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INTERVAL BOUNDARY ELEMENT METHOD FOR TRANSIENT DIFFUSION PROBLEM IN TWO-LAYERED DOMAIN

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In the paper, the description of an unsteady heat transfer for one-dimensional problem proceeding in a two-layered domain is presented. It is assumed that all thermophysical parameters appearing in the mathematical description of the problem analysed are given as directed interval values. The problem discussed has been solved using the 1st scheme of the interval boundary element method. The interval Gauss elimination method has been applied to solve the obtained interval system of equations. In the final part of the paper, results of numerical computations are shown.

Key words: directed interval arithmetic, interval boundary element method, heat transfer

1. Introduction

Heat transfer problems are usually solved using equations with deterministic parameters. However, in most cases of the engineering practice, values of these parameters cannot be defined with a high precision, and in such cases it is much more convenient to define these parameters as intervals.

In the available literature we can find examples of papers using the interval arithmetic (Neumaier, 1990) and the theory of fuzzy sets (Zadeh, 1965) allowing one to solve problems taking into account "uncertainties" in the mathematical model. We can also find papers dedicated to interval boundary element method (Burczyński and Skrzypczyk, 1997) and the interval finite element method (Muhanna *et al.*, 2005). However, most of these papers are related to boundary problems, and we can hardly find any examples of papers dealing with boundary-initial problems.

In this paper, the Interval Boundary Element Method (IBEM) for solving non-steady heat transfer problems with directed interval thermophysical parameters of both sub-domains has been presented with the approach of the directed interval arithmetic (Markov, 1995, Popova, 2001). This assumption is closer to real physical conditions of the process considered because it is difficult to estimate the thermophysical parameters appearing in the mathematical model. The main advantage of the directed interval arithmetic upon the classical interval arithmetic is that the obtained temperature intervals are much narrower.

In theory as well as in practice, it is valuable to develop the Interval Boundary Element Method (IBEM).

2. Directed interval arithmetic

Let us consider a directed interval $\widetilde{a}$ which can be defined as a set $\boldsymbol{D}$ of all directed pairs of real numbers defined as follows (Kużelewski, 2008; Markov, 1995; Popova, 2001)

$$\widetilde{a} = \langle a^-, a^+ \rangle := \{\widetilde{a} \in \boldsymbol{D} |\ a^-, a^+ \in \boldsymbol{R}\} \tag{2.1}$$

where a^- and a^+ denote the beginning and the end of the interval, respectively.

The left or the right endpoint of the interval $\widetilde{a}$ can be denoted as a^s, $s \in \{+, -\}$, where s is a binary variable. This variable can be expressed as a product of two binary variables and is defined as

$$\begin{aligned} ++ = -- = + \\ +- = -+ = - \end{aligned} \tag{2.2}$$

An interval is called proper if $a^- \leqslant a^+$, improper if $a^- \geqslant a^+$ and degenerate if $a^- = a^+$. The set of all directed interval numbers can be written as $\boldsymbol{D} = \boldsymbol{P} \cup \boldsymbol{I}$, where $\boldsymbol{P}$ denotes the set of all directed proper intervals and $\boldsymbol{I}$ denotes the set of all improper intervals.

Additionally, a subset $\boldsymbol{Z} = \boldsymbol{Z}_P \cup \boldsymbol{Z}_I \in \boldsymbol{D}$ should be defined, where

$$\boldsymbol{Z}_P = \{\widetilde{a} \in \boldsymbol{P} |\ a^- \leqslant 0 \leqslant a^+\} \qquad\qquad \boldsymbol{Z}_I = \{\widetilde{a} \in \boldsymbol{I} |\ a^+ \leqslant 0 \leqslant a^-\} \tag{2.3}$$

For directed interval numbers two binary variables are defined. The first of them is the direction variable

$$\tau(\widetilde{a}) = \begin{cases} + & \text{if} \quad a^- \leqslant a^+ \\ - & \text{if} \quad a^- > a^+ \end{cases} \tag{2.4}$$

and the other is the sign variable

$$\sigma(\widetilde{a}) = \begin{cases} + & \text{if} \quad a^- > 0, \quad a^+ > 0 \\ - & \text{if} \quad a^- < 0, \quad a^+ < 0 \end{cases} \qquad \widetilde{a} \in \boldsymbol{D}\backslash\boldsymbol{Z} \tag{2.5}$$

For the zero argument $\sigma(\langle 0,0\rangle) = \sigma(0) = +$, for all intervals including the zero element $\widetilde{a} \in \boldsymbol{Z}$, $\sigma(\widetilde{a})$ is not defined.

The sum of two directed intervals $\widetilde{a} = \langle a^-, a^+\rangle$ and $\widetilde{b} = \langle b^-, b^+\rangle$ can be written as

$$\widetilde{a} + \widetilde{b} = \langle a^- + b^-, a^+ + b^+\rangle \qquad \widetilde{a}, \widetilde{b} \in \boldsymbol{D} \tag{2.6}$$

The difference is of the form

$$\widetilde{a} - \widetilde{b} = \langle a^- - b^+, a^+ - b^-\rangle \qquad \widetilde{a}, \widetilde{b} \in \boldsymbol{D} \tag{2.7}$$

The product of the directed intervals is described by the formula

$$\widetilde{a} \cdot \widetilde{b} = \begin{cases} \left\langle a^{-\sigma(\widetilde{b})} \cdot b^{-\sigma(\widetilde{a})}, a^{\sigma(\widetilde{b})} \cdot b^{\sigma(\widetilde{a})} \right\rangle & \widetilde{a}, \widetilde{b} \in \boldsymbol{D}\backslash\boldsymbol{Z} \\ \left\langle a^{\sigma(\widetilde{a})\tau(\widetilde{b})} \cdot b^{-\sigma(\widetilde{a})}, a^{\sigma(\widetilde{a})\tau(\widetilde{b})} \cdot b^{\sigma(\widetilde{a})} \right\rangle & \widetilde{a} \in \boldsymbol{D}\backslash\boldsymbol{Z}, \quad \widetilde{b} \in \boldsymbol{Z} \\ \left\langle a^{-\sigma(\widetilde{b})} \cdot b^{\sigma(\widetilde{b})\tau(\widetilde{a})}, a^{\sigma(\widetilde{b})} \cdot b^{\sigma(\widetilde{b})\tau(\widetilde{a})} \right\rangle & \widetilde{a} \in \boldsymbol{Z}, \quad \widetilde{b} \in \boldsymbol{D}\backslash\boldsymbol{Z} \\ \langle \min(a^- \cdot b^+, a^+ \cdot b^-), \max(a^- \cdot b^-, a^+ \cdot b^+)\rangle & \widetilde{a}, \widetilde{b} \in \boldsymbol{Z}_P \\ \langle \max(a^- \cdot b^-, a^+ \cdot b^+), \min(a^- \cdot b^+, a^+ \cdot b^-)\rangle & \widetilde{a}, \widetilde{b} \in \boldsymbol{Z}_I \\ 0 \qquad (\widetilde{a} \in \boldsymbol{Z}_P, \widetilde{b} \in \boldsymbol{Z}_I) \cup (\widetilde{a} \in \boldsymbol{Z}_I, \widetilde{b} \in \boldsymbol{Z}_P) \end{cases} \tag{2.8}$$

The quotient of two directed intervals can be written as

$$\widetilde{a}/\widetilde{b} = \begin{cases} \left\langle a^{-\sigma(\widetilde{b})}/b^{\sigma(\widetilde{a})}, a^{\sigma(\widetilde{b})}/b^{-\sigma(\widetilde{a})} \right\rangle & \widetilde{a}, \widetilde{b} \in \boldsymbol{D}\backslash\boldsymbol{Z} \\ \left\langle a^{-\sigma(\widetilde{b})}/b^{-\sigma(\widetilde{b})\tau(\widetilde{a})}, a^{\sigma(\widetilde{b})}/b^{-\sigma(\widetilde{b})\tau(\widetilde{a})} \right\rangle & \widetilde{a} \in \boldsymbol{Z}, \quad \widetilde{b} \in \boldsymbol{D}\backslash\boldsymbol{Z} \end{cases} \tag{2.9}$$

In the directed interval arithmetic, two extra operators are defined – inversion of summation

$$-_D \widetilde{a} = \langle -a^-, -a^+\rangle \qquad \widetilde{a} \in \boldsymbol{D} \tag{2.10}$$

and inversion of multiplication

$$1/_D \widetilde{a} = \langle 1/a^-, 1/a^+\rangle \qquad \widetilde{a} \in \boldsymbol{D}\backslash\boldsymbol{Z} \tag{2.11}$$

