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LIE ALGEBRA APPROACH IN THE STUDY OF THE STABILITY OF STOCHASTIC LINEAR HYBRID SYSTEMS

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The problem of the stability of a class of stochastic linear hybrid systems with a special structure of matrices and a multiplicative excitation is considered. Sufficient conditions of the exponential p-th mean stability and the almost sure stability for a class of stochastic linear hybrid systems with the Markovian switching are derived. Also, sufficient conditions of the exponential mean-square stability for a class of stochastic linear hybrid systems satisfying the Lie algebra conditions with any switching are found. The obtained results are illustrated by examples and simulations.

Key words: hybrid systems, nonlinear systems, asymptotic stability

1. Introduction

Starting with 50's, there has been an increasing interest in the analysis of switching systems, for instance, Itkis (1983), Loparo and Aslanis (1987), Utkin (1978), and variable structures systems (Kazakow and Artemiev, 1980). In mechanical engineering, the typical example are vibration systems with impacts (Dimentberg and Iourtchenko, 2004). It concerns both a deterministic and a stochastic case.

The generalization of these systems are hybrid systems (Liberzon, 2003; Boukas, 2005), which are dynamic systems consisting of several structures described by deterministic or stochastic differential equations. In the successive moments of time their structure can change according to the given switching rule, thereupon creates the hybrid system. One of the most important problems in the analysis of hybrid systems is the determination of stability conditions. In the case of stochastic systems, usually the almost sure and the p-mean stability is considered, see for instance Mao *et al.* (2008), Boukas (2006). It is a well known fact that even when all subsystems are stable, the whole hybrid system can be unstable (Liberzon, 2003). Therefore, researchers are interested in finding sufficient conditions of the stability of hybrid systems. Such conditions based on Lie algebra properties of system matrices were found in a particular case for linear deterministic systems in Zhai *et al.* (2006), Liberzon (2009).

The Lie algebra approach was also used in the study of control mechanical systems such that planar bodies, satellites and underwater vehicles (Bullo *et al.*, 2000), in the analysis of the controllability of dynamic systems with nonholonomic constrains (Sussman and Liu, 1993; He and Lu, 2006). Unfortunately, a nonholonomic system can not be stabilized by a continuous time invariant state feedback control (Brockett, 1983; Bloch and Crouch, 1992). Consistently, there appears a question if the Lie algebra properties can be used in the study of stability of hybrid linear mechanical systems. It will be shown in this paper that the answer in some cases is positive and in others is negative.

The paper is organized as follows. Section 2 describes mathematical preliminaries. A class of the multi-dimmensional linear hybrid system that satisfies Lie algebra conditions is introduced in Section 3, followed by sufficient conditions of the stability for systems with a Markovian switching rule and with any switching in Section 4 and 5, respectively. In Section 6, the Lee algebra approach is used in the study of practical stability of stochastic linear hybrid systems. The obtained results are illustrated by examples and simulations.

2. Mathematical preliminaries

Throughout this paper, we use the following notation. Let $|\cdot|$ and $<\cdot>$ be the Euclidean norm and the inner product in $\mathbb{R}^n$, respectively. By $\lambda(\mathbf{A})$ we denote the eigenvalue of the matrix $\mathbf{A}$, $\lambda_{min}(\mathbf{A})$ and $\lambda_{max}(\mathbf{A})$ denotes the smallest and the biggest eigenvalue of the matrix $\mathbf{A}$, respectively. We mark $\mathbb{R}_+ = [0,\infty)$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying usual conditions. Let $w(t)$, $t \geqslant 0$ be the m-dimensional Wiener process defined on the probability space. Let $r(t)$, $t \geqslant 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$ with the generator $\mathbf{\Gamma} = [\gamma_{ij}]_{N\times N}$, i.e.

$$\mathbb{P}\{r(t+\delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if} \quad i \neq j \\ 1 + \gamma_{ii}\delta + o(\delta) & \text{if} \quad i = j \end{cases} \tag{2.1}$$

where $\delta > 0$, $\gamma_{ij} \geqslant 0$ is the transition rate from i to j if $i \neq j$, $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$. We assume that the Markov chain is irreducible i.e. $\text{rank}(\mathbf{\Gamma}) = N-1$, and has a unique stationary distribution $\mathcal{P} = [p_1, p_2, \dots, p_N]^\top \in \mathbb{R}^N$ which can be determined by solving

$$\begin{cases} \mathcal{P}\mathbf{\Gamma} = \mathbf{0} \\ \text{subject to} \quad \displaystyle\sum_{i=1}^{N} p_i = 1 \quad \text{and} \quad p_i > 0 \ \text{ for all } \ i \in \mathbb{S} \end{cases} \tag{2.2}$$

We consider a linear hybrid system with multiplicative excitations described by the vector Itô differential equation

$$d\boldsymbol{z}(t) = \mathbf{A}(\sigma(t))\boldsymbol{z}(t)\, dt + \sum_{k=1}^{m} \mathbf{B}_{\boldsymbol{k}}(\sigma(t))\boldsymbol{z}(t) dw_k(t) \qquad \boldsymbol{z}(0) = \boldsymbol{z}_0 \tag{2.3}$$

$t \geqslant 0$, $\sigma(0) = \sigma_0 \in \mathbb{S}$, $\boldsymbol{z} \in \mathbb{R}^n$, $\mathbf{A}, \mathbf{B}_{\boldsymbol{k}} : \mathbb{S} \to \mathbb{R}^{n \times n}$, $k = 1, \dots, m$, $z_0 \in \mathbb{R}^n$ is the initial condition, the process $\sigma(t) : \mathbb{R}^+ \to \mathbb{S}$ is a switching signal. We assume that the solution $z(t)$ is everywhere continuous.

Processes $w_k(t)$ are normalized standard Wiener processes with

$$\begin{aligned} &\mathbb{E}[w_k(t)] = 0 \\ &\mathbb{E}[w_k(t)w_k(s)] = q^2 \min(t,s) \qquad i = 1, \dots, m \end{aligned} \tag{2.4}$$

where q is a constant parameter.

Throughout the paper, we assume that the processes $w_k(t)$, the process $\sigma(t)$ and the initial condition are mutually independent. The processes $w_k(t)$ and $\sigma(t)$ are $\{\mathcal{F}_t\}_{t \geqslant 0}$ adapted.

We use the following definitions of the stability (Mao, 1994):

Definition 1. The null solution to (2.3) is said to be almost surely exponentially stable if there exists a constant $\alpha > 0$ such that for each pair of $(\boldsymbol{x}_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$ there is a finite random variable C such that

$$|\boldsymbol{x}(t, \boldsymbol{x}_0, t_0)| \leqslant C \exp\{-\alpha(t - t_0)\} \qquad \text{a.s. for all } t \geqslant t_0 \tag{2.5}$$

or if

$$\limsup_{t \to \infty} \frac{1}{t} \log |\boldsymbol{x}(t, \boldsymbol{x}_0, t_0)| \leqslant -\alpha \qquad \text{a.s.} \tag{2.6}$$

The left hand side of (2.6) is called the almost sure Lyapunov exponent of the solution to (2.3).

Definition 2. The null solution of (2.3) is said to be p-th mean exponentially stable if there exists a pair of positive scalars α, c such that $\forall(\boldsymbol{x}_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$

$$\mathbb{E}[|\boldsymbol{x}(t, \boldsymbol{x}_0, t_0)|^p] \leqslant c\mathbb{E}|\boldsymbol{x}_0|^p \exp\{-\alpha(t-t_0)\} \qquad t \geqslant t_0 \tag{2.7}$$

or if

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}[|\boldsymbol{x}(t, \boldsymbol{x}_0, t_0)|^p]) \leqslant -\alpha \tag{2.8}$$

The left hand side of (2.8) is called the p-th mean Lyapunov exponent of the solution to (2.3). In the case of $p = 2$, it is usually called the mean-square exponential stability.

Let $\mathbf{P}$ be a non-singular $n \times n$ dimensional quadratic matrix such that matrices

$$\widetilde{\mathbf{A}}(i) = \mathbf{P}\mathbf{A}(i)\mathbf{P}^{-1} \qquad \widetilde{\mathbf{B}}_{\boldsymbol{k}}(i) = \mathbf{P}\mathbf{B}_{\boldsymbol{k}}(i)\mathbf{P}^{-1} \qquad \begin{matrix} k = 1, \dots, m \\ i = 1, \dots, N \end{matrix} \tag{2.9}$$

are upper triangular.

Multiplying equation (2.3) by the matrix $\mathbf{P}$ and introducing $\mathbf{I} = \mathbf{P}^{-1}\mathbf{P}$, we find

$$d(\mathbf{P}\boldsymbol{z}(t)) = \mathbf{P}\mathbf{A}(\sigma(t))\mathbf{P}^{-1}\mathbf{P}\boldsymbol{z}(t)dt + \sum_{k=1}^{m} \mathbf{P}\mathbf{B}_{\boldsymbol{k}}(\sigma(t))\mathbf{P}^{-1}\mathbf{P}\boldsymbol{z}(t)dw_k(t) \tag{2.10}$$

Introducing a new vector variable $\boldsymbol{x} = \mathbf{P}\boldsymbol{z}$, $\boldsymbol{x} \in \mathbb{R}^n$ in equations (2.10), we obtain a system with upper triangular matrices $\widetilde{\mathbf{A}}(\sigma(t)) = \mathbf{P}\mathbf{A}(\sigma(t))\mathbf{P}^{-1}$, $\widetilde{\mathbf{B}}_{\boldsymbol{k}}(\sigma(t)) = \mathbf{P}\mathbf{B}_{\boldsymbol{k}}(\sigma(t))\mathbf{P}^{-1}$

$$d\boldsymbol{x}(t) = \widetilde{\mathbf{A}}(\sigma(t))\boldsymbol{x}(t)dt + \sum_{k=1}^{m} \widetilde{\mathbf{B}}_{\boldsymbol{k}}(\sigma(t))\boldsymbol{x}(t)dw_k(t) \tag{2.11}$$

on $t \geqslant 0$ with the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0 = \mathbf{P}\boldsymbol{z}_0 \in \mathbb{R}^n$, $\sigma(0) = \sigma_0 \in \mathbb{S}$, $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}_{\boldsymbol{k}} : \mathbb{S} \to \mathbb{R}^{n\times n}$, the process $\sigma(t) : \mathbb{R}_+ \to \mathbb{S}$ is the switching signal.

We quote a definition and some basic facts for a Lie algebra:

Definition 3. A Lie algebra over a field $\mathcal{F}$ is a triple $(V, +, [\cdot,\cdot])$ where $(V, +)$ is a vector space over $\mathcal{F}$ and where $[\cdot,\cdot]$ is a bilinear map from $V \times V$ into V such that

1. $[v_1, v_2] = -[v_2, v_1]$ (antisymmetry)
2. $[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$ (Jacobi identity)

For example, the vector space of all $n \times n$ matrices over the field with $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ is a Lie algebra. We mark by $\mathbb{L}(A_1, A_2, \ldots, A_n)$ the Lie algebra generated by $A_1, A_2, \ldots, A_n$.

Definition 4. Let us assign for every Lie algebra $\mathbb{L}$ the following sequence

$$\begin{aligned} &\mathbb{L}^0 = \mathbb{L} \\ &\vdots \\ &\mathbb{L}^{n+1} = [\mathbb{L}^n, \mathbb{L}^n] = \{[\mathbf{A}, \mathbf{B}] \mid \mathbf{A}, \mathbf{B} \in \mathbb{L}^n\} \qquad n \geqslant 0 \end{aligned} \tag{2.12}$$

The Lie algebra $\mathbb{L}$ is abelian if $\mathbb{L}^{(1)} = \{0\}$ and is solvable if $\mathbb{L}^n = \{0\}$ for some n.

The following result plays a key rule in further considerations.

Lemma 1. (Samelson, 1969) A matrix Lie algebra is solvable iff there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PMP}^{-1}$ is upper triangular for all matrices $\mathbf{M}$ in the Lie algebra.

Note that if the Lie algebra $\mathbb{L}(\mathbf{A}, \mathbf{B})$ is solvable, then eigenvalues of a linear combination of $\mathbf{A}$ and $\mathbf{B}$ are equal to the same linear combination of the corresponding eigenvalues

$$\lambda_j(\mathbf{A} + \mu\mathbf{B}) = a_j + \mu b_j \qquad j = 1, \ldots, n \tag{2.13}$$

where a_j and b_j are eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, respectively; μ is a constant.

In the case of diagonal matrices $\mathbf{A}$, $\mathbf{B}_k$, the stability analysis of system (2.3) is equivalent to the same properties for n first order systems.

This result can be applied if the matrices $\mathbf{A}$, $\mathbf{B}_k$ can be diagonalized by means of the same transformation matrix $\mathbf{P}$. Furthermore, this result can be extended to the cases when the matrices $\mathbf{A}$, $\mathbf{B}_k$ are upper triangular or when they can be transformed into such a form by the same transformation matrix $\mathbf{P}$. In this case, the following result is obtained for the nonhybrid system (a special case of (2.3) with $N = 1$)

Theorem 1. (Willems, 1976) The null solution is p-th mean exponentially stable, if the mapping $\mathbf{Q} \to L(\mathbf{Q})$ in the space of symmetric matrices of order n, defined by

$$L(\mathbf{Q}) = \Big(\mathbf{A} + \frac{p-2}{4}\sum_{k=1}^{m}\mathbf{B}_k^2\Big)\mathbf{Q} + \mathbf{Q}\Big(\mathbf{A} + \frac{p-2}{4}\sum_{k=1}^{m}\mathbf{B}_k^2\Big)^\top + \frac{p}{2}\sum_{k=1}^{m}\mathbf{B}_k\mathbf{Q}\mathbf{B}_k^\top \tag{2.14}$$

has only eigenvalues with negative real parts. If all eigenvalues of matrices $\mathbf{B}_{\boldsymbol{k}}$ are real, then this is equivalent to the Hurwitz character of the matrix

$$\mathbf{A}+\frac{p-1}{2}\sum_{k=1}^{m}\mathbf{B}_{\boldsymbol{k}}^2 \tag{2.15}$$

The null solution is then almost surely exponentially stable if the matrix

$$\mathbf{A}-\frac{1}{2}\sum_{k=1}^{m}\mathbf{B}_{\boldsymbol{k}}^2 \tag{2.16}$$

is Hurwitz.

3. Multi-dimensional linear hybrid systems

Let us consider a multi-dimensional linear hybrid system driven by Wiener processes of the form

$$d\boldsymbol{z}(t)=\mathbf{A}(\sigma(t))\boldsymbol{z}(t)dt+\sum_{k=1}^{m}\mathbf{B}_{\boldsymbol{k}}(\sigma(t))\boldsymbol{z}(t)dw_k(t) \tag{3.1}$$

on $t\geqslant 0$ with the initial condition $\boldsymbol{z}(0)=\boldsymbol{z}_0\in\mathbb{R}^n$, $\sigma(0)=\sigma_0\in\mathbb{S}$, $\boldsymbol{z}\in\mathbb{R}^n$, $\mathbf{A},\mathbf{B}:\mathbb{S}\to\mathbb{R}^{n\times n}$, $k=1,\dots,m$, the Wiener processes $w_k(t)$ satisfy conditions $(2.4)_1$, the process $\sigma(t)$ is the switching signal. Let us assume that the Lie algebra

$$\mathbb{L}(\mathbf{A}(i),\ \mathbf{B}_{\boldsymbol{k}}(i),\ k=1,\dots,m,\ i=1,\dots,N) \tag{3.2}$$

is solvable.

The solvability of the Lie algebra $\mathbb{L}$ implies (Lemma 1) the existence of a similarity transformation $\mathbf{P}$ which brings matrices $\mathbf{A}(i)$, $\mathbf{B}_{\boldsymbol{k}}(i)$, $k=1,\dots,m$, $i=,\dots,N$ into an upper triangular form, and hybrid system (3.1) can be for every $i\in\mathbb{S}$ transformed into

$$d\boldsymbol{x}(t)=\begin{bmatrix} a_1(i) & & * \\ & \ddots & \\ 0 & & a_n(i)\end{bmatrix}\boldsymbol{x}(t)+\sum_{k=1}^{m}\begin{bmatrix} b_{k1}(i) & & * \\ & \ddots & \\ 0 & & b_{kn}(i)\end{bmatrix}\boldsymbol{x}(t)dw_k(t) \tag{3.3}$$

where $\boldsymbol{x}=\mathbf{P}\boldsymbol{z}$. Elements above the main diagonal $(*)$ are not essential for further analysis, and all elements below the main diagonal vanish. It can be proved (Willems and Aeyels, 1976) that the p-th mean exponential stability and the almost sure exponential stability for system (3.3) are equivalent to the same properties for n first order systems

$$dx_j(t) = a_j(i)x_j(t) + \sum_{k=1}^{m} b_{kj}(i)x_j(t)dw_k(t) \qquad \begin{array}{l} j = 1, 2, \dots, n \\ i = 1, \dots, N \end{array} \tag{3.4}$$

where $a_j(i), b_{1j}(i), \dots, b_{mj}(i)$, $j = 1, \dots, n$ are eigenvalues of matrices $\mathbf{A}(i)$, $\mathbf{B}_{\boldsymbol{k}}(i)$, $i = 1, \dots, N$, $k = 1, \dots, m$, $t \geqslant 0$, with the initial values $x_j(0) = x_{j0} \in \mathbb{R}$ determined from x_0, $x_j \in \mathbb{R}$.

Consider in place of (3.1) n first order systems

$$dx_j(t) = a_j(\sigma(t))x_j(t) + \sum_{k=1}^{m} b_{kj}(\sigma(t))x_j(t)dw_k(t) \qquad j = 1, 2, \dots, n \tag{3.5}$$

where $a_j(i), b_{1j}(i), \dots, b_{mj}(i)$, $j = 1, \dots, n$ are eigenvalues of matrices $\mathbf{A}(i)$, $\mathbf{B}_{\boldsymbol{k}}(i)$, $i = 1, \dots, N$, $k = 1, \dots, m$, $t \geqslant 0$, with the initial values $x_j(0) = x_{j0} \in \mathbb{R}$, $x_j \in \mathbb{R}$.

4. Hybrid systems with the Markovian switching rule

For the one-dimensional hybrid system of the form

$$dx(t) = a(r(t))x(t) + b(r(t))x(t)dw(t) \tag{4.1}$$

where $t \geqslant 0$ with the initial value $x(0) = x_0 \in \mathbb{R}$, $r(0) = r_0 \in \mathbb{S}$, $w(t)$ is the standard Wiener process, $x \in \mathbb{R}$, $a, b : \mathbb{S} \to \mathbb{R}$, the switching rule is given by (2.1) and (2.2), we have the following lemma.

Lemma 2. (Mao *et al.*, 2008) The sample Lyapunov exponent of (4.1) is

$$\lim_{t\to\infty} \frac{1}{t} \log(|x(t)|) = \sum_{i=1}^{N} p_i\Big(a(i) - \frac{1}{2}b^2(i)\Big) \qquad \text{a.s.} \tag{4.2}$$

for $x_0 \neq 0$. Hence (4.1) with the switching rule given by (2.1) and (2.2) is almost surely exponentially stable if

$$\sum_{i=1}^{N} p_i\Big(a(i) - \frac{1}{2}b^2(i)\Big) < 0 \tag{4.3}$$

Using Lemma 2, we find that the null solution to (3.1) for complex values a_j, b_{kj} is almost surely exponentially stable if for $j = 1, \dots, n$

$$\sum_{i=1}^{N} p_i \Big(\mathrm{Re}(a_j(i)) - \frac{1}{2} \sum_{k=1}^{m} [\mathrm{Re}(b_{kj}(i))]^2 \Big) < 0 \tag{4.4}$$

Hence, we obtain the theorem for the vector linear hybrid system.

Theorem 2. The trivial solution of hybrid system (3.1) with switching rule (2.1) and (2.2) is almost surely exponentially stable if condition (4.4) is satisfied.

Remark 1. If all eigenvalues of matrices $\mathbf{B}_{\boldsymbol{k}}(i)$ are real $i = 1, \dots, N$, $k = 1, \dots, m$ condition (4.4) is equivalent to the Hurwitz character of the matrix

$$\sum_{i=1}^{N} p_i \Big(\mathbf{A}(i) - \frac{1}{2} \sum_{k=1}^{m} \mathbf{B}_k^2(i) \Big) \tag{4.5}$$

Sufficient conditions of the p-th mean exponential stability for one--dimensional hybrid system (4.1) are given by the following lemma.

Lemma 3. (Mao *et al.*, 2008) The p-th mean Lyapunov exponent of (4.1) is

$$\lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}[|x(t)|^p]) = \sum_{i=1}^{N} p_i p \Big(a(i) + \frac{1}{2}(p-1) b^2(i) \Big) \qquad \text{a.s.} \tag{4.6}$$

for $x_0 \neq 0$. Hence, (4.1) with switching rule (2.1) and (2.2) is p-th mean exponentially stable if

$$\sum_{i=1}^{N} p_i \Big(a(i) + \frac{1}{2}(p-1) b^2(i) \Big) < 0 \tag{4.7}$$

Using Lemma 3, we find that the null solution to (3.1) is p-th mean exponentially stable if for $j = 1, \dots, n$

$$\sum_{i=1}^{N} p_i \Big(\mathrm{Re}(a_j(i)) + \frac{p-1}{2} \sum_{k=1}^{m} [\mathrm{Re}(b_{kj}(i))]^2 \Big) < 0 \tag{4.8}$$

and the null solution of (3.1) is p-th mean exponentially stable if the mapping $\mathbf{Q} \to L(\mathbf{Q})$ in the space of symmetric matrices of order n defined as

$$
\begin{aligned}
L(\mathbf{Q}) = &\Big(\sum_{i=1}^{N} p_i\Big(\mathbf{A}(i) + \frac{p-2}{4}\sum_{k=1}^{m}\mathbf{B}_k^2(i)\Big)\Big)\mathbf{Q} + \\
&+\mathbf{Q}\Big(\sum_{i=1}^{N} p_i\Big(\mathbf{A}(i) + \frac{p-2}{4}\sum_{k=1}^{m}\mathbf{B}_k^2(i)\Big)\Big)^{\top} + + \sum_{i=1}^{N} p_i\Big(\frac{p}{2}\sum_{k=1}^{m}\mathbf{B}_{\boldsymbol{k}}(i)\mathbf{Q}\mathbf{B}_{\boldsymbol{k}}^{\top}(i)\Big)
\end{aligned}
\tag{4.9}
$$

has only eigenvalues with negative real parts.

From these considerations follows the next theorem.

Theorem 3. The trivial solution to hybrid system (3.1) with the switching rule (2.1) and (2.2) is p-th mean exponentially stable if the mapping $Q \to L(Q)$ in the space of symmetric matrices of order n defined by (4.9) has only eigenvalues with negative real parts.

Remark 2. If all eigenvalues of matrices $\mathbf{B}_{\boldsymbol{k}}(i)$ are real $i = 1, \dots, N$, $k = 1, \dots, m$ condition (4.4) is equivalent to the Hurwitz character of the matrix

$$\sum_{i=1}^{N} p_i\Big(\mathbf{A}(i) + \frac{p-1}{2}\sum_{k=1}^{m}\mathbf{B}_k^2(i)\Big) \tag{4.10}$$

Example 1. Let us consider a particular case of hybrid system (3.1) with the switching rule given by (2.1) and (2.2) and two structures: almost surely exponentially and mean-square exponentially stable $(i = 1)$ and unstable $(i = 2)$, $m = 1$, where

$$\mathbf{A}(1) = \begin{bmatrix} -1.5 & 0.5 \\ 1.0 & -1.0 \end{bmatrix} \qquad \mathbf{B}_1(1) = \begin{bmatrix} -0.01 & 0.05 \\ 0.05 & -0.01 \end{bmatrix} \tag{4.11}$$

and

$$\mathbf{A}(2) = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{B}_1(2) = \begin{bmatrix} 0.05 & 0.01 \\ 0.01 & 0.05 \end{bmatrix} \tag{4.12}$$

Matrices $\mathbf{A}(1) + \frac{1}{2}\mathbf{B}_1^2(1)$ and $\mathbf{A}(1) - \frac{1}{2}\mathbf{B}_1^2(1)$ are Hurwitz while $\mathbf{A}(2) + \frac{1}{2}\mathbf{B}_1^2(2)$ and $\mathbf{A}(2) - \frac{1}{2}\mathbf{B}_1^2(2)$ are unstable.

We find that the Lie algebra $\mathbb{L}(\mathbf{A}(1), \mathbf{A}(2), \mathbf{B}_1(1), \mathbf{B}_1(2))$ is solvable with the matrix $\mathbf{P}$ given by

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{4.13}$$

Eigenvalues of $\mathbf{A}(i) + \frac{1}{2}\mathbf{B}_1^2(i)$ are $\{-1.9982, -0.4992\}$ for $i = 1$ and $\{1.0008, 4.0018\}$ for $i = 2$, respectively, and from criterion (4.10), hybrid system (3.1) with matrices given by (4.11) and (4.12) is mean-square exponentially stable if the probabilities p_i, $i = 1, 2$ satisfy the following inequalities

$$\begin{cases} -1.9982p_1 + 1.0008p_2 < 0 \\ -0.4992p_1 + 4.0018p_2 < 0 \\ p_2 = 1 - p_1 \end{cases} \Rightarrow \begin{cases} p_1 > 0.89 \\ p_2 = 1 - p_1 \end{cases} \tag{4.14}$$

Eigenvalues of $\mathbf{A}(i) - \frac{1}{2}\mathbf{B}_1^2(i)$ are $\{-2.0018, -0.5008\}$ for $i = 1$ and $\{0.9992, 3.9982\}$ for $i = 2$, respectively, and from criterion (4.5), hybrid system (3.1) with matrices given by (4.11) and (4.12) is almost surely exponentially stable if the probabilities p_i, $i = 1, 2$ satisfy the following inequalities

$$\begin{cases} -2.0018p_1 + 0.9992p_2 < 0 \\ -0.5008p_1 + 3.9982p_2 < 0 \\ p_2 = 1 - p_1 \end{cases} \Rightarrow \begin{cases} p_1 > 0.89 \\ p_2 = 1 - p_1 \end{cases} \tag{4.15}$$

Exemplary simulations are shown in Fig. 1.

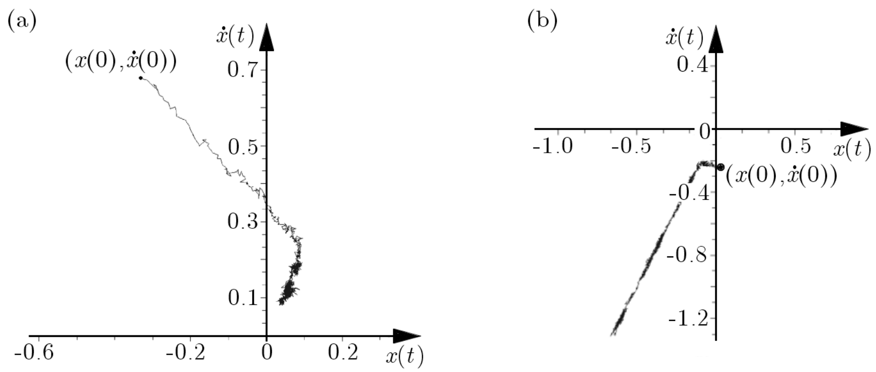

Fig. 1. Exemplary samples for system (3.1) with matrices (4.11) and (4.12). (a) A stable sample of system (3.1) for parameters: $p_1 = 0.9$, $p_2 = 0.1$; (b) an unstable sample of (3.1) for parameters: $p_1 = 0.87$, $p_2 = 0.13$

5. Hybrid systems with any switching

Let us consider a particular case of linear hybrid system (3.1) described by

$$d\boldsymbol{x}(t) = \mathbf{A}(\sigma(t))\boldsymbol{x}(t)dt + \mathbf{B}(\sigma(t))\boldsymbol{x}(t)dw(t) \qquad i = 1, \dots, N \tag{5.1}$$

on $t \geqslant 0$ with the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$, $\boldsymbol{x} \in \mathbb{R}^n$, $\sigma(0) = \sigma_0 \in \mathbb{S}$, $\mathbf{A}, \mathbf{B} : \mathbb{S} \to \mathbb{R}^{n\times n}$ and with any switching rule $\sigma(t)$, $w(t)$ is a scalar standard Wiener process.

Definition 5. If there exists a common definite matrix $\mathbf{H}$ satisfying

$$\mathbf{A}(i)^\top \mathbf{H} + \mathbf{H}\mathbf{A}(i) + \mathