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IMPROVED METHOD FOR SIMULATING TRANSIENTS OF TURBULENT PIPE FLOW

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The paper presents the problem of modelling and simulation of transients during turbulent fluid flow in hydraulic pipes. The instantaneous wall shear stress on a pipe wall is presented in the form of integral convolution of a weighting function and local acceleration of the liquid. This weighting function depends on the dimensionless time and Reynolds number. Its original, very complicated mathematical structure is approximated to a simpler form which is useful for practical engineering calculations. The paper presents an efficient way to solve the integral convolution based on the method given by Trikha (1975) for laminar flow. An application of an improved method with the use of the Method of Characteristic for the case of unsteady flow (water hammer) is presented. This method is characterised by high efficiency compared to traditional numerical schemes.

Key words: unsteady pipe flow, transients, waterhammer, efficient numerical simulation

Notation

c_0 [ms^{-1}] – acoustic wave speed
p [kg m^{-1}s^{-2}] – pressure
s [s^{-1}] – Laplace operator
t [s] – time
v [ms^{-1}] – instantaneous mean flow velocity
$\widehat{t} = \nu t/R^2$ [–] – dimensionless time

v [ms^{-1}] – average value of velocity in cross-section of pipe
v_z [ms^{-1}] – axial velocity
w [–] – weighting function
z [m] – distance along pipe axis
L [m] – pipe length
R [m] – radius of pipe
$\mathrm{Re} = 2Rv/\nu$ [–] – Reynolds number
μ [kg m^{-1}s^{-1}] – dynamic viscosity
ν [m^{2}s^{-1}] – kinematic viscosity
λ [–] – Darcy-Weisbach friction coefficient
ρ_0 [kg m^{-3}] – fluid density (constant)
$\tau_w, \tau_{wq}, \tau_{wu}$ [kg m^{-1}s^{-2}] – wall shear stress, wall shear stress for quasi-steady flow, unsteady wall shear stress, respectively
$\Omega = \omega R^2/\nu$ [–] – dimensionless frequency

1. Introduction

There are many problems concerning the issue of unsteady flow in pipes, and they include such phenomena as: energy dissipation, fluid-structure interactions, cavitation (for example column separation) and many others. It would be very difficult, if not impossible, to account for all the phenomena in just one mathematical model. This paper is devoted to the problem of energy dissipation and it concerns unsteady friction modelling of the liquid in pipes during turbulent liquid pipe flow.

The commonly used quasi-steady, one-dimensional (1D) model in which the wall shear stress is approximated by means of Darcy-Weisbach formula is correct for slow changes of the liquid velocity field in the pipe cross-section, i.e. either for small accelerations or for low frequencies (if the flow is pulsating). The formula is also correct for the waterhammer effect for its first wave cycle. This model does not take into account the gradient of speed changing in time, which is directed into the radial direction, and so, as a result, the changing energy dissipation is not taken into account either.

Models which represent unsteady friction losses can be divided into two groups. The first are those in which the shear stress connected with the unsteady friction term is proportional to the instantaneous local acceleration (Daily *et al.*, 1956; Carrtens and Roller, 1959; Sawfat and Polder, 1973). This group contains a popular model developed by Brunone *et al.* (1994, 1995) in which the unsteady friction term is a sum of two terms: the first of which is proportional to the instantaneous local acceleration and the other is proportional to the instantaneous convective acceleration. Vitkovsky *et al.* (2000) improved Brunone's model by introducing a sign to the convective acceleration.

In the second group of models, the unsteady wall shear stress depends on the past flow accelerations. These models are based on a two-dimensional (2D) equation of motion and they take into account the velocity field which can be changeable in time in the cross-section of the pipe. Zielke (1968) proposed for the first time a laminar flow model in which the instantaneous shear stress depends on the instantaneous pipe flow acceleration and the weighting function of the past velocity changes. In the case of unsteady turbulent flow, models of friction losses are mainly based on the eddy viscosity distribution in the cross-section of the pipe determined for a steady flow. The following models should be listed: two and three region models (Vardy, 1980; Vardy and Brown, 1995, 1996, 2003, 2004, 2007; Vardy *et al.* 1993) and as well as Zarzycki's four-layer model (Zarzycki, 1993, 1994, 2000). Lately, Vardy and Brown (2004) used an idealised viscosity distribution model for a rough pipe flow.

Unsteady friction models are used for simulation of transients in pipes. The traditional numerical method of unsteady friction calculation proposed by Zielke (1968), requires a large amount of computer memory and is time consuming. Therefore, Trikha (1975) and then Suzuki (1991) and Schohl (1993) improved Zielke's method making it more effective.

In the case of unsteady turbulent flow, Rahl and Barlamond (1996) and then Ghidaoui and Mansour (2002) used Vardy's model to simulate the water-hammer effect. These methods provide only approximated results. They are not very effective and cannot be considered to be satisfactory.

The aim of the present paper is to develop an effective method of simulating shear stress for unsteady turbulent pipe flow, a method that would be similar to that proposed by Trikha and Schoohl for laminar flow. That also involves developing an approximated model of the weighting function, which can be useful for effective simulation of transients.

2. Equations of unsteady turbulent flow

Unsteady, axisymmetric turbulent flow of a Newtonian fluid in a long pipe with a constant internal radius and rigid walls is considered. In addition, the following assumptions are made:

- constant distribution of pressure in the pipe cross-section,
- body forces and thermal effects are negligible,
- mean velocity in the pipe cross-section is considerably smaller than the speed of sound in the fluid.

The following approximate equation, presented in the previous papers by Ohmi *et al.* (1980–1981, 1985), Zarzycki (1993), which omits small terms in the fundamental equation for unsteady flow and in which the Reynolds stress term is described by using eddy viscosity ν_t, is used

$$\frac{\partial \overline{v}_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} + \nu_\Sigma \frac{\partial^2 \overline{v}_z}{\partial r^2} + \Big(r\frac{\partial \nu_\Sigma}{\partial r} + \frac{\nu_\Sigma}{r}\Big)\frac{\partial \overline{v}_z}{\partial r} \tag{2.1}$$

where $\overline{v}_z$, $\overline{p}$ – averaged in time, respectively: velocity component in the axial direction and pressure, ρ_0 – density of the liquid (constant), t – time, z – distance along the pipe axis, r – radial distance from the pipe axis.

The modified eddy viscosity coefficient ν_Σ is the sum of the kinematic viscosity coefficient ν and the eddy viscosity coefficient ν_t, that is

$$\nu_\Sigma = \nu + \nu_t \tag{2.2}$$

It is assumed that changes of the coefficient ν_t resulting from the flow change in time are negligible, which means that ν_t is only a function of the radial distance r. In practice, this limits the analysis to relatively short periods of time.

2.1. Turbulent kinematics viscosity distributions

In order to describe the distribution of the turbulent viscosity coefficient ν_t throughout the pipe cross-section, the flow region is divided into several layers and, for each of them, the function $\nu_t(r)$ has to be determined. By analogy to steady pipe flow, the following regions of flow can be distinguished: VS (Viscous Sublayer), BL (Buffer Layer), DTL (Developed Turbulent Layer) and TC (Turbulent Core) (Hussain and Reynolds (1975)). This is shown in Fig. 1. The distribution of eddy viscosity presented in Fig. 1 was created on the basis of experimental data of the coefficient ν_t provided by Laufer (for $\mathrm{Re} = 4.1 \cdot 10^5$) and Nunner (for $\mathrm{Re} = 3 \cdot 10^4$) and published by Hinze (1953) as well as data about a region close to the wall, which was provided by Schubauer and published by Hussain and Reynolds (1975). That was presented in depth by Zarzycki (1993, 1994).

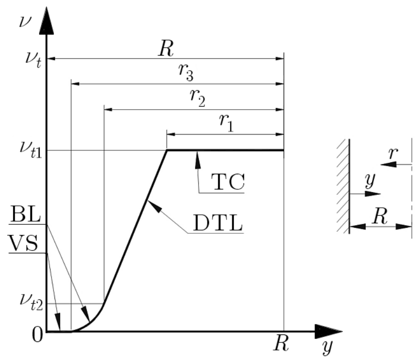

Fig. 1. Schematic representation of the four-region model of ν_t

The radial distances r_1, r_2 and r_3 from the pipe centre line are as follows

$$\begin{aligned} & r_j = R - y(y_j^+) && j = 1,2,3 \\ & y_1^+ = 0.2R^+ && y_2^+ = 35 \qquad y_3^+ = 5 \end{aligned} \tag{2.3}$$

where: $y^+ = yv^*/\nu$ is the dimensionless distance y, y – distance from the pipe wall, $v^* = \sqrt{\tau_{ws}/\rho_0}$ – dynamic velocity, τ_{ws} – wall shear stress for steady flow, $R^+ = Rv^*/\nu$ – dimensionless radius R of the pipe.

The coefficient of turbulent viscosity ν_t is expressed for particular layers (except the buffer layer) as follows

$$\begin{aligned} & \nu_t = \alpha_i r + \beta_i && i = 1,3,4 \\ & \alpha_1 = 0 && \alpha_3 = -\frac{\nu_{t1} - \nu_{t2}}{r_2 - r_1} && \alpha_4 = 0 \\ & \beta_1 = 0 && \beta_2 = \nu_{t2} - \alpha_3 r_2 && \beta_4 = \nu_{t1} \end{aligned} \tag{2.4}$$

For the buffer layer $(r_2 \leqslant r < r_3)$, ν_t is expressed as follows

$$\nu_t = 0.01(y^+)^2\nu \tag{2.5}$$

Expressions, which define the quantities ν_{t1}, ν_{t2}, r_1, r_2 and r_3 have the following forms

$$\nu_{t1} = 0.016\frac{\widehat{v}^*}{\sqrt{2}}\nu \qquad \nu_{t2} = 12.25\nu \tag{2.6}$$

and

$$r_1 = 0.8 \qquad r_2 = \Big(1 - \frac{140\sqrt{2}}{\widehat{v}^*}\Big)R \qquad r_3 = \Big(1 - \frac{20\sqrt{2}}{\widehat{v}^*}\Big)R \tag{2.7}$$

The quantities ν_{t1}, r_2 and r_3, which define the distribution of the coefficient ν_t, depend on a dimensionless dynamic velocity $\widehat{v}^*$, which is determined in the following way

$$\widehat{v}^* = \frac{4\sqrt{2}R}{\nu} v^* = \sqrt{\lambda_s}\mathrm{Re}_s \tag{2.8}$$

where: λ_s is the friction coefficient for steady flow, $\mathrm{Re}_s = 2Rv_s/\nu$ – Reynolds number for steady flow, v_s – mean fluid velocity in the pipe cross-section.

Since we have $\lambda_s = \lambda_s(\mathrm{Re}_s)$ then, for smooth pipes, the profile of the coefficient ν_t in the pipe cross-section depends only on the Re_s number.

However, if – instead of the Re_s – we assume the instantaneous Reynolds number $\mathrm{Re} = 2Rv/\nu$ (v – instantaneous, averaged in the pipe cross-section fluid velocity) and – instead of λ_s – we assume a quasi-steady friction coefficient $\lambda_q = \lambda_q(\mathrm{Re})$, then the presented distribution of the coefficient ν_t will become a quasi-steady one. In this paper λ_q is determined from the Prandtl formula

$$\frac{1}{\sqrt{\lambda_q}} = 0.869\ln(\mathrm{Re}\sqrt{\lambda_q}) - 0.8 \tag{2.9}$$

2.2. Momentum equations for particular layers

For the four-layer model of the flow region, equation (2.1) breaks up into four equations. Introducing the differential operator $D = \partial/\partial t$ into equation (2.1), we obtain:

— for the viscous sublayer

$$D\overline{v}_{z1} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} + \nu\Big(\frac{\partial^2 \overline{v}_{z1}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v}_{z1}}{\partial r}\Big) \tag{2.10}$$

— for the remaining layers of the flow region

$$D\overline{v}_{zi} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} + \nu_\Sigma\frac{\partial^2 \overline{v}_{zi}}{\partial r^2} + \Big(\frac{\partial \nu_\Sigma}{\partial r} + \frac{\nu_\Sigma}{r}\Big)\frac{\partial \overline{v}_{zi}}{\partial r} \tag{2.11}$$

where: $i = 2, 3, 4$ ($i = 2$ for the BL, $i = 3$ for the DTL and $i = 4$ for the TC), $\overline{v}_{zi}$ is the axial component of the velocity for particular layers of the flow region, $\nu_\Sigma = \nu + \nu_t$ – effective coefficient of turbulent viscosity.

Equations (2.10) and (2.11) (for $i = 3, 4$) can be resolved using relationship (2.4) into the modified Bessel equations. Equation (2.11) (for $i = 2$), taking into account the fact that

$$\frac{r}{R-r} \gg \frac{1}{2} \qquad \text{for} \qquad r_2 \leqslant r < r_3 \tag{2.12}$$

assumes the form of a differential equation of Euler type.

The solutions to the momentum equations for particular layers are shown in papers (Zarzycki, 1993, 1994, 2000).

3. Wall shear stress and weighting function

Integrating equation (2.1) over the pipe cross-section (from $r = 0$ to $r = R$) and considering that $\partial/\partial t = D$, we obtain

$$\rho_0 D v + \frac{\partial p}{\partial z} + \frac{2}{R}\tau_w = 0 \tag{3.1}$$

By determining the wall shear stress at the pipe wall τ_w as follows

$$\tau_w = -\rho_0 \nu \frac{\partial \overline{v}_{z1}}{\partial r}\Big|_{r=R} \tag{3.2}$$

we obtain a relationship of the following type

$$\tau_w = f(D, \mathrm{Re})\frac{\partial p}{\partial z} \tag{3.3}$$

Equations (3.1) and (3.3) enable one to determine transfer function $\widehat{G}_{\tau v}$ that describes hydraulic resistance, relating the pipe wall shear stress and the mean velocity

$$\widehat{G}_{\tau v}(\widehat{D}, \mathrm{Re}) = \frac{\widehat{\tau}_w}{\widehat{v}} \tag{3.4}$$

where $\widehat{v} = (R/\nu)v$, $\widehat{\tau}_w = [R^2/(\rho_0\nu^2)]\tau_w$, $\widehat{p} = [R^2/(\rho_0\nu^2)]p$ denote dimensionless velocity, wall shear stress, and pressure, respectively.

A precise mathematical form of the transfer function $\widehat{G}_{\tau v}$ is very complex and was presented by Zarzycki (1994).

The quantity $\widehat{D} = (R^2/\nu)\partial/\partial t$ presents the dimensionless differential operator.

Taking into account transfer function (3.4) for $\widehat{D} = \widehat{s}$ (where $\widehat{s} = sR^2/\nu$ is the dimensionless Laplace operator) one can express the stress τ_w as a sum of quasi-steady wall shear stress τ_{wq} and integral convolution of the fluid acceleration $\partial v/\partial t$ and the weighting function $w(t)$, i.e. as a convolution of the fluid acceleration $\partial v/\partial t$ and the step response $g(t)$ (Zielke, 1968; Zarzycki, 1993, 2000), i.e.

$$\tau_w(t) = \tau_{wq} + \frac{2\mu}{R}\int\limits_0^t \frac{\partial v}{\partial t}(u)w(t-u)\;du \tag{3.5}$$

where τ_{wq} is described by the following expression

$$\tau_{wq} = \frac{1}{8}\lambda_q \rho_0 v|v| \tag{3.6}$$

The graph of weighting function $w(t)$ for different Reynolds numbers and for the laminar flow is presented in Fig. 2.

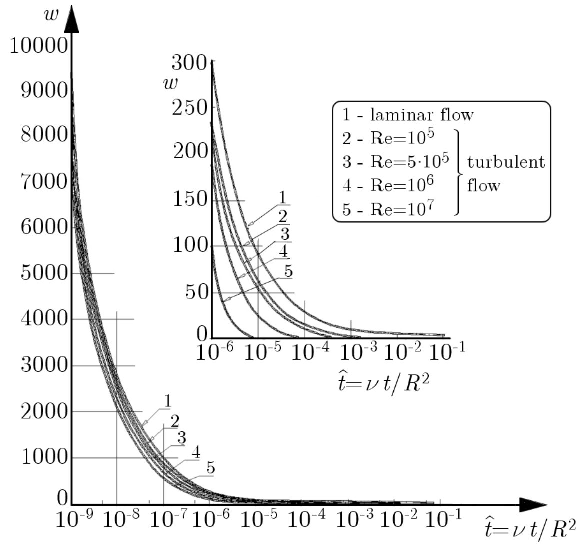

Fig. 2. The weighting functions for turbulent and laminar flow

The weighting function approaches zero when time tends to infinity. This tendency becomes stronger for higher Reynolds numbers.

4. Approximated weighting function model

The weighting function has a complex mathematical form and it is not very useful to calculate and simulate unsteady turbulent flow (Zarzycki, 1994, 2000). Zarzycki (1994) approximated it to the following form

$$w_{apr} = C\frac{1}{\sqrt{\widehat{t}}}\mathrm{Re}^n \tag{4.1}$$

where $C = 0.299635$, $n = -0.005535$.

However, numerical calculations that include this function lead to some complications (dividing by zero) as time and the Reynolds number are to be found in the denominator. Hence, in order to improve the effectiveness of calculations, the function will be approximated with a simpler form. Additionally, it is suggested that the approximated function should meet the following assumptions:

1) its form should be similar to Trikha's (Trikha, 1975) or Schohl's (Schohl, 1993) models (methods of simulating shear stress based on these models are very effective),
2) its mathematical form should allow an easy way of carrying out Laplace transformation. This is important for simulating oscillating or pulsating flow by the matrix method (Wylie and Streeter, 1993).

In order to meet the above presented criteria, a number of possible forms of the function were investigated. The form which was eventually selected can be given by

$$w_N(\widehat{t}) = (c_1\mathrm{Re}^{|c_2|} + c_3)\sum_{i=1}^{n} A_i \mathrm{e}^{-|b_i|\widehat{t}} \tag{4.2}$$

where $i = 1, 2, \dots, n$.

Determining the coefficients of the weighting function was carried out in two stages in the following way:

a) **The first stage.** In the range of the Reynolds numbers $\mathrm{Re} \in \langle 2{\cdot}10^3, 10^7\rangle$, a several equally distributed points were chosen, and for each of them weighting functions were estimated using Statistica programme. Estimation was conducted for the range of dimensional time $\widehat{t} \in \langle 10^{-6}, 10\rangle$.
b) **The second stage.** Two-dimensional weighting function $w(\widehat{t}, \mathrm{Re})$ was estimated using Matlab's module for optimization. For the optimization procedure, the following assumption was formulated:
   - to determine the decision variables vector $\boldsymbol{x} = [x_1, x_2, \dots, x_{19}]$, where: $x_1 = c_1$, $x_2 = c_2$, $x_3 = c_3$, $x_4 = A_1$, $x_5 = b_1$, $x_6 = A_2$, $x_7 = b_2$, ..., $x_{18} = A_8$, $x_{19} = b_8$ (are coefficients of (4.2))
   - which minimizes the vector objective function

$$\min f(\mathbf{x}) = [\mathbf{f}(\mathbf{x})] \tag{4.3}$$

where

$$f(\mathbf{x}) = \sum_{\mathbf{i=1}}^{\mathbf{r}} \Big(\frac{\mathbf{w_{N(i)}}(\widehat{\mathbf{t}}, \mathrm{Re}) - \mathbf{w_i}(\widehat{\mathbf{t}}, \mathrm{Re})}{\mathbf{w_i}(\widehat{\mathbf{t}}, \mathrm{Re})} \Big)^{\mathbf{2}} \tag{4.4}$$

and satisfies the following limitation

$$g(\mathbf{x}) = \int\limits_{\mathbf{2 \cdot 10^3}}^{\mathbf{10^7}} \Big[\int\limits_{\mathbf{10^{-5}}}^{\mathbf{1}} \mathbf{w_N}(\widehat{\mathbf{t}}, \mathrm{Re})\ \mathbf{d}\widehat{\mathbf{t}} \Big]\ \mathbf{d}\mathrm{Re} - \int\limits_{\mathbf{2 \cdot 10^3}}^{\mathbf{10^7}} \Big[\int\limits_{\mathbf{10^{-5}}}^{\mathbf{1}} \mathbf{w}(\widehat{\mathbf{t}}, \mathrm{Re})\ \mathbf{d}\widehat{\mathbf{t}} \Big]\ \mathbf{d}\mathrm{Re} = \mathbf{0} \tag{4.5}$$

In relation (4.4) the arguments $\widehat{t}$ and Re were selected so that the determined points (pairs of numbers) on the plane $\widehat{t}$–Re would uniformly cover the interval $\widehat{t} \in \langle 10^{-5}, 10^{-1} \rangle$ and $\mathrm{Re} = \langle 2 \cdot 10^3, 10^7 \rangle$, and that they would be different from the points selected at the first stage. A total of $r = 1000$ points were used in the calculations. The values of coefficients in Eq. (4.2), determined at the first stage, were used as the starting points (the values of elements of the decision variables vector) for the optimization.

Once the optimization problems were formulated in the way described above, calculations were conducted using Matlab and its function "*fgoalattain*".

Details of the whole process of a new weighting function estimation were described by Zarzycki and Kudźma (2004).

On the basis of the above assumptions and calculations that were conducted later, the final form of the weighting function was established

$$w_N(\widehat{t}) = (c_1 \mathrm{Re}^{c_2} + c_3) \sum_{i=1}^{8} A_i \mathrm{e}^{-b_i \widehat{t}} \tag{4.6}$$

$c_1 = -13.27813$; $c_2 = 0.000391$; $c_3 = 14.27658$; $A_1 = 0.224$, $A_2 = 1.644$, $A_3 = 2.934$, $A_4 = 5.794$, $A_5 = 11.28$, $A_6 = 19.909$, $A_7 = 34.869$, $A_8 = 63.668$; $b_1 = 0.10634$, $b_2 = 8.44$, $b_3 = 88.02$, $b_4 = 480.5$, $b_5 = 2162$, $b_6 = 8425$, $b_7 = 29250$, $b_8 = 96940$.

The present form of the weighting function contrary to the one put forward in (Zarzycki and Kudźma, 2004) contains eight terms. As it turned out, the effective method of calculating wall shear stress presented later in this work is more sensitive to inaccurate fit between the proposed function and the original. Therefore, in order to ensure correct mapping, the number of elements of the new weighting function was increased from six to eight.

In addition, a comparison was made of new expression (4.6) course and Zarzycki's model with the weighting function of Vardy and Brown (2003)

$$w(\widehat{t}) = \frac{1}{2\sqrt{\pi\widehat{t}}}\exp\Big(-\frac{\widehat{t}}{C^*}\Big) \tag{4.7}$$

where: $C^* = 12.86/\mathrm{Re}^{\kappa}$; $\kappa = \log_{10}(15.29/\mathrm{Re}^{0.0567})$.

The accuracy of Zarzycki's mapping of the weighting function by the new function is as follow:

- For a range of Reynolds numbers between $2\cdot 10^3 \leqslant \mathrm{Re} \leqslant 10^7$ and dimensionless time between $10^{-5} \leqslant \widehat{t} \leqslant 10^{-1}$, the relative error satisfies condition

$$\Big|\frac{w_N(\widehat{t}) - w(\widehat{t})}{w(\widehat{t})}\Big| \leqslant 0.05$$

Graphical comparison of the newly estimated function and models of weighing function by Zarzycki and Vardy-Brown are presented in Fig. 3. The relative error of the new function and Zarzycki's model is presented in Fig. 4. As it is shown, the error in some areas is up to 5% but it has to be mentioned that the run of the new function is between the models of Vardy-Brown and Zarzycki weighting functions. There is no clear evidence of better consistency with the experiment for a wide range of Reynolds numbers up to now, so it is ambiguous which models should be mapped more precisely by the new function. The authors did not want to complicate the form of weighting expression through an increase of the number of exponential components thus left it in form (4.6). But, it is worth to mention that it is possible to increase accuracy of mapping the original weight in a wider range of Reynolds numbers or dimensionless time in the case of new more precise and experimentally tested models.

The lowest limit of usage of the new weighting function for dimensionless time is $\widehat{t} = 10^{-5}$. For the presented method of transients simulations it is a satisfactory value from the point of engineering view. It permit simulations of low viscosity fluids in quite large pipe radius (e.g. for fluid which viscosity $\nu = 1\,\mathrm{mm}^2/\mathrm{s}$ acceptable diameter $= 0.3\,\mathrm{m}$). It has to be also mentioned that such a big radius of the pipeline has usually long length, what means, in consequence, low frequency of transients pressure oscillations. It allows one to take a quite big time step in numerical calculations, which is in the acceptable range of dimensionless time.

In order to quantitatively assess a mapping of Zarzycki's model by the new expression for selected Reynolds numbers, a relative error is calculated according to the expression

$$\text{relative error} = \Big|\frac{w_N(\widehat{t}) - w(\widehat{t})}{w(\widehat{t})}\Big| \tag{4.8}$$

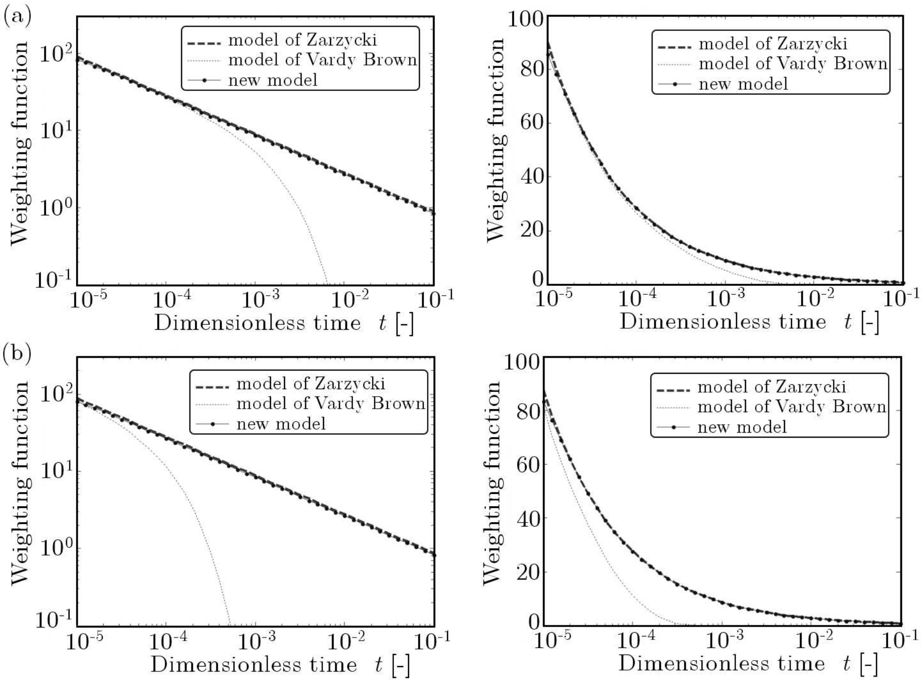

Fig. 3. Comparison of Vardy-Brown's, Zarzycki's and new weighting function runs for different Reynolds numbers: (a) $\mathrm{Re} = 10^4$, (b) $\mathrm{Re} = 10^6$

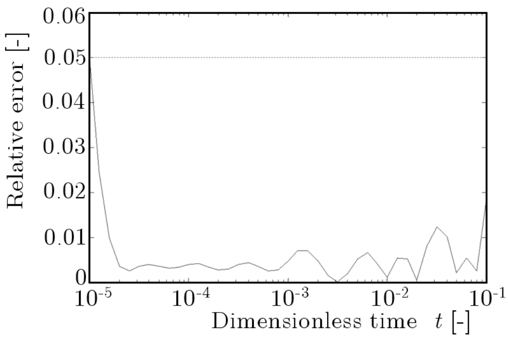

Fig. 4. The relative error between the new function and Zarzycki's weighting function model for $\mathrm{Re} = 10^4$

Results of its usage are shown for the case of Reynolds number $\mathrm{Re} = 10^4$ in Fig. 4.

In the case of harmonically pulsating flow, a transfer matrix method is usually used (Wylie and Streeter, 1993). For this method, an adequate (laminar or turbulent) weighting function transformed into Laplace domain is needed.

Next, the Laplace variable is substitutet by $\widehat{s} = j\Omega$.

Conducting the above mentioned transformations, the new weighting function takes the following form

$$w_N(j\Omega) = (c_1 \mathrm{Re}^{c_2} + c_3) \sum_{i=1}^{8} \frac{A_i}{j\Omega + b_i} \tag{4.9}$$

where: $\Omega = \omega R^2/\nu$ is the dimensionless angular frequency.

In Fig. 5, a comparison of real and imaginary parts of new weighting function (4.9) with Zarzycki and Vardy-Brown models is presented.

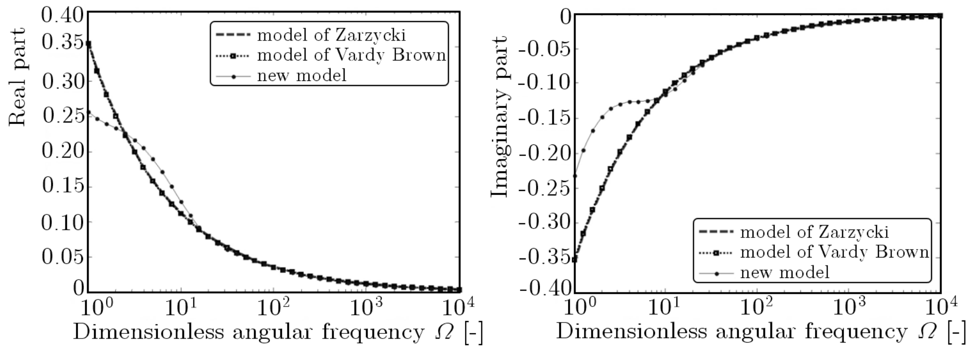

Fig. 5. Comparison of the new weighting function with models by Zarzycki and Vardy-Brown for $\mathrm{Re} = 10^4$

To sum up, it should be pointed out that new approximating function (5.3) well reflects the weight of the unsteady friction model for turbulent flow for $\widehat{t} \geqslant 10^{-5}$ and $2000 \leqslant \mathrm{Re} \leqslant 10^7$. However, as far as frequency is concerned, the presented relation is right for a dimensionless frequency $\Omega \geqslant 10$. These particular ranges are most important as far as engineering applications and precision of numerical simulations are concerned.

5. Improvement of the approximation method for simulations of wall shear stress

Numerical calculations of the unsteady wall shear stress (convolution of the local liquid acceleration and the weighting function) in the traditional approach is both time-consuming and takes a lot of computer memory space (Suzuki *et al.*, 1991). In papers by Trikha (1975) and Schohl (1993) another, more efficient method of calculating unsteady shear stress for laminar flow was presented.

This section presents the implementation of Trikha's method into calculations of wall shear stress for turbulent flow using the developed here form of weighting function (4.6).

Weighting function (4.6) can be given by

$$w_N(t) = \sum_{i=1}^{8} w_i \tag{5.1}$$

where $w_i = (c_1 \mathrm{Re}^{c_2} + c_3) A_i \mathrm{e}^{-b_i \nu t / R^2}$.

Using the above form of the weighting function, the unsteady term of pipe wall shear stress τ_{wn} can be written as

$$\tau_{wu} = \frac{2\mu}{R} \sum_{i=1}^{8} y_i(t) \tag{5.2}$$

where

$$y_i(t) = (c_1 \mathrm{Re}^{c_2} + c_3) A_i \mathrm{e}^{-b_i \frac{\nu}{R^2} t} \int_0^t \mathrm{e}^{b_i \frac{\nu}{R^2} u} \frac{\partial v}{\partial u}(u)\, du \tag{5.3}$$

The Reynolds number was used in the above expressions as a variable to be updated at every iterative step. This approach was also suggested by Vardy *et al.* (1993).

Because in the characteristics method calculations are carried out for some fixed time intervals, then

$$\begin{aligned} y_i(t+\varDelta t) &= \int_0^{t+\varDelta t} w_i(t+\varDelta t - u) \frac{\partial v}{\partial u}(u)\, du = \\ &= \mathrm{e}^{-b_i \frac{\nu}{R^2}\varDelta t} \underbrace{(c_1 \mathrm{Re}^{c_2} + c_3) A_i \mathrm{e}^{-b_i \frac{\nu}{R^2} t} \int_0^t \mathrm{e}^{b_i \frac{\nu}{R^2} u} \frac{\partial v}{\partial u}(u)\, du}_{y_i(t)} + \\ &\underbrace{+ (c_1 \mathrm{Re}^{c_2} + c_3) A_i \mathrm{e}^{-b_i \frac{\nu}{R^2}(t+\varDelta t)} \int_t^{t+\varDelta t} \mathrm{e}^{b_i \frac{\nu}{R^2} u} \frac{\partial v}{\partial u}(u)\, du}_{\varDelta y_i(t)} \end{aligned} \tag{5.4}$$

While comparing relations (5.4) and (5.3), we obtain

$$y_i(t+\varDelta t) = y_i(t) \mathrm{e}^{-b_i \frac{\nu}{R^2}\varDelta t} + \varDelta y_i(t) \tag{5.5}$$

where

$$\Delta y_i(t) = (c_1 \mathrm{Re}^{c_2} + c_3) A_i \mathrm{e}^{-b_i \frac{\nu}{R^2}(t+\Delta t)} \int\limits_{t}^{t+\Delta t} \mathrm{e}^{b_i \frac{\nu}{R^2} u} \frac{\partial v}{\partial u}(u)\, du \tag{5.6}$$

If in the above given expression describing $\Delta y_i(t)$, we assume that $v(u)$ is a linear function $[v(u) = au + b]$ in the interval from t to $t + \Delta t$, then its derivative calculated after time given by $\partial v(u)/\partial u$ can be treated as a constant whose value can be determined from the following expression: $[v(t + \Delta t) - v(t)]/\Delta t$. On the strength of this assumption, we can write

$$\begin{aligned}
\Delta y_i(t) &\approx (c_1 \mathrm{Re}^{c_2} + c_3) A_i \mathrm{e}^{-b_i \frac{\nu}{R^2}(t+\Delta t)} \frac{[v(t+\Delta t) - v(t)]}{\Delta t} \int\limits_{t}^{t+\Delta t} \mathrm{e}^{b_i \frac{\nu}{R^2} u}\, du = \\
&= (c_1 \mathrm{Re}^{c_2} + c_3) A_i \mathrm{e}^{-b_i \frac{\nu}{R^2}(t+\Delta t)} \frac{[v(t+\Delta t) - v(t)]}{\Delta t} \frac{R^2}{b_i \nu} \Big(\mathrm{e}^{b_i \frac{\nu}{R^2}(t+\Delta t)} - \mathrm{e}^{b_i \frac{\nu}{R^2} t} \Big) = \\
&= (c_1 \mathrm{Re}^{c_2} + c_3) \frac{A_i R^2}{\Delta t b_i \nu} \Big(1 - \mathrm{e}^{-b_i \frac{\nu}{R^2} \Delta t} \Big) [v(t+\Delta t) - v(t)]
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
y_i(t+\Delta t) &\approx y_i(t) \mathrm{e}^{-b_i \frac{\nu}{R^2} \Delta t} + \\
&\quad + (c_1 \mathrm{Re}^{c_2} + c_3) \frac{A_i R^2}{\Delta t b_i \nu} \Big(1 - \mathrm{e}^{-b_i \frac{\nu}{R^2} \Delta t} \Big) [v(t+\Delta t) - v(t)]
\end{aligned} \tag{5.8}$$

$$\tau_{wu}(t+\Delta t) = \frac{2\mu}{R} \sum_{i=1}^{8} y_i(t+\Delta t) \tag{5.9}$$

$$\begin{aligned}
\tau_{wu}(t+\Delta t) &\approx \frac{2\mu}{R} \sum_{i=1}^{8} y_i(t) \mathrm{e}^{-b_i \frac{\nu}{R^2} \Delta t} + \\
&\quad + (c_1 \mathrm{Re}^{c_2} + c_3) \frac{A_i R^2}{\Delta t b_i \nu} \Big(1 - \mathrm{e}^{-b_i \frac{\nu}{R^2} \Delta t} \Big) [v(t+\Delta t) - v(t)]
\end{aligned} \tag{5.10}$$

At the beginning of simulations (i.e. at the first iterative step), the value of all terms of $y_i(t)$ equals zero. At every subsequent time step, these values change according to Eq. (5.5).

6. Results of waterhammer simulations and efficiency of the method

6.1. Governing equations

The unsteady flow, accompanying the water hammer effect, may be described by a set of the following partial differential equations (Wylie and Streeter, 1993):
— equation of continuity

$$\frac{\partial p}{\partial t} + c_0^2 \rho_0 \frac{\partial v}{\partial x} = 0 \tag{6.1}$$

— equation of motion

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{2\tau}{\rho_0 R} = 0 \tag{6.2}$$

where $v = v(x,t)$ is the average velocity of liquid in the pipe cross-section, $p = p(x,t)$ – average pressure in the pipe cross-section, R – inner radius of the tube, τ – wall shear sterss, ρ_0 – density of fluid, c_0 – velocity of pressure wave propagation, t – time, x – axial location along the pipe.

A waterhammer simulation is carried out by the method of characteristics (MOC). This method (for small Mach number flow) transform equations (6.1), (6.2) into the following systems of equations (Wylie and Streeter, 1993)

$$\pm dp \pm \rho_0 c_0 dv + \frac{2\tau_w a}{R} dt = 0 \tag{6.3}$$

if

$$dz = \pm c_0 dt \tag{6.4}$$

Details of the further procedure of calculation may be found in papers by Zarzycki and Kudźma (2004) or Zarzycki *et al.* (2007).

6.2. Numerical simulation

The investigated system (tank-pipeline-valve) is presented in Fig. 6.

For calculations, the following data were assumed (Wichowski, 1984):

- length of hydraulic line $L = 3600\,\mathrm{m}$
- inner diameter of tube $d = 0.62\,\mathrm{m}$
- thickness of tube wall $g = 0.016\,\mathrm{m}$
- modulus of elasticity of tube wall material (cast iron) $E = 1.0 \cdot 10^{11}\,\mathrm{Pa}$
- bulk modulus of liquid $E_L = 2.07 \cdot 10^{9}\,\mathrm{Pa}$

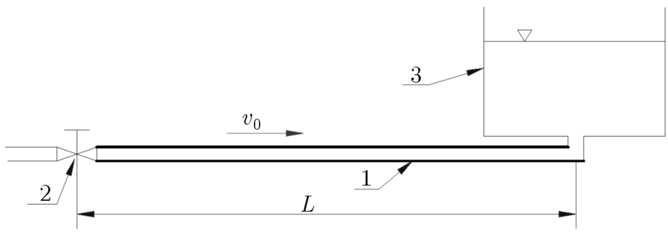

Fig. 6. Scheme of the pipeline system; 1 – hydraulic pipeline, 2 – cut-off valve, 3 – tank, v_0 – direction of fluid flow (initial condition)

- velocity of pressure wave propagation $c_0 = 1072\,\mathrm{m/s}$
- initial velocity of fluid flow $v_0 = 1.355\,\mathrm{m/s}$
- density of water $\rho_0 = 1000\,\mathrm{kg/m^3}$
- kinematic viscosity of fluid $\nu = 1.31 \cdot 10^{-6}\,\mathrm{m^2/s}$

The critical Reynolds number is calculated according to the following equation (Ohmi *et al.*, 1985)

$$\mathrm{Re}_c = 800\sqrt{\Omega} \tag{6.5}$$

where $\Omega = \omega R^2/\nu$ is the dimensionless angular frequency; $\omega = 2\pi/T$ – angular frequency; $T = 4L/c_0$ – period of pressure pulsation, L – length of pipe.

The value of Reynolds number at the initial condition is $\mathrm{Re} = 2Rv_0/\nu = 6.41 \cdot 10^5$, the critical Reynolds number according to Eq. (6.5) $\mathrm{Re}_c = 1.48 \cdot 10^5$, so $(\mathrm{Re} > \mathrm{Re}_c)$ and the flow is regarded as turbulent.

The unsteady state of the system was achieved by a complete closure of the valve for a time longer than a half of the wave's period in the pipe, i.e. $t_z = 90\,\mathrm{s} > 2L/c_0 = 6.72\,\mathrm{s}$, consequently it can be regarded as an example of a complex waterhammer (Wylie and Streeter, 1993). Closure of the valve proceeded according to the characteristics presented in Fig. 7 (Wichowski, 1984). The presented characteristics illustrate the variable liquid flow rate Q_z through the valve due to its slow shutdown.

Two simulation cases were investigated:

- in the first one, the unsteady stress τ_{wu} was calculated by means of Eq. (5.10) by making use of new weighting function (4.6),
- in the second one, the unsteady stress τ_{wu} was calculated using the traditional method (Zielke, 1968).

The results of numerical simulation at the valve carried out for a given number of time steps is presented in Fig. 8. As one can see, after the valve

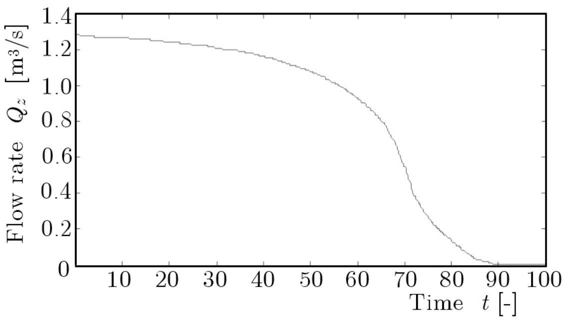

Fig. 7. Flow characteristics of closing of the water supply valve

closure which lasts 90s, there are still evident significant pressure pulsations. This effect is caused by very large inertia of the liquid (high value of the Reynolds number and a substantial length of the liquid line). Therefore, despite the long time of valve closure, disappearing pressure pulsations are still present in the line.

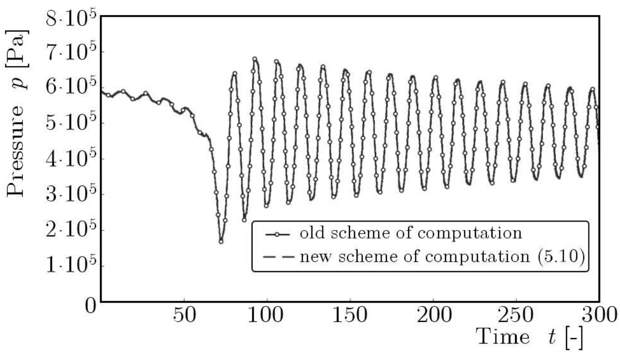

Fig. 8. Pressure fluctuation at the valve after its closure

The comparison of two simulations, using the old and new method of calculations of unsteady frictional losses presented in Fig. 8 shows a good agreement of the results obtained, and thus confirms that the model of efficient computation satisfactorily replaces the traditional method.

However, an important difference that can be observed in the simulations is the time of numerical calculations. Figure 9 presents the dependence of time necessary to carry out calculations on the number of time steps used for the simulations. As it can be clearly seen, the time of numerical calculations carried out by means of the traditional method using Zarzycki's weighting function (4.1) is much longer than that used to calculate it using the presented

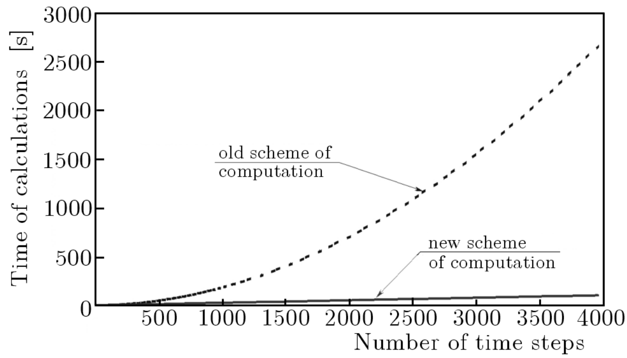

Fig. 9. Comparison of numerical computation times

method. Moreover, the time necessary to carry out the calculations conducted for the traditional method increases exponentially, along with the increase of the number of time steps. However, for the effective method the time increases slowly in a linear way.

On the basis of the above presented investigations, it can be stated that the presently developed method of simulating pressure changes and flow velocity, while taking into account the unsteady friction loss expressed by (5.10), is an effective method which can replace the tradition method and which offers satisfactory accuracy.

7. Conclusions

The paper presented the transfer function relating the pipe wall shear stress and the mean velocity. It is based on a four-region distribution of eddy viscosity at the cross-section of the pipe. This model was used to determine the unsteady wall shear stress as integral convolution of acceleration and weighting function similarly as it was done for laminar flow (see e.g. Zielke, 1968). However, the weighting function has a complex mathematical form and it is not very convenient to use it for simulating transient flows. Therefore, it was approximated to a simpler form, which was a sum of eight exponential summands. For this form of the weighting function and approximated in this way, an effective method of simulating wall shear stress was developed.

The most important conclusions which can be drawn from this study are as follows:

- The weighting function depends on the dimensionless time $\widehat{t} = \nu t/R^2$ (as in the case for laminar flow) and additionally on the Reynolds number Re.

- The weighting function approaches zero as the time increases. For a given value of dimensionless time it is greater in the case of laminar flow and it decreases along with the increase of Reynolds number Re in the case of turbulent flow. It means that the hydraulic resistance with an increase in the Reynolds number Re goes faster to a quasi-steady value, which is associated with increase of damping.
- Approximated weighting function (4.6) proves to be a precise expression in the range $10^{-5} \leqslant \widehat{t} \leqslant 10^{-1}$ and $2 \cdot 10^3 \leqslant \mathrm{Re} \leqslant 10^7$ (relative error smaller than 5%).
- The present approximated method of calculating wall shear stress (which incidentally is similar to Trikha's method developed for laminar flow) makes it possible to significantly reduce the time necessary to carry out calculations as compared with the traditional method.
- The weighting function developed in this study shows a good fit with Vardy's model for short times (high frequencies). However, the discrepancy between the values of weighting function obtained in both methods increase together with the increase of Reynolds number. It should be pointed out that both functions were obtained on the basis of determined distribution of eddy viscosity. This is justified for high frequencies (Brown *et al.*, 1969). It seems that for the mean band of frequencies another model would be more appropriate. A model in which eddy viscosity, except for viscous sublayer, would be determined using turbulent kinetic energy models (e.g. $k-\varepsilon$, $k-l$).
- The weighting function has a form which can easily undergo the first Laplace transformation and then Fourier transform. This is useful for determination a frequency characteristics of a long hydraulic pipeline systems.

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Ulepszona metoda symulacji przepływów przejściowych w przewodach hydraulicznych

Streszczenie

Artykuł przedstawia zagadnienie modelowania i symulacji przebiegów przejściowych podczas turbulentnego przepływu cieczy w przewodach ciśnieniowych. Chwilowe naprężenie styczne na ściance przewodu przedstawiono w postaci całki splotowej z funkcji wagi i przyspieszenia cieczy. Funkcja wagi dla naprężenia stycznego na ściance przewodu zależy od czasu bezwymiarowego i liczby Reynoldsa. Ma ona zawiłą postać matematyczną, dlatego aproksymowano ją do prostszej postaci, przydatnej do praktycznych obliczeń inżynierskich. Przedstawiono efektywny sposób rozwiązania całki splotowej, opierając się na metodzie podanej przez Trikha (1975) dla przepływu laminarnego. Podano zastosowanie ulepszonej metody symulacji naprężenia stycznego do metody charakterystyk podczas uderzenia hydraulicznego. Charakteryzuje się ona dużą efektywnością w stosunku do metody tradycyjnej.

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