrix

$$
\begin{aligned}
T_f &= 1 + \sum_{k=1}^{N} w_k R^{(2k-1)} \cos[(2k-1)\theta] \\
T_m &= 1 + \sum_{k=1}^{N} \frac{w_k}{2} \Big[(F+1+\gamma_k) R^{(2k-1)} - (F-1-\gamma_k) \frac{E^{2(2k-1)}}{R^{(2k-1)}} \Big] \cos[(2k-1)\theta]
\end{aligned}
\tag{2.5}
$$

where $\gamma_k = (2k-1)/(\mathrm{Bi}E)$.

The constants w_k are determined by fulfilment of the condition of collocation points on the upper Γ_2 and on the right Γ_1 edge of the concerned region

$$
\begin{aligned}
T_m &= 0 \qquad \text{for} \qquad X = 1 \\
\frac{\partial T_m}{\partial Y} &= 0 \qquad \text{for} \qquad Y = 1
\end{aligned}
\tag{2.6}
$$

Condition $(2.6)_2$ can be written for polar coordinates

$$
\frac{\partial T_m}{\partial Y} = \frac{\partial T_m}{\partial R} \sin\theta + \frac{1}{R} \frac{\partial T_m}{\partial \theta} \cos\theta = 0 \tag{2.7}
$$

Choosing $N1$ points on the right boundary Γ_1 and $N2$ points on the upper boundary Γ_2 and collocating conditions $(2.6)_1$ and (2.7), we obtain a system of linear equations $N1 + N2$ with N unknown constants w_k, $k = 1, \dots, N$:

— $(R_j, \theta_j) \in \Gamma_1$, $j = 1, \dots, N1$

$$
\sum_{k=1}^{N} w_k \Big[(F+1+\gamma_k) R_j^{(2k-1)} - (F-1-\gamma_k) \frac{E^{2(2k-1)}}{R_j^{(2k-1)}} \Big] \cos[(2k-1)\theta_j] = -2 \tag{2.8}
$$

— $(R_j, \theta_j) \in \Gamma_2$, $j = 1, \dots, N2$

$$
\begin{aligned}
&\sum_{k=1}^{N} w_k (2k-1) \Big[(F+1+\gamma_k) R_j^{2(k-1)} \sin[2(k-1)\theta_j] \\
&\qquad - (F-1-\gamma_k) \frac{E^{2(2k-1)}}{R_j^{2k}} \sin(2k\theta_j) \Big] = 0
\end{aligned}
\tag{2.9}
$$

The constants w_k obtained by the Gaussian elimination method provide estimates of the value of the global heat flux through the unit region of the considered element

$$
q = \frac{1}{b} \Big(-\lambda_f \int_0^a \frac{\partial \widetilde{T}_f}{\partial x}\Big|_{x=0} dy - \lambda_m \int_a^b \frac{\partial \widetilde{T}_m}{\partial x}\Big|_{x=0} dy \Big) \tag{2.10}
$$

The transverse effective thermal conductivity is defined by the formula

$$\lambda_z = \frac{qb}{\Delta\widetilde{T}} \tag{2.11}$$

where b is the distance between the isothermal boundaries, $\Delta\widetilde{T} = \widetilde{T}_L - \widetilde{T}_R$ is the difference of temperature at the isothermal edges.

After taking into consideration in formula (2.11) the definition of the non-dimensional temperature and coordinates, the value of effective thermal conductivity in relation to the thermal conductivity of the matrix can be calculated from the relationship

$$\frac{\lambda_z}{\lambda_m} = -F\int\limits_0^E \frac{1}{R}\frac{\partial T_f}{\partial\theta}\Big|_{\theta=\frac{\pi}{2}} dR - \int\limits_E^1 \frac{1}{R}\frac{\partial T_m}{\partial\theta}\Big|_{\theta=\frac{\pi}{2}} dR \tag{2.12}$$

or

$$\frac{\lambda_z}{\lambda_m} = \sum_{k=1}^N \frac{w_k}{2}(-1)^k[(F+1+\gamma_k) + (F-1-\gamma_k)E^{2(2k-1)}] \tag{2.13}$$

3. Inverse problem: determination of the volume fraction of fibres in composite for a given effective thermal conductivity

When designing composites with specific properties of fibres and matrix, we must estimate the fraction of volume of fibres to obtain effective thermal conductivity values. Assuming that $E = a/b$ is unknown, we use the known value of effective thermal conductivity in relation to the thermal conductivity of the matrix λ_z/λ_m. From the collocation of the boundary conditions in $N1+N2$ points on the right Γ_1 and upper Γ_2 edge of the consideration region and from condition (2.13), we obtain a system of $N1+N2+1$ non-linear equations with $N+1$ the unknowns w_k, E:

— $(R_j,\theta_j) \in \Gamma_1,\ j = 1,\dots,N1$

$$\begin{aligned} f_j(w_k,E) = 1 + \sum_{k=1}^N \frac{w_k}{2}\Big[(F+1+\gamma_k)R_j^{(2k-1)} \\ - (F-1-\gamma_k)\frac{E^{2(2k-1)}}{R_j^{(2k-1)}}\Big]\cos[(2k-1)\theta_j] = 0 \end{aligned} \tag{3.1}$$

— $(R_j,\theta_j) \in \Gamma_2,\ j = 1,\dots,N2$

$$
\begin{aligned}
f_{N1+j}(w_k,E) = \sum_{k=1}^{N} \frac{w_k}{2}(2k-1)\Big[(F+1+\gamma_k)R_j^{2(k-1)} \sin[2(k-1)\theta_j] \\
-(F-1-\gamma_k)\frac{E^{2(2k-1)}}{R_j^{2k}} \sin(2k\theta_j)\Big] = 0
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
f_{N1+N2+1}(w_k,E) = \sum_{k=1}^{N} \frac{w_k}{2}(-1)^k[(F+1+\gamma_k) \\
+(F-1-\gamma_k)E^{2(2k-1)} - \frac{\lambda_z}{\lambda_m} = 0
\end{aligned}
\tag{3.3}
$$

The non-linear system of $N+1$ equations $f(w_k,E)$ with $N1+N2+1$ unknowns $\mathbf{w} = [w_1,\dots,w_N,E]^\top$ is solved by using the Newton iteration method

$$
\begin{bmatrix} w_1 \\ \vdots \\ w_N \\ E \end{bmatrix}^{(i+1)} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \\ E \end{bmatrix}^{(i)} - \begin{bmatrix} f_1^{(i)} \\ \vdots \\ f_{N1+N2}^{(i)} \\ f_{N1+N2+1}^{(i)} \end{bmatrix} \begin{bmatrix} J_{1,1}^{(i)} & \dots & J_{1,N+1}^{(i)} \\ \vdots & \dots & \vdots \\ J_{N1+N2,1}^{(i)} & \dots & \vdots \\ J_{N1+N2+1,1}^{(i)} & \dots & J_{N1+N2+1,N+1}^{(i)} \end{bmatrix}^{-1}
$$

$$
\begin{aligned}
&\mathbf{w}^{(i+1)} = \mathbf{w}^{(i)} - \mathbf{f}(\mathbf{w}^{(i)})\mathbf{J}(\mathbf{w}^{(i)})^{-1} \rightarrow \mathbf{w}^{(i+1)} = \mathbf{w}^{(i)} - \mathbf{Y}(\mathbf{w}^{(i)}) \\
&\mathbf{Y}(\mathbf{w}^{(i)}) = \mathbf{f}(\mathbf{w}^{(i)})\mathbf{J}(\mathbf{w}^{(i)})^{-1} \rightarrow \mathbf{J}(\mathbf{w}^{(i)})\mathbf{Y}(\mathbf{w}^{(i)}) = \mathbf{f}(\mathbf{w}^{(i)})
\end{aligned}
\tag{3.4}
$$

The functions f_i are described by equations (3.1)-(3.3), while Jacobi elements have the following form:
— $k = 1,\dots,N,\ j = 1,\dots,N1$

$$
J_{j,k} = \frac{\partial f_j}{\partial w_k} = \frac{1}{2}\Big[(F+1+\gamma_k)R_j^{(2k-1)} - (F-1-\gamma_k)\frac{E^{2(2k-1)}}{R_j^{(2k-1)}}\Big] \cos[(2k-1)\theta_j]
\tag{3.5}
$$

— $j = 1,\dots,N1$

$$
\begin{aligned}
J_{j,N+1} = \frac{\partial f_j}{\partial E} = \sum_{k=1}^{N} \frac{w_k(1-2k)}{2} \\
\cdot \Big[\frac{R_j^{(2k-1)}}{E^2\mathrm{Bi}} + \Big(2(F-1) - \frac{4k-3}{E\mathrm{Bi}}\Big)\frac{E^{(4k-3)}}{R_j^{(2k-1)}}\Big] \cos[(2k-1)\theta_j]
\end{aligned}
\tag{3.6}
$$

— $k = 1, \dots, N,\ j = 1, \dots, N2,\ i = j + N1$

$$J_{i,k} = \frac{\partial f_i}{\partial w_k} = \frac{(2k-1)}{2}\Big[(F+1+\gamma_k)R_j^{2(k-1)}\sin[2(1-k)\theta_j] + (F-1-\gamma_k)\frac{E^{2(2k-1)}}{R_j^{2k}}\sin(2k\theta_j)\Big] \tag{3.7}$$

— $j = 1, \dots, N2,\ i = j + N1$

$$J_{i,N+1} = \frac{\partial f_i}{\partial E} = \sum_{k=1}^{N} w_k \frac{(2k-1)^2}{2}\Big[\frac{R_j^{2(k-1)}}{E^2\mathrm{Bi}}\sin[2(k-1)\theta_j] + \Big(2(F-1) - \frac{4k-3}{E\mathrm{Bi}}\Big)\frac{E^{4k-3}}{R_j^{2k}}\sin(2k\theta_j)\Big] \tag{3.8}$$

— $k = 1, \dots, N,\ i = N1 + N2 + 1$

$$J_{i,k} = \frac{\partial f_i}{\partial w_k} = \frac{1}{2}(-1)^k[(1+F+\gamma_k) + (F-1-\gamma_k)E^{2(2k-1)}] \tag{3.9}$$

— $i = N1 + N2 + 1$

$$J_{i,N+1} = \frac{\partial f_i}{\partial E} = \sum_{k=1}^{N} w_k \frac{2k-1}{2}(-1)^k\Big[\Big(2(F-1) - \frac{4k-3}{E\mathrm{Bi}}\Big)E^{4k-3} - \frac{1}{E^2\mathrm{Bi}}\Big] \tag{3.10}$$

To start the Newton iteration, we need to know $\mathbf{w}^{(0)} = [w_1^0, \dots, w_N^0, E^0]^\top$ as the initial condition. As the initial value of the constants $\{w_k^0\}_{k=1}^N$, the solution to the linear problem for $E = 0.1$ has been adopted. The condition for the end of iteration has been set at: $\delta_{Newton} = \|\mathbf{w}^{(i+1)} - \mathbf{w}^{(i)}\|_{max} \leqslant 10^{-7}$, where $\|\cdot\|_{max}$ means the maximum norm.

4. Results of the numerical experiment

For the direct problem Fig. 2 shows the values of the effective thermal conductivity in relation to the thermal conductivity of the matrix $\lambda_z/\lambda_m = \lambda_z/\lambda_m(\varphi)|_{F,\mathrm{Bi}}$ as a function of the volume fraction of fibres φ for the assumed value of the coefficient $F \in \{0.5, 1, 2, 10, 20\}$ and the resistance number $\mathrm{Bi} \in \{0.001, 0.01, 0.1, 1, 10, 1000\}$.

For a very small value of the resistance number $\mathrm{Bi} = 0.001$ and $\mathrm{Bi} = 0.01$ with the increasing volume fraction of fibres φ in the matrix for all tested

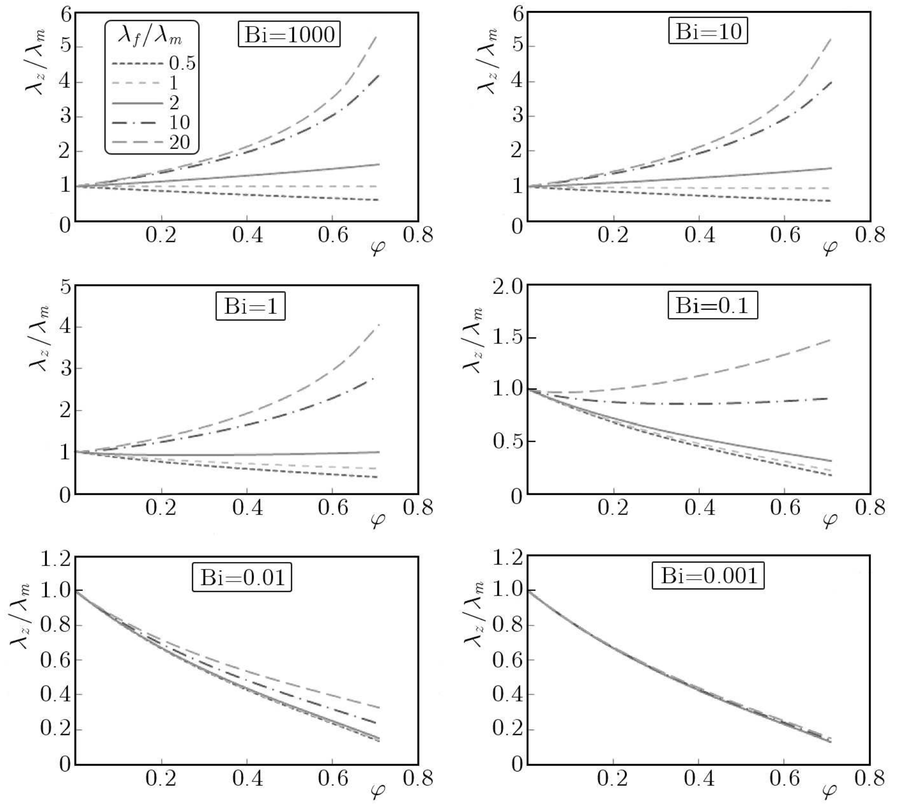

Fig. 2. The effective thermal conductivity in relation to the thermal conductivity of the matrix as a function of the volume fraction of fibres in the matrix for different values of the ratio of thermal conductivity fibres to the matrix and different values of resistance number

values of F, the effective thermal conductivity decreases and the composite becomes a perfect insulator. For $\text{Bi} = 1000$ the graph obtained corresponds to the conditions close to ideal thermal contact between the components of the composite.

For the components of the composite with the identical values of the thermal conductivity $F = 1$, one can notice that for $\text{Bi} = 10$ the obtained value of the effective thermal conductivity is close to unity, which indicates the correctness of the results.

The value of the effective thermal conductivity in relation to the thermal conductivity of the matrix λ_z/λ_m, except for the constants characterizing the composite F, E, Bi, also depends on the constants w_k arising from the approximate fulfilment of the boundary conditions at the collocation points.

Table 1 shows the influence of the number of the collocation points on the maximum error fulfilling the collocation boundary conditions calculated at the control points (between collocation points).

Table 1. Impact of the number of collocation points on the effective thermal conductivity of the composite and the maximal error of fulfilling the boundary conditions at control points

$N1$	$N2$	$E=0.5,\ F=10,\ \mathrm{Bi}=1$			$E=0.9,\ F=10,\ \mathrm{Bi}=0.01$		
		λ_z/λ_m	$\delta_{max}\big\vert_{T_m=0}$	$\delta_{max}\big\vert_{\frac{\partial T_m}{\partial Y}=0}$	λ_z/λ_m	$\delta_{max}\big\vert_{T_m=0}$	$\delta_{max}\big\vert_{\frac{\partial T_m}{\partial Y}=0}$
5	4	1.2365	7.084E-05	8.834E-04	0.2857	4.238E-03	9.794E-02
6	5	1.2365	1.542E-05	2.613E-04	0.2865	2.940E-03	8.168E-02
7	6	1.2365	2.888E-07	4.599E-05	0.2861	9.246E-04	2.783E-02
8	7	1.2365	5.729E-06	2.352E-04	0.2862	6.949E-05	5.838E-03
9	8	1.2365	1.216E-05	5.404E-04	0.2863	1.475E-03	6.737E-02
10	9	1.2365	3.057E-05	1.524E-03	0.2860	4.651E-03	2.332E-01
11	10	1.2365	4.499E-05	2.455E-03	0.2867	7.300E-03	3.991E-01
12	11	1.2365	1.398E-04	8.409E-03	0.2851	2.287E-02	1.376E+00
13	12	1.2365	1.154E-04	7.488E-03	0.2872	1.924E-02	1.249E+00
14	13	1.2366	1.222E-03	8.621E-02	0.2778	1.918E-01	1.354E+01
15	14	1.2364	2.727E-04	2.055E-02	0.2883	4.592E-02	3.460E+00
$N1$	$N2$	$E=0.5,\ F=0.5,\ \mathrm{Bi}=10$			$E=0.9,\ F=0.5,\ \mathrm{Bi}=0.1$		
		λ_z/λ_m	$\delta_{max}\big\vert_{T_m=0}$	$\delta_{max}\big\vert_{\frac{\partial T_m}{\partial Y}=0}$	λ_z/λ_m	$\delta_{max}\big\vert_{T_m=0}$	$\delta_{max}\big\vert_{\frac{\partial T_m}{\partial Y}=0}$
5	4	0.8504	4.570E-05	6.018E-04	0.2381	4.694E-03	1.082E-01
6	5	0.8504	1.009E-05	1.822E-04	0.2389	3.259E-03	9.036E-02
7	6	0.8504	1.204E-07	2.532E-05	0.2386	1.024E-03	3.074E-02
8	7	0.8504	3.599E-06	1.474E-04	0.2386	8.080E-05	6.646E-03
9	8	0.8504	7.689E-06	3.417E-04	0.2388	1.642E-03	7.498E-02
10	9	0.8504	1.935E-05	9.645E-04	0.2384	5.171E-03	2.593E-01
11	10	0.8504	2.848E-05	1.554E-03	0.2391	8.117E-03	4.438E-01
12	11	0.8504	8.848E-05	5.322E-03	0.2375	2.541E-02	1.529E+00
13	12	0.8504	7.302E-05	4.739E-03	0.2397	2.140E-02	1.390E+00
14	13	0.8503	7.732E-04	5.456E-02	0.2300	2.122E-01	1.498E+01
15	14	0.8504	1.726E-04	1.301E-02	0.2408	5.110E-02	3.851E+00

The best results are obtained for 7 collocation points at the right edge Γ_1 and for 6 points at the upper edge Γ_2 of the repeated element for $E = 0.5$, 8 points on the right-hand side and 7 points at the upper edge for $E = 0.9$. The increase in the number of the collocation points does not improve the quality of the results.

The results of the iterative calculation of the volume fraction of fibres for the composite are shown in Fig. 3. The value of volume fraction of fibres φ in composites is presented as a function of the effective thermal conductivity in relation to the thermal conductivity of the matrix $\varphi = \varphi(\lambda_z/\lambda_m)\big|_{F,\mathrm{Bi}}$ for specific values of the coefficient $F \in \{0.5, 1, 2, 10, 20\}$ and resistance number $\mathrm{Bi} = \{0.001, 0.01, 0.1, 1, 10, 1000\}$.

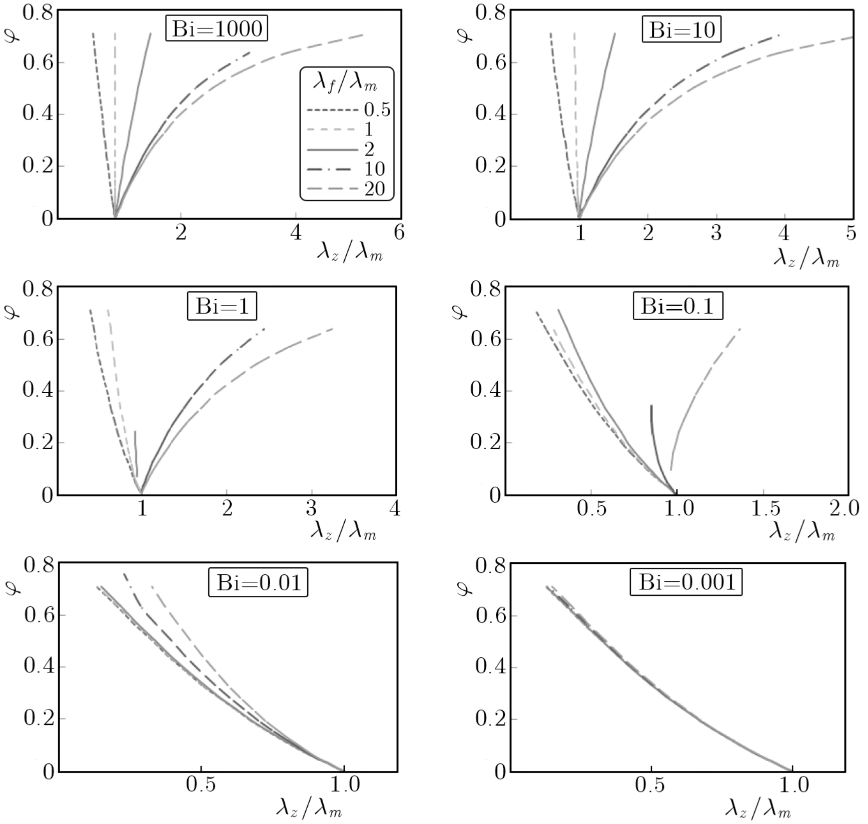

Fig. 3. The volume fraction of fibres in the matrix as a function of the effective thermal conductivity in relation to the thermal conductivity of the matrix of the composite for different relative values of the thermal conductivity of the fibre and matrix and different values of the resistance number

Table 2. Impact of the number of collocation points on the fraction of fibres and the maximum error of fulfilling the boundary conditions at control points

$N1$	$N2$	$\lambda_z/\lambda_m = 1.24,\ F = 10,\ \mathrm{Bi} = 1$				$\lambda_z/\lambda_m = 0.285,\ F = 10,\ \mathrm{Bi} = 0.01$			
		φ	E	$\delta_{max}\big\vert_{T_m=0}$	$\delta_{max}\big\vert_{\frac{\partial T_m}{\partial Y}=0}$	φ	E	$\delta_{max}\big\vert_{T_m=0}$	$\delta_{max}\big\vert_{\frac{\partial T_m}{\partial Y}=0}$
5	4	0.1985	0.5027	7.19E-05	8.96E-04	0.6370	0.9006	4.30E-03	9.95E-02
6	5	0.1985	0.5027	1.56E-05	2.65E-04	0.6381	0.9013	3.07E-03	8.52E-02
7	6	–	–	–	–	0.6376	0.9010	9.61E-04	2.89E-02
8	7	–	–	–	–	97.0018	11.1133	3.31E-01	1.35E+01
9	8	–	–	–	–	–	–	–	–

$N1$	$N2$	$\lambda_z/\lambda_m = 0.85,\ F = 0.5,\ \mathrm{Bi} = 10$				$\lambda_z/\lambda_m = 0.24,\ F = 0.5,\ \mathrm{Bi} = 0.1$			
		φ	E	$\delta_{max}\big\vert_{T_m=0}$	$\delta_{max}\big\vert_{\frac{\partial T_m}{\partial Y}=0}$	φ	E	$\delta_{max}\big\vert_{T_m=0}$	$\delta_{max}\big\vert_{\frac{\partial T_m}{\partial Y}=0}$
5	4	0.1969	0.5007	4.58E-05	6.04E-04	0.6340	0.8985	4.51E-03	1.04E-01
6	5	0.1969	0.5008	1.01E-05	1.83E-04	0.6349	0.8991	3.17E-03	8.79E-02
7	6	0.1969	0.5008	1.20E-07	2.54E-05	0.6345	0.8988	9.80E-04	2.94E-02
8	7	0.1969	0.5008	3.61E-06	1.48E-04	0.9252	1.0854	1.14E+00	4.94E+01
9	8	0.1969	0.5008	7.71E-06	3.43E-04	–	–	–	–
10	9	0.1969	0.5007	1.94E-05	9.67E-04	–	–	–	–
11	10	0.1969	0.5008	2.86E-05	1.56E-03	–	–	–	–

Similarly, as in the problem of identification of λ_z/λ_m, the iterative identification of the volume fraction of fibres φ in a composite, the number of collocation points $N1 + N2$, where the boundary condition is fulfilled in an approximate manner, affects the accuracy of the calculations. Table 2 shows the impact of the number of collocation points on the value of volumetric fraction of fibres φ in the composite and the maximum error of fulfilment of boundary conditions at the control points.

In the inverse problem, the impact of the number of collocation points on the quality of the results is greater than in the direct problem. Exceeding a certain value of the number of collocation points (different for different examples – see Table 2) leads to lack of convergence of the calculations (bad conditioning of the collocation matrix) or to obtaining a very large error of boundary conditions.

While solving the inverse problem determining the volume fraction of fibres using iterative methods, the convergence of the method should be examined. Table 3 shows the convergence of the used Newton iterative method for four test examples. For the composite with a surface resistance of the fibre-matrix boundary used in determining the volume fraction of fibres, the Newton method is not fast. Nevertheless it should be noted that the method is acceptably convergent because after a maximum of 10 iterations for the test examples the correct results were obtained with an error less than 10^{-7}.

Table 3. Convergence of the Newton method for the four test examples

Iteration	$F = 10$				$F = 0.5$			
	$\lambda_z/\lambda_m = 1.24$, Bi = 1		$\lambda_z/\lambda_m = 0.285$, Bi = 0.01		$\lambda_z/\lambda_m = 0.85$, Bi = 10		$\lambda_z/\lambda_m = 0.24$, Bi = 0.1	
	δ_{Newton}	E	δ_{Newton}	E	δ_{Newton}	E	δ_{Newton}	E
1	4.28E+00	0.1000	2.33E+00	0.1000	2.71E+00	0.1000	2.44E+00	0.1000
2	1.84E+00	4.3809	1.22E+00	2.4306	2.38E+00	0.9689	9.92E-01	2.5430
3	1.83E+00	4.1689	5.21E-01	1.2077	2.85E-01	0.8492	1.16E+00	1.5510
4	1.11E+00	2.3362	1.81E-01	0.6864	6.01E-02	0.5646	2.54E-01	0.3907
5	6.11E-01	1.2267	3.23E-02	0.8670	3.80E-03	0.5046	1.67E-01	0.6444
6	8.16E-02	0.6153	1.67E-03	0.8993	1.69E-05	0.5008	8.18E-02	0.8112
7	3.00E-02	0.5337	2.62E-06	0.9010	3.35E-10	0.5008	5.83E-03	0.8930
8	9.83E-04	0.5037	9.04E-12	0.9010	–	–	2.53E-06	0.8988
9	1.64E-06	0.5027	–	–	–	–	2.91E-10	0.8988
10	3.87E-12	0.5027	–	–	–	–	–	–

5. Conclusions

The presented method of determining the volume fraction of fibres of a composite or the effective thermal conductivity, except for the cases of the maximum fibre density and the excellent thermal resistance at the border of the components, is easy to implement and efficient. It can be easily applied to other configurations of the regular arrangement of fibres in the matrix, for example to a triangular or hexagonal mesh. The study compared the influence of the ratio of the thermal conductivity of fibres to the thermal conductivity of the matrix F and the influence of the resistance number Bi – a parameter of the surface resistance on the border of fibre-matrix, on the value of the volumetric fraction of fibres and the value of the effective thermal conductivity of the composite. It was also shown that the increasing of the number of collocation points does not reduce the error of approximation of the boundary conditions while it leads to the ill-conditioning of the system of equations.

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Odwrotne określenie objętościowego udziału włókien we wzmocnionym kompozycie z niedoskonałym kontaktem termicznym pomiędzy składnikami

Streszczenie

W pracy rozważa się problem określenia objętościowego udziału włókien w jednokierunkowo wzmocnionym kompozycie w celu uzyskania odpowiedniego efektywnego współczynnika przewodzenia ciepła. Problem sformułowany w ten sposób jest traktowany jako odwrotny problem przewodzenia ciepła. Współczynniki przewodzenia ciepła składników (włókien i matrycy) oraz sposób ułożenia włókien są znane. Obliczenia są wykonane dla niedoskonałego kontaktu termicznego pomiędzy włóknami i matrycą.

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