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DYNAMIC ANALYSIS OF TORSIONAL DISCRETE-CONTINUOUS SYSTEMS WITH POSITION-DEPENDENT VARIABLE INERTIA

Amalia Pielorz

Kielce University of Technology, Faculty of Management and Computer Modelling, Kielce, Poland

e-mail: apielorz@tu.kielce.pl

In the paper, dynamic investigations of multi-mass discrete-continuous systems torsionally deformed are performed taking into account the position-dependent mass moment of inertia of one of rigid bodies. It is assumed that the rigid body representing a motor moves with a variable velocity. An originally nonlinear problem is linearized, and the wave approach leading to solving equations with a retarded argument having variable coefficients is used for the solution. Exemplary numerical calculations are done for a three-mass system.

Key words: variable inertia, discrete-continuous systems, wave approach

1. Introduction

Variable inertia problems play an important role in dynamics of various mechanisms and machines undergoing torsional deformations. It appears that in rotating and vibrating systems, the mass moment of inertia of rigid bodies can depend on the angular displacement. For example, in the crankshaft of a multi-cylinder engine the total moment of inertia of the rotating parts reduced to the axis of the crankshaft is not constant but it depends on the location of the cranks, see Hesterman and Stone (1995), Koser and Pasin (1995, 1997), Pasricha and Carnegie (1979), Sinha and Butcher (1997), Turhan and Koser (2004), Zajączkowski (1987). Systems with position-dependent inertia in the technical literature are analyzed using discrete models, cf Pasricha and Carnegie (1979), Sinha and Butcher (1997), Zajączkowski (1987), discrete-continuous models, cf Koser and Pasin (1995, 1997), Pielorz and Skóra (2006, 2009a) or a finite element method, cf Turhan and Koser (2004). Two-mass discrete-continuous systems were studied by Koser and Pasin (1995, 1997) while multi-mass systems were discussed by Pielorz and Skóra (2006, 2009a). In the papers by Koser and Pasin (1995), Pielorz and Skóra (2006, 2009a) it is assumed that the rigid body representing a motor has a constant velocity.

The aim of the present paper is to generalize the considerations performed by Koser and Pasin (1995, 1997), Pielorz and Skóra (2006, 2009a). This is done for multi-mass systems with the assumptions that the rigid body representing the motor moves with a variable velocity. In the discussion, a wave approach is applied similarly to studies of discrete-continuous systems, see e.g. Nadolski and Pielorz (2001), Pielorz (1999, 2003, 2007, 2010), Pielorz and Skóra (2006, 2009b). In the numerical calculations, the effect of variable inertia on the behavior of a three-mass system is considered, including parameters describing the motor.

2. Governing equations for a multi-mass system

Consider the multi-mass discrete-continuous system torsionally deformed as shown in Fig. 1. It consists of rigid bodies connected by means of the shafts. The central axis of shafts, together with elements settled on them, coincide with the main axis of the system. The x-axis is parallel to the main axis of the system, and its origin coincides with the position of the left-hand end of the first shaft at time instant $t = 0$. The equations of motion for the shafts are classical wave equations.

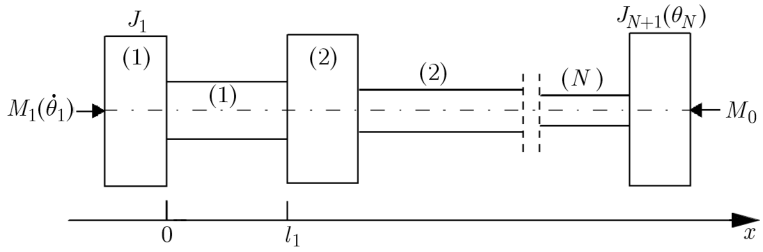

Fig. 1. Discrete-continuous multi-mass torsional system

The i-th shaft, $i = 1, 2, \ldots, N$, is characterized by the length l_i, density ρ, shear modulus G and polar moment of inertia I_{0i}. The i-th rigid body of the model is characterized by the mass moment of inertia J_i. The mass moment of inertia of the last rigid body depends on its angular displacement, i.e., $J_{N+1} = J_{N+1}(\theta_N)$. The considered system can represent shafts connected to mechanisms with position-dependent inertia.

The first rigid body, having the constant mass moment of inertia J_1, represents an electrical motor. It is loaded by the motor torque $M_1 = M_0 + K(\varOmega_0 - \partial\theta_1/\partial t)$, where M_0 is the nominal torque, $\varOmega_0$ is the nominal angular velocity and K is the slope of the motor characteristic. The last rigid body is loaded by an equivalent resistance torque equal to M_0. Damping in the shafts is neglected. The study is concentrated in a forced response resulting from the variable inertia of (N+1)-th rigid body.

On the above assumptions, the determination of displacements θ_i of the elastic elements of the considered system is reduced to solving N equations

$$\frac{\partial^2\theta_i}{\partial t^2} - c^2\frac{\partial^2\theta_i}{\partial x^2} = 0 \qquad i = 1, 2, \ldots, N \tag{2.1}$$

with the following boundary conditions

$$\begin{aligned}
&J_1\frac{\partial^2\theta_1}{\partial t^2} - GI_{01}\frac{\partial\theta_1}{\partial x} = M_0 + K\Big(\varOmega_0 - \frac{\partial\theta_1}{\partial t}\Big) \qquad \text{for} \quad x = 0\\
&\theta_i(x,t) = \theta_{i+1}(x,t) \qquad \text{for} \quad x = \sum_{k=1}^{i} l_k \qquad i = 1, 2, \ldots, N-1\\
&-J_{i+1}\frac{\partial^2\theta_i}{\partial t^2} - GI_{0i}\frac{\partial\theta_i}{\partial x} + GI_{0,i+1}\frac{\partial\theta_{i+1}}{\partial x} = 0 \qquad \text{for} \quad x = \sum_{k=1}^{i} l_k \qquad i = 1, 2, \ldots, N-1\\
&J_{N+1}\frac{\partial^2\theta_N}{\partial t^2} + \frac{1}{2}\frac{dJ_{N+1}}{d\theta_N}\Big(\frac{\partial\theta_N}{\partial t}\Big)^2 + GI_{0N}\frac{\partial\theta_N}{\partial x} = -M_0 \qquad \text{for} \quad x = \sum_{k=1}^{N} l_k
\end{aligned} \tag{2.2}$$

where $c^2 = G/\rho$ is the wave speed. The condition in the cross-section $x = l_1 + l_2 + \ldots + l_N$ can have various forms. Here it is assumed in the form suggested by Koser and Pasin (1995, 1997). Boundary conditions (2.2) are more general than those in the papers by Pielorz and Skóra (2006, 2009a). They differ in the condition for $x = 0$.

The above problem is nonlinear. The paper is a generalization of the studies presented by Koser and Pasin (1995, 1997), Pielorz and Skóra (2006, 2009a), so in analogy to these papers, it is linearized by introducing the following new unknown functions

$$\theta_i(x,t) = \varOmega_0 t + \alpha_i(x,t) - \frac{M_0}{GI_{0i}}\Big(x - \sum_{k=1}^{i} l_k\Big) \qquad i = 1, 2, \ldots, N \tag{2.3}$$

the variable mass moment of inertia $J_{N+1}(\theta_N) = J_{N+1}(\Omega_0 t + \alpha_N)$ with its first derivative

$$\frac{dJ_{N+1}}{d\theta_N} = \frac{J_{N+1}}{d\theta_N}(\Omega_0 t + \alpha_N)$$

are expanded in the Taylor series around $\Omega_0 t$, and the second order together with higher order terms are neglected.

Then the determination of functions α_i is reduced to solving N wave equations

$$\frac{\partial^2 \alpha_i}{\partial t^2} - c^2 \frac{\partial^2 \alpha_i}{\partial x^2} = 0 \qquad i = 1, 2, \ldots, N \tag{2.4}$$

with linear boundary conditions

$$\begin{aligned}
&J_1 \frac{\partial^2 \alpha_1}{\partial t^2} + K \frac{\partial \alpha_1}{\partial t} - GI_{01} \frac{\partial \alpha_1}{\partial x} = 0 \qquad \text{for} \quad x = 0 \\
&\alpha_i(x,t) = \alpha_{i+1}(x,t) + \frac{M_0}{GI_{0,i+1}} l_{i+1} \qquad \text{for} \quad x = \sum_{k=1}^{i} l_k \qquad i = 1, 2, \ldots, N-1 \\
&-J_{i+1} \frac{\partial^2 \alpha_i}{\partial t^2} - GI_{0i} \frac{\partial \alpha_i}{\partial x} + GI_{0,i+1} \frac{\partial \alpha_{i+1}}{\partial x} = 0 \qquad \text{for} \quad x = \sum_{k=1}^{i} l_k \qquad i = 1, 2, \ldots, N-1 \\
&J_{N+1} \frac{\partial^2 \alpha_N}{\partial t^2} + \frac{dJ_{N+1}}{dt} \frac{\partial \alpha_N}{\partial t} + GI_{0N} \frac{\partial \alpha_N}{\partial x} + \frac{1}{2} \frac{d^2 J_{N+1}}{dt^2} \alpha_N \\
&\qquad = -\frac{1}{2} \Omega_0 \frac{dJ_{N+1}}{dt} \qquad \text{for} \quad x = \sum_{k=1}^{N} l_k
\end{aligned} \tag{2.5}$$