So, two additional mathematical operations can be defined as follows

$$\widetilde{a} -_D \widetilde{b} = \langle a^- - b^-, a^+ - b^+\rangle \qquad \widetilde{a}, \widetilde{b} \in \boldsymbol{D} \tag{2.12}$$

and

$$\widetilde{a}/_D\widetilde{b} = \begin{cases} \left\langle a^{-\sigma(\widetilde{b})}/b^{-\sigma(\widetilde{a})}, a^{\sigma(\widetilde{b})}/b^{\sigma(\widetilde{a})} \right\rangle & \widetilde{a}, \widetilde{b} \in \boldsymbol{D}\backslash\boldsymbol{Z} \\ \left\langle a^{-\sigma(\widetilde{b})}/b^{\sigma(\widetilde{b})}, a^{\sigma(\widetilde{b})}/b^{\sigma(\widetilde{b})} \right\rangle & \widetilde{a} \in \boldsymbol{Z}, \quad \widetilde{b} \in \boldsymbol{D}\backslash\boldsymbol{Z} \end{cases} \tag{2.13}$$

Now, it is possible to obtain the number zero by subtraction of two identical intervals $\widetilde{a} -_D \widetilde{a} = 0$ and the number one as the result of the division $\widetilde{a}/_D\widetilde{a} = 1$, which was impossible when applying classical interval arithmetic (Neumayer, 1990).

3. Heat transfer model in two-layered domain

Let us consider a two-layered domain of dimension $L = L_1 + L_2$. The heat conduction process in the first sub-domain is described by the following energy equation (Mochnacki and Suchy, 1995; Majchrzak, 2001)

$$x \in (0, L_1): \qquad \langle c_1^-, c_1^+ \rangle \frac{\partial T_1(x,t)}{\partial t} = \langle \lambda_1^-, \lambda_1^+ \rangle \frac{\partial^2 T_1(x,t)}{\partial x^2} \tag{3.1}$$

where $\langle c_1^-, c_1^+ \rangle$ is the directed interval volumetric specific heat for the first sub-domain, $\langle \lambda_1^-, \lambda_1^+ \rangle$ is the directed interval thermal conductivity, T_1, x, t denote temperature, spatial co-ordinate and time, respectively.

Equation (3.1) can be expressed as follows

$$x \in (0, L_1): \qquad \frac{\partial T_1(x,t)}{\partial t} = \langle a_1^-, a_1^+ \rangle \frac{\partial^2 T_1(x,t)}{\partial x^2} \tag{3.2}$$

where $\langle a_1^-, a_1^+ \rangle = \langle \lambda_1^-, \lambda_1^+ \rangle / \langle c_1^-, c_1^+ \rangle$ is the directed interval diffusion coefficient, and its beginning and end can be defined according to the rules of the directed interval arithmetic (Markov, 1995).

Taking into account the assumption that $\widetilde{\lambda}_1, \widetilde{c}_1 \in \boldsymbol{D}\backslash\boldsymbol{Z}$. one obtains the following formula

$$\widetilde{\lambda}_1/_D\widetilde{c}_1 = \left\langle \lambda_1^{-\sigma(\widetilde{c}_1)}/c_1^{-\sigma(\widetilde{\lambda}_1)}, \lambda_1^{\sigma(\widetilde{c}_1)}/c_1^{\sigma(\widetilde{\lambda}_1)} \right\rangle \tag{3.3}$$

For example, for the interval coefficients $\widetilde{\lambda}_1 = \langle 34, 35 \rangle$ and $\widetilde{c}_1 = \langle 4900000, 5400000 \rangle$ the sign variables are $\sigma(\widetilde{\lambda}_1) = +$, $\sigma(\widetilde{c}_1) = +$, so the quotient of $\widetilde{\lambda}_1$ and $\widetilde{c}_1$ can be calculated according to the formula

$$\widetilde{\lambda}_1/_D\widetilde{c}_1 = \langle \lambda_1^{-+}/c_1^{-+}, \lambda_1^{+}/c_1^{+} \rangle = \langle \lambda_1^-/c_1^-, \lambda_1^+/c_1^+ \rangle \tag{3.4}$$

and the directed interval diffusion coefficient $\widetilde{a}_1$ is computed as follows

$$
\begin{aligned}
\widetilde{a}_1 &= \frac{\widetilde{\lambda}_1}{\widetilde{c}_1} = \frac{\langle 34, 36\rangle}{\langle 4900000, 5400000\rangle} = \langle 34, 36\rangle /_D \langle 4900000, 5400000\rangle = \\
&= \left\langle \frac{34}{4900000}, \frac{36}{5400000} \right\rangle \approx \langle 0.0000069, 0.0000066\rangle
\end{aligned}
\tag{3.5}
$$

As a result, the interval obtained is improper.

The temperature field in the other sub-domain is determined by the energy equation

$$
x \in (L_1, L_2): \qquad \langle c_2^-, c_2^+\rangle \frac{\partial T_2(x,t)}{\partial t} = \langle \lambda_2^-, \lambda_2^+\rangle \frac{\partial^2 T_2(x,t)}{\partial x^2} \tag{3.6}
$$

where $\langle c_2^-, c_2^+\rangle$, $\langle \lambda_2^-, \lambda_2^+\rangle$ are the directed interval values of volumetric specific heat and thermal conductivity, respectively, and T_2 denotes temperature for the second sub-domain.

The above equation, (3.6), can be transformed as follows

$$
x \in (L_1, L_2): \qquad \frac{\partial T_2(x,t)}{\partial t} = \langle a_2^-, a_2^+\rangle \frac{\partial^2 T_2(x,t)}{\partial x^2} \tag{3.7}
$$

where $\langle a_2^-, a_2^+\rangle = \langle \lambda_2^-, \lambda_2^+\rangle / \langle c_2^-, c_2^+\rangle$ is the directed interval diffusion coefficient for the second layer.

Equations (3.2) and (3.7) must be supplemented by the boundary-initial conditions of the following form

$$
\begin{aligned}
&x = 0: &&\widetilde{q}(0,t) = -\langle \lambda_1^-, \lambda_1^+\rangle \frac{\partial T_1(x,t)}{\partial x} = \widetilde{q}_L \\
&x = L_2: &&\widetilde{q}(L_2,t) = -\langle \lambda_2^-, \lambda_2^+\rangle \frac{\partial T_2(x,t)}{\partial x} = \widetilde{q}_R \\
&t = 0: &&T_1(x,0) = T_{10}(x) \qquad T_2(x,0) = T_{20}(x)
\end{aligned}
\tag{3.8}
$$

and the continuity condition on the contact surface between two layers

$$
x = L_1: \qquad \begin{cases} -\langle \lambda_1^-, \lambda_1^+\rangle \dfrac{\partial T_1(x,t)}{\partial x} = -\langle \lambda_2^-, \lambda_2^+\rangle \dfrac{\partial T_2(x,t)}{\partial x} \\ T_1(x,t) = T_2(x,t) \end{cases} \tag{3.9}
$$

where $\widetilde{q}_L$, $\widetilde{q}_R$ are the given interval boundary heat fluxes, T_{10} and T_{20} are the initial temperatures for the first and second layer, respectively.

4. Interval boundary element method

In this paper, the 1st scheme of the interval boundary element method is used (Brebbia *et al.*, 1984; Majchrzak, 1998, 2001). At first, the time grid must be introduced

$$0 = t^0 < t^1 < t^2 < \ldots < t^{f-1} < t^f < \ldots < t^F < \infty \tag{4.1}$$

with a constant time step $\Delta t = t^f - t^{f-1}$.

Let us consider the constant elements with respect to time (Majchrzak, 2001, Majchrzak and Mochnacki, 1996). The boundary integral equation corresponding to the transition $t^{f-1} \to t^f$ for the first layer is following

$$\begin{aligned}
&\widetilde{T}_1(\xi, t^f) + \Big[\frac{1}{\widetilde{c}_1}\widetilde{q}_1(x, t^f)\int\limits_{t^{f-1}}^{t^f} \widetilde{T}_1^*(\xi, x, t^f, t)\; dt\Big]_{x=0}^{x=L_1} = \\
&= \Big[\frac{1}{\widetilde{c}_1}\widetilde{T}_1(x, t^f)\int\limits_{t^{f-1}}^{t^f} \widetilde{q}_1^*(\xi, x, t^f, t)\; dt\Big]_{x=0}^{x=L_1} + \int\limits_0^{L_1} \widetilde{T}_1^*(\xi, x, t^f, t^{f-1})\widetilde{T}_1(x, t^{f-1})\; dx
\end{aligned} \tag{4.2}$$

where ξ is the observation point, $\widetilde{q}_1(x, t^f)$ is the directed interval heat flux. The directed interval fundamental solution $T_1^*(\xi, x, t^f, t)$ is of the following form (Brebbia *et al.*, 1984; Majchrzak, 2001)

$$\widetilde{T}_1^*(\xi, x, t^f, t) = \frac{1}{2\sqrt{\pi\widetilde{a}_1(t^f - t)}}\exp\Big[-\frac{(x-\xi)^2}{4\widetilde{a}_1(t^f - t)}\Big] \tag{4.3}$$

and should be calculated according to the formula

$$\langle T^{*-}, T^{*+}\rangle = \exp\Big\{\Big[-\frac{(x-\xi)^2}{4(t^f - t)}\Big]/_D\langle a_1^-, a_1^+\rangle\Big\}/_D\Big[2\sqrt{\pi\langle a_1^-, a_1^+\rangle(t^f - t)}\Big] \tag{4.4}$$

Because $\sigma\{-(x-\xi)^2/[4(t^f - t)]\} = -$, $\sigma(\widetilde{a}_1) = +$, the interval fundamental solution can be calculated as follows (see Eq. (2,13)) (Markov, 1995; Moore and Bierbaum, 1979)

$$\begin{aligned}
&\langle T^{*-}, T^{*+}\rangle = \\
&= \Big\langle \exp\Big[-\frac{(x-\xi)^2}{4a_1^+(t^f - t)}\Big], \exp\Big[-\frac{(x-\xi)^2}{4a_1^-(t^f - t)}\Big]\Big\rangle/_D\Big[2\sqrt{\pi\langle a_1^-, a_1^+\rangle(t^f - t)}\Big]
\end{aligned} \tag{4.5}$$

The directed interval heat flux resulting from the interval fundamental solution should be found in an analytical way, and then

$$
\begin{aligned}
\widetilde{q}_1^*(\xi,x,t^f,t) &= -\langle\lambda_1^-,\lambda_1^+\rangle \frac{\partial \widetilde{T}_1^*(\xi,x,t^f,t)}{\partial n} = \langle\lambda_1^-,\lambda_1^+\rangle(x-\xi)\cdot \\
&\cdot\Big\langle \exp\Big[-\frac{(x-\xi)^2}{4a_1^+(t^f-t)}\Big], \exp\Big[-\frac{(x-\xi)^2}{4a_1^-(t^f-t)}\Big]\Big\rangle /_D \{4\sqrt{\pi}[\langle a_1^-,a_1^+\rangle(t^f-t)]^{3/2}\}
\end{aligned}
\tag{4.6}
$$

The boundary integral equation corresponding to the transition $t^{f-1}\to t^f$ for the other layer can be expressed as follows

$$
\begin{aligned}
&\widetilde{T}_2(\xi,t^f) + \Big[\frac{1}{\widetilde{c}_2}\widetilde{q}_2(x,t^f)\int\limits_{t^{f-1}}^{t^f} \widetilde{T}_2^*(\xi,x,t^f,t)\;dt\Big]_{x=L_1}^{x=L_2} = \\
&= \Big[\frac{1}{\widetilde{c}_2}\widetilde{T}_2(x,t^f)\int\limits_{t^{f-1}}^{t^f} \widetilde{q}_2^*(\xi,x,t^f,t)\;dt\Big]_{x=L_1}^{x=L_2} + \int\limits_{L_1}^{L_2} \widetilde{T}_2^*(\xi,x,t^f,t^{f-1})\widetilde{T}_2(x,t^{f-1})\;dx
\end{aligned}
\tag{4.7}
$$

where $\widetilde{T}_2^*(\xi,x,t^f,t)$ is the directed interval fundamental solution for the second sub-domain.

The numerical approximation of interval equations (4.2) and (4.7) leads to the system of interval equations

$$
\begin{bmatrix}
-\widetilde{H}_{11}^1 & -\widetilde{H}_{12}^1 & \widetilde{G}_{12}^1 & 0\\
-\widetilde{H}_{21}^1 & -\widetilde{H}_{22}^1 & \widetilde{G}_{22}^1 & 0\\
0 & -\widetilde{H}_{11}^2 & \widetilde{G}_{11}^2 & -\widetilde{H}_{12}^2\\
0 & -\widetilde{H}_{21}^2 & \widetilde{G}_{21}^2 & -\widetilde{H}_{22}^2
\end{bmatrix}
\begin{bmatrix}
\widetilde{T}_1(0,t^f)\\ \widetilde{T}_1(L_1,t^f)\\ \widetilde{q}(L_1,t^f)\\ \widetilde{T}_2(L_2,t^f)
\end{bmatrix}
=
\begin{bmatrix}
-\widetilde{G}_{11}^1\cdot\widetilde{q}_L\\ -\widetilde{G}_{21}^1\cdot\widetilde{q}_L\\ -\widetilde{G}_{12}^2\cdot\widetilde{q}_R\\ -\widetilde{G}_{22}^2\cdot\widetilde{q}_R
\end{bmatrix}
+
\begin{bmatrix}
\widetilde{P}_1(0,t^{f-1})\\ \widetilde{P}_1(L_1,t^{f-1})\\ \widetilde{P}_2(L_1,t^{f-1})\\ \widetilde{P}_2(L_2,t^{f-1})
\end{bmatrix}
\tag{4.8}
$$

where

$$
\begin{aligned}
\widetilde{G}_{11}^e &= -\widetilde{G}_{22}^e = -\frac{\sqrt{\varDelta t}}{\sqrt{\widetilde{\lambda}_e\widetilde{c}_e\pi}} \qquad\qquad L_0 = 0\\
\widetilde{G}_{12}^e &= -\widetilde{G}_{21}^e = \frac{\sqrt{\varDelta t}}{\sqrt{\widetilde{\lambda}_e\widetilde{c}_e\pi}}\exp\Big[-\frac{(L_e-L_{e-1})^2}{4\widetilde{a}_e\varDelta t}\Big] - \frac{L_e-L_{e-1}}{2\widetilde{\lambda}_e}\operatorname{erfc}\Big(\frac{L_e-L_{e-1}}{2\sqrt{\widetilde{a}_e\varDelta t}}\Big)
\end{aligned}
\tag{4.9}
$$

and

$$
\widetilde{H}_{11}^e = \widetilde{H}_{22}^e = -\frac{1}{2} \qquad\qquad \widetilde{H}_{12}^e = \widetilde{H}_{21}^e = \frac{1}{2}\operatorname{erfc}\Big(\frac{L_e-L_{e-1}}{2\sqrt{\widetilde{a}_e\varDelta t}}\Big)
\tag{4.10}
$$

while

$$
\begin{aligned}
\widetilde{P}_1(0,t^{f-1}) &= \frac{1}{2\sqrt{\pi\widetilde{a}_1\Delta t}}\int\limits_0^{L_1}\exp\Big[-\frac{x^2}{4\widetilde{a}_1\Delta t}\Big]\widetilde{T}_1(x,t^{f-1})\,dx\\
\widetilde{P}_1(L_1,t^{f-1}) &= \frac{1}{2\sqrt{\pi\widetilde{a}_1\Delta t}}\int\limits_0^{L_1}\exp\Big[-\frac{(x-L_1)^2}{4\widetilde{a}_1\Delta t}\Big]\widetilde{T}_1(x,t^{f-1})\,dx\\
\widetilde{P}_2(L_1,t^{f-1}) &= \frac{1}{2\sqrt{\pi\widetilde{a}_2\Delta t}}\int\limits_{L_1}^{L_2}\exp\Big[-\frac{(x-L_1)^2}{4\widetilde{a}_2\Delta t}\Big]\widetilde{T}_2(x,t^{f-1})\,dx\\
\widetilde{P}_2(L_2,t^{f-1}) &= \frac{1}{2\sqrt{\pi\widetilde{a}_2\Delta t}}\int\limits_{L_1}^{L_2}\exp\Big[-\frac{(x-L_2)^2}{4\widetilde{a}_2\Delta t}\Big]\widetilde{T}_2(x,t^{f-1})\,dx
\end{aligned}
\tag{4.11}
$$

where $e=1$ denotes the first layer, $e=2$ denotes the other one.

For example, for the interval coefficients $\widetilde{\lambda}_1 = \langle 34, 36\rangle$ and $\widetilde{c}_1 = \langle 4900000, 5400000\rangle$, the product of $\widetilde{\lambda}_1$ and $\widetilde{c}_1$ is calculated according to the formula

$$
\begin{aligned}
\widetilde{\lambda}_1\cdot\widetilde{c}_1 &= \langle \lambda_1^{-+}\cdot c_1^{-+}, \lambda_1^{+}\cdot c_1^{+}\rangle = \langle \lambda_1^{-}\cdot c_1^{-}, \lambda_1^{+}\cdot c_1^{+}\rangle =\\
&= \langle 34\cdot 4900000, 36\cdot 5400000\rangle = \langle 1.666\cdot 10^8, 1.944\cdot 10^8\rangle
\end{aligned}
\tag{4.12}
$$

The interval Gauss elimination method (Neumayer, 1990; Piasecka-Belkhayat, 2008) has been used to solve the interval system of equations (4.8). After determining the 'missing' boundary values for both layers, the interval temperatures $\widetilde{T}_e^f$ at the internal points ξ^i can be calculated using the formulas:

— for the first layer $(\xi\in(0,L_1))$

$$
\begin{aligned}
\widetilde{T}_1(\xi,t^f) &= \frac{1}{2}\exp\Big(-\frac{L_1-\xi}{\sqrt{\widetilde{a}_1\Delta t}}\Big)\widetilde{T}_1(L_1,t^f) + \frac{1}{2}\exp\Big(-\frac{\xi}{\sqrt{\widetilde{a}_1\Delta t}}\Big)\widetilde{T}_1(0,t^f) +\\
&+\frac{\sqrt{\Delta t}}{2\sqrt{\widetilde{\lambda}_1\widetilde{c}_1}}\exp\Big(-\frac{L_1-\xi}{\sqrt{\widetilde{a}_1\Delta t}}\Big)\widetilde{q}(L_1,t^f) +\\
&+\frac{\sqrt{\Delta t}}{2\sqrt{\widetilde{\lambda}_1\widetilde{c}_1}}\exp\Big(-\frac{\xi}{\sqrt{\widetilde{a}_1\Delta t}}\Big)q_L(0,t^f) + \widetilde{P}_1(\xi,t^{f-1})
\end{aligned}
\tag{4.13}
$$

— for the other layer $(\xi \in (L_1, L_2))$

$$
\begin{aligned}
\widetilde{T}_2(\xi, t^f) &= \frac{1}{2}\exp\Big(-\frac{L_2-\xi}{\sqrt{\widetilde{a}_2 \Delta t}}\Big)\widetilde{T}_2(L_2, t^f) + \frac{1}{2}\exp\Big(-\frac{\xi - L_1}{\sqrt{\widetilde{a}_2 \Delta t}}\Big)\widetilde{T}_2(L_1, t^f) + \\
&+ \frac{\sqrt{\Delta t}}{2\sqrt{\widetilde{\lambda}_2 \widetilde{c}_2}}\exp\Big(-\frac{L_2-\xi}{\sqrt{\widetilde{a}_2 \Delta t}}\Big) q_R(L_2, t^f) + \\
&+ \frac{\sqrt{\Delta t}}{2\sqrt{\widetilde{\lambda}_2 \widetilde{c}_2}}\exp\Big(-\frac{\xi - L_1}{\sqrt{\widetilde{a}_2 \Delta t}}\Big) \widetilde{q}(L_1, t^f) + \widetilde{P}_2(\xi, t^{f-1})
\end{aligned}
\tag{4.14}
$$

5. Numerical example

In the paper, an example of one-dimensional heat transient transfer in a two-layered domain of dimensions $L_1 = 0.02\,\mathrm{m}$ and $L_2 = 0.02\,\mathrm{m}$ is presented. On both sides the boundary condition of the 2nd type of the form $\widetilde{q}_L = \langle 9800, 10200\rangle\ \mathrm{W/m^2}$ and $\widetilde{q}_R = 0\,\mathrm{W/m^2}$ has been assumed. The first and other layer have been divided into 20 constant internal cells, respectively.

The following input data have been introduced: $\lambda_1 = 90\,\mathrm{W/(m\,K)}$, $c_1 = 3.916\,\mathrm{MJ/(m^3K)}$, $\lambda_2 = 35\,\mathrm{W/(m\,K)}$, $c_2 = 5.175\,\mathrm{MJ/(m^3K)}$. The initial temperature of the first and other sub-domain is $T_{01} = T_{02} = 20°\mathrm{C}$, time step $\Delta t = 1\,\mathrm{s}$.

All thermophysical parameters of the two-layered domain are assumed to be directed interval values:

$$
\begin{aligned}
\widetilde{\lambda}_1 &= \langle \lambda_1 - 0.05\lambda_1, \lambda_1 + 0.05\lambda_1\rangle \\
\widetilde{c}_1 &= \langle c_1 - 0.05c_1, c_1 + 0.05c_1\rangle \\
\widetilde{\lambda}_2 &= \langle \lambda_2 - 0.05\lambda_2, \lambda_2 + 0.05\lambda_2\rangle \\
\widetilde{c}_2 &= \langle c_2 - 0.05c_2, c_2 + 0.05 \cdot c_2\rangle
\end{aligned}
$$

Figure 1 illustrates the temperature distribution in the domain analysed obtained for chosen times. The dashed and solid lines denote the lower and the upper bounds of the temperature intervals, respectively.

Figure 2 shows a comparison between the temperature distribution obtained for the time 80 s and the results obtained with classical BEM for thermophysical parameters defined without intervals (dotted line).

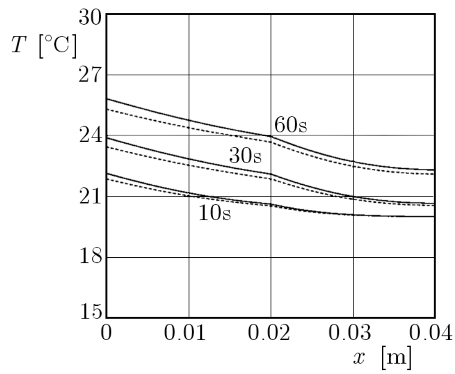

Fig. 1. Temperature distribution for chosen times

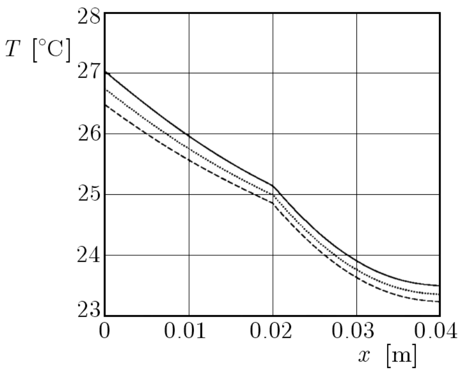

Fig. 2. Comparison of temperature distribution for the time 80 s

6. Conclusions

In this paper, the description of an unsteady heat transfer for 1D problem for the two-layered domain has been presented. All the thermophysical parameters appearing in the mathematical model of the domain analysed have been considered as directed interval values. The problem discussed has been solved using the 1st scheme of the interval boundary element method according to the rules of the directed interval arithmetic.

The main advantage of the directed interval arithmetic upon the classical interval arithmetic is that the obtained temperature intervals are much narrower (Piasecka-Belkhayat, 2007). The problem analysed can be extended to multi-layered domains.

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Przedziałowa metoda elementów brzegowych dla zadań dyfuzji w obszarach dwuwarstwowych

Streszczenie

W pracy przedstawiono opis nieustalonego przepływu ciepła dla zadań jednowymiarowych w obszarach dwuwarstwowych. Założono, że wszystkie parametry termofizyczne pojawiające się w opisie matematycznym analizowanego zadania są wartościami przedziałowymi skierowanymi. Omawiane zagadnienie zostało rozwiązane za pomocą pierwszego schematu przedziałowej metody elementów brzegowych. Do rozwiązania otrzymanego interwałowego układu równań zastosowano przedziałową metodę eliminacji Gaussa. W końcowej części pracy pokazano wyniki obliczeń numerycznych.

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