The last boundary condition has now such a form that the direct effect of variable inertia can be investigated. It should be pointed out that after transformation (2.3), the nominal torque moment M_0 appears only in the boundary condition for displacements in cross-sections $x = l_1 + l_2 + \ldots + l_i$, $i = 1, 2, \ldots, N-1$. However, when the two-mass system with $N = 1$ is considered, no effect of M_0 is observed, cf Koser and Pasin (1995, 1997).

Boundary conditions (2.5) differ from the corresponding boundary conditions in the papers by Pielorz and Skóra (2006, 2009a) by the condition in the cross-section $x = 0$. This is connected with the fact that the velocity of the first rigid body of the multi-mass system in those papers was constant while in the present paper it is variable.

Upon the introduction of the following dimensionless quantities

$$\begin{array}{llll}
\overline{x} = \dfrac{x}{l_0} & \overline{t} = \dfrac{ct}{l_0} & \overline{\alpha}_i = \dfrac{\alpha_i}{\alpha_0} & K_r = \dfrac{I_{01}\rho l_0}{J_0} \\
E_i = \dfrac{J_0}{J_i} & \overline{\Omega}_0 = \dfrac{\Omega_0 l_0}{\alpha_0 c} & \overline{K} = \dfrac{K l_0}{J_0 c} & \overline{M}_0 = \dfrac{M_0 l_0^2}{J_0 c^2 \alpha_0} \\
\overline{J}_{N+1}(\overline{t}) = \dfrac{J_{N+1}(t)}{J_0} & \overline{l}_i = \dfrac{l_i}{l_0} & B_i = \dfrac{I_{0i}}{I_{01}} &
\end{array} \tag{2.6}$$

the determination of displacements $\alpha_i(x,t)$ is reduced to solving N equations

$$\frac{\partial^2 \alpha_i}{\partial t^2} - \frac{\partial^2 \alpha_i}{\partial x^2} = 0 \qquad i = 1, 2, \ldots, N \tag{2.7}$$

with the following boundary conditions

$$
\begin{aligned}
&\frac{\partial^2\alpha_1}{\partial t^2} + KE_1\frac{\partial\alpha_1}{\partial t} - K_rE_1\frac{\partial\alpha_1}{\partial x} = 0 \qquad \text{for} \quad x=0\\
&\alpha_i(x,t) = \alpha_{i+1}(x,t) + C_{i+1} \qquad \text{for} \quad x=\sum_{k=1}^{i} l_k \qquad i=1,2,\dots,N-1\\
&-\frac{\partial^2\alpha_i}{\partial t^2} - K_rB_iE_{i+1}\frac{\partial\alpha_i}{\partial x} + K_rB_{i+1}E_{i+1}\frac{\partial\alpha_{i+1}}{\partial x} = 0\\
&\qquad \text{for} \quad x=\sum_{k=1}^{i} l_k \qquad i=1,2,\dots,N-1\\
&J_{N+1}(t)\frac{\partial^2\alpha_N}{\partial t^2} + \frac{dJ_{N+1}}{dt}\frac{\partial\alpha_N}{\partial t} + K_rB_N\frac{\partial\alpha_N}{\partial x} + \frac{1}{2}\frac{d^2J_{N+1}}{dt^2}\alpha_N\\
&\qquad = -\frac{1}{2}\Omega_0\frac{dJ_{N+1}}{dt} \qquad \text{for} \quad x=\sum_{k=1}^{N} l_k
\end{aligned}
\tag{2.8}
$$

where $C_i = M_0(K_rB_i)^{-1}l_i$, α_0 is a fixed angular displacement, J_0 is a fixed mass moment of inertia and l_0 is a fixed length, correspondingly. Moreover, the bars denoting dimensionless quantities are omitted for convenience.

We are interested in torsional vibrations, so for simplicity, we assume zero initial conditions for α_i, i.e.

$$
\alpha_i(x,t) = \frac{\partial\alpha_i}{\partial t}(x,t) = 0 \qquad \text{for} \quad t=0 \tag{2.9}
$$

The solutions to equations (2.7), similarly to problems discussed by Pielorz and Skóra (2006), are sought in the form

$$
\alpha_i(x,t) = f_i\Big(t - x + 2\sum_{k=1}^{i} l_k - \sum_{k=1}^{N} l_k\Big) + g_i\Big(t + x - \sum_{k=1}^{N} l_k\Big) \qquad i=1,2,\dots,N \tag{2.10}
$$

The functions f_i and g_i in (2.10) represent the waves caused by variable inertia, propagating in the i-th shaft in the positive and negative senses of the x-axis, respectively. These functions are continuous and equal to zero for negative arguments. In (2.10), it is taken into account that the first disturbance appears in the i-th element in the cross-section $x_{0i} = l_1 + l_2 + \ldots + l_i$ at the time instant $t_{0i} = l_{i+1} + l_{i+2} + \ldots + l_N$.

Substituting postulated solutions (2.10) into boundary conditions (2.8) and denoting the largest argument in each boundary condition separately by z, we obtain the following set of ordinary differential equations for the unknown functions f_i and g_i

$$
\begin{aligned}
&f_1''(z) + r_{01}f_1'(z) = -g_1''(z-2l_1) + r_{02}g_1'(z-2l_1)\\
&f_i(z) = f_{i-1}(z-2l_i) + g_{i-1}(z-2l_i) - g_i(z-2l_i) - C_i \qquad i=2,3,\dots,N\\
&r_{N1}g_N''(z) + r_{N2}g_N'(z) + r_{N3}g_N(z) = F(z) + r_{N4}f_N''(z) + r_{N5}f_N'(z) + r_{N6}f_N(z)\\
&g_i''(z) + r_{i1}g_i'(z) = -f_i''(z) + r_{i2}f_i'(z) + r_{i3}g_{i+1}'(z) \qquad i=N-1,N-2,\dots,1
\end{aligned}
\tag{2.11}
$$

where

$$
\begin{array}{lll}
r_{01} = E_1(K_r+K) & r_{02} = E_1(K_r-K) & r_{N1}(z) = J_{N+1}(z)\\
r_{N2}(z) = K_rB_N + J_{N+1}'(z) & r_{N3}(z) = \dfrac{1}{2}J_{N+1}''(z) & r_{N4}(z) = -r_{N1}(z)\\
r_{N5}(z) = K_rB_N - J_{N+1}'(z) & r_{N6}(z) = -r_{N3}(z) & r_{i1} = K_rE_{i+1}(B_i+B_{i+1})\\
r_{i2} = K_rE_{i+1}(B_i-B_{i+1}) & r_{i3} = 2K_rE_{i+1}B_{i+1} & F(z) = -\dfrac{1}{2}J_{N+1}'(z)\Omega_0\\
i=1,2,\dots,N-1 & &
\end{array}
\tag{2.12}
$$

Equations (2.11) are differential equations with a retarded argument. Coefficients in (2.11) are variable. Equations (2.11) are solved numerically by means of the Runge-Kutta method. It should be pointed out that functional equations having shifted arguments are investigated in the literature, see e.g. Cherepennikov (2008), Cherepennikov and Ermolaeva (2006), Hale and Verduyn Lunen (1993).

The mass moment of inertia of the last rigid body, after expanding in the Taylor series around $\Omega_0 t$, depends on $\Omega_0 t$ and now it can be described by an arbitrary function containing $\Omega_0 t$. This function may also have the form of the Fourier series

$$J_{N+1}(t) = a_0 + \sum_{m=1}^{\infty} (a_m \cos m\Omega_0 t + b_m \sin m\Omega_0 t) \tag{2.13}$$

where a_i and b_i are constants.

3. Numerical results

Exemplary numerical results are performed for the three-mass system. The variable moment of inertia of the last rigid bodies is described by the formula

$$J_{N+1}(\Omega_0 t) = a_0 + a_2 \cos(2\Omega_0 t) \qquad N = 2 \tag{3.1}$$

by analogy to the papers by Koser and Pasin (1995, 1997). Comparing relations (2.13) and (3.1), it is seen that function (3.1) is the Fourier series with $m = 2$ and with constants a_0 and a_2 being the only nonzero constants. Such a form of the mass moment of inertia was proposed by Koser and Pasin (1995, 1997) for the analysis of a two-mass discrete-continuous system using an analytical approach.

Diagrams presented below concern mainly solutions in the steady state of motion. They are plotted out after solving equations (2.11) with

$$\begin{array}{llll} N = 2 & K_r = 0.05 & B_1 = B_2 = 1 & E_2 = 0.8 \\ l_1 = l_2 = 1 & a_0 = 1 & a_2 = 0.05 & C_2 = 0 \end{array} \tag{3.2}$$

The parameters E_1 and K, characterizing the motor, can vary. The first four natural frequencies of the linear system with $E_1 = 1.0$ are $\omega_1 = 0.222$, $\omega_2 = 0.359$, $\omega_3 = 3.158$, $\omega_4 = 3.183$. We are interested in vibrations caused by the variable inertia, so the constant C_2 is neglected in the numerical analysis.

The wave approach enables us to determine the required quantities in various cross-sections of the considered discrete-continuous systems in transient and in steady states of motion.

In Figs. 2 and 3, numerical results for angular displacements in the cross-sections $x = 0$, 0.5, 1.0, 1.5, 2.0 are presented for $E_1 = 1$, $K = 0.1$ with $\Omega_0 = 0.2$ and $\Omega_0 = 0.5$, respectively. The diagrams in Fig. 2, show the results in the transient state of motion while the diagrams in Figs. 3a and 3b show the angular displacements in the steady state. In the steady state, the solution behaves as a harmonic vibration with the period equal to the period of function (3.1) describing the variable mass moment of inertia J_3, i.e., $T = \pi/\Omega_0$. One can notice that the amplitude is steady for $t > 8000$ when $\Omega_0 = 0.2$ and for $t > 3500$ when $\Omega_0 = 0.5$. In the case of $\Omega_0 = 0.2$, the highest displacement amplitude occurs in the cross-section $x = 2$ and the smallest one in $x = 0.5$. For $\Omega_0 = 0.5$, the displacement amplitudes increase with the increase of x. From Figs. 2 and 3, it follows that the steady state is gained earlier for a larger value of Ω_0.

The displacement amplitudes α_A versus Ω_0 shown in Fig. 4, plotted for $E_1 = 1$, $K = 0.1$, contain four resonant regions ($\omega_1 = 0.222$, $\omega_2 = 0.359$, $\omega_3 = 3.158$, $\omega_4 = 3.183$). The resonances

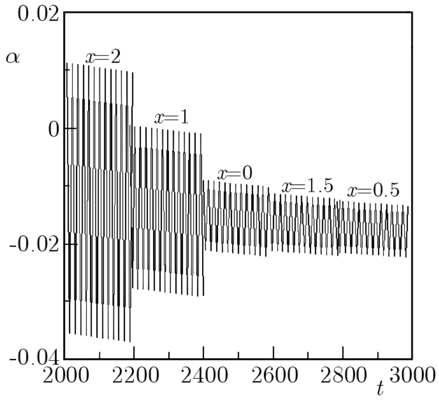

Fig. 2. Angular displacements in the transient state with $\Omega_0 = 0.2$

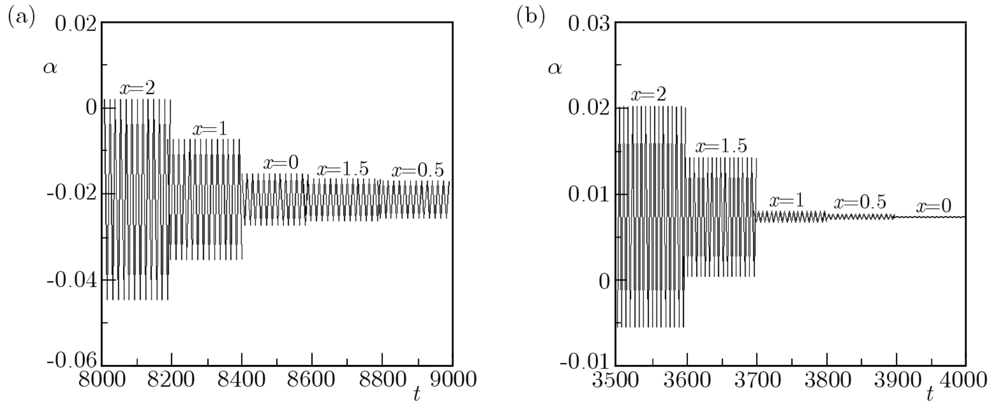

Fig. 3. Angular displacements in the steady state and with $\Omega_0 = 0.2$ (a) and with $\Omega_0 = 0.5$ (b)

correspond to $\Omega_0 = \omega_i/2$. In the third and fourth resonances, the displacement amplitudes are much higher than in the first and second resonant regions. In the first resonant region, the highest amplitude occurs in the cross-section $x = 2$ and the smallest one in $x = 1$. In the second resonant region the highest amplitude occurs in the cross-section $x = 1$ and the smallest one in $x = 0.5$. From Fig. 4, it follows that for the assumed parameters (3.2), the displacement amplitudes are small except for resonant regions. Besides, near $p = 0.78$, the amplitude of the displacement in the cross-section $x = 1.5$ is equal to 0.0419 while the displacement amplitudes in the remaining cross-sections are smaller than 0.013.

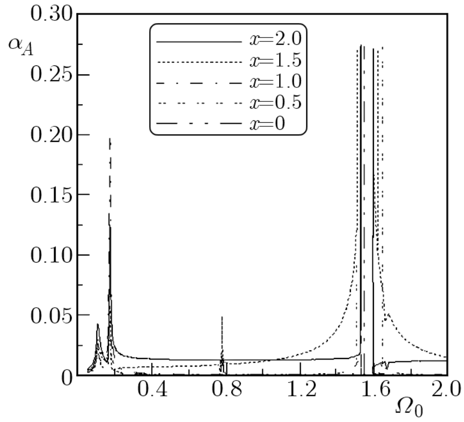

Fig. 4. Displacement amplitudes versus Ω_0

The first rigid body representing the motor in the model shown in Fig. 1 is characterized by the parameter E_1 (in the dimensional quantities by the mass moment of inertia J_1) and by the parameter K (the slope of characteristics). The effect of these parameters on the amplitudes α_A in the cross-section $x = 2$ for $0.003 < \Omega_0 < 0.245$ is shown in the next two figures. The diagrams contain the first two resonant regions ($\omega_1 = 0.222$, $\omega_2 = 0.359$). For bigger values of Ω_0, the displacement amplitudes are small except for resonances where they become high. In these figures, apart from numerical results obtained by using equations (2.11) with (3.2), the curves marked by broken lines are plotted using the analytical solution for the three-mass system derived in the paper by Pielorz and Skóra (2009a) when $K \to \infty$ and $E_1 = 1$.

The diagrams for displacement amplitudes α_A with $E_1 = 0.2$, 0.5, 1.0, 1.5 are given in Fig. 5a. From these diagrams, it follows that the natural frequencies increase, the resonant regions become wider and the maximal amplitudes decrease with the increase of E_1. Besides, for small values of E_1, i.e., for large J_1, the numerical results approach the analytical solution.

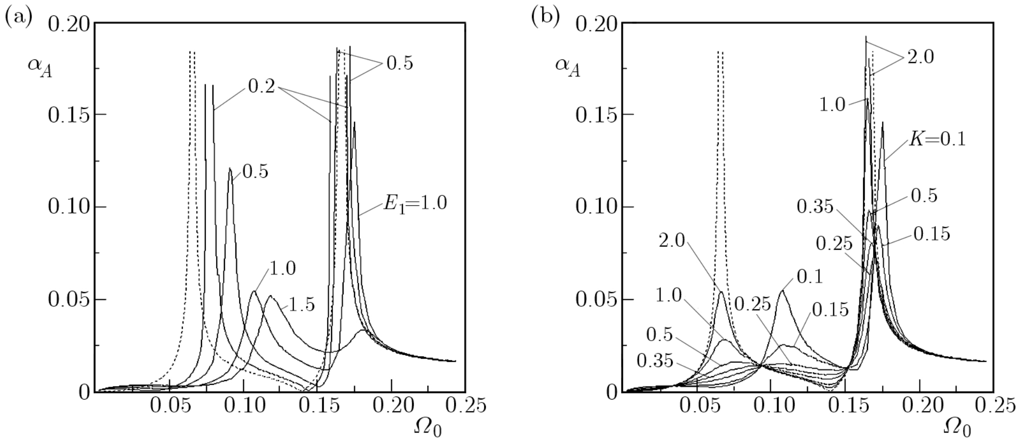

Fig. 5. (a) Effect of E_1 in $x = 2$ with $K = 0.1$; (b) effect of K in $x = 2$ with $E_1 = 1.0$

The effect of the parameter K is discussed for $K = 0.1$, 0.15, 0.25, 0.35, 0.5, 1.0, 2.0. In Fig. 5b, four resonant regions are shown: two concern the analytical solution derived in the papers by Pielorz and Skóra (2006, 2009a) with the natural frequencies $\omega_1 = 2\Omega_0 = 0.132$, $\omega_2 = 2\Omega_0 = 0.332$ and two regions for the three-mass system with a variable velocity of the motor with $\omega_1 = 0.222$, $\omega_2 = 0.359$.

The behavior of the system is different for small and larger values of K. Namely, the parameter K with smaller values (up to 0.35) plays the role of a damping coefficient which decreases the amplitudes while for $K > 0.35$ the maximal amplitudes increase with the increase of K approaching the analytical solution. Besides, it is seen that the numerical solutions with smaller values of K are in the resonant regions corresponding to the system under considerations while numerical solutions with $K > 0.35$ lay in the resonant regions determined for the system with the motor working with a constant velocity as studied by Pielorz and Skóra (2009a).

From Figs. 4 and 5, it follows that an additional resonant region may occur for Ω_0 close to 0.05. It concerns $E_1 = 1.0$, 1.5 or $K < 0.35$. For $\Omega_0 < 4.0$, no other additional resonant regions, except for those connected with the linear system, were found assuming $\Delta\Omega_0 = 0.001$.

It is interesting to compare the results obtained by means of the wave approach with the results obtained using another method. It should be pointed out that the wave method enables us to determine solutions in transient as well as in steady states of motion.

Some comparisons could be done for the discrete-continuous torsional system shown in Fig. 1 with the first rigid body representing a motor working with a constant velocity as investigated by Pielorz and Skóra (2006, 2009a). Then, the boundary condition for $x = 0$ in (2.8) is simpler and has the form

$$\frac{\partial \alpha_1}{\partial t} = 0 \tag{3.3}$$

This case corresponds to $K \to \infty$. For the problem with this boundary condition, it was possible to derive analytical solutions. These solutions are valid in steady states of motion.

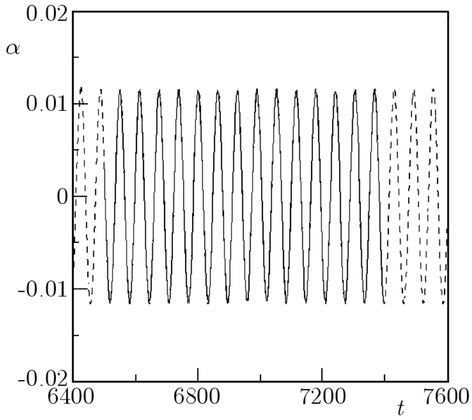

Fig. 6. Comparison of numerical (dashed line) and analytical solutions (continuous line) for displacements of the three-mass system in $x = 2$ in a steady state with boundary condition (3.3) and $\Omega_0 = 0.05$

Comparative results concerning angular displacements are plotted out in Fig. 6 with $\Omega_0 = 0.05$ using appropriate analytical solutions for the three-mass system derived by Pielorz and Skóra (2009a).

4. Final remarks

Problems of discrete-continuous systems torsionally deformed with variable inertia, taking into account variable velocity of the rigid body representing an electrical motor, can be described by classical wave equations with boundary conditions having variable coefficients. These equations can be solved applying the method using the dAlembert approch to the solution of equations of motion which leads to equations with a retarded argument.

The presented numerical calculations show that the variable motor velocity has a significant influence on the behavior of the considered systems. For a large mass moment of the motor J_1 and a large slope of its characteristics K, the numerical solutions approach the analytical solutions derived in the paper by Pielorz and Skóra (2009a) for a simpler case of boundary conditions, i.e. for the system with the motor moving with a constant velocity.

From the comparison of the results given in the paper by Pielorz and Skóra (2006) and in the present paper, it follows that the wave approach, i.e., applying the solution of dAlembert type, is more effective in the case of the motor with a variable velocity. This is connected with the fact that the parameter K plays partly the role of a damping coefficient.

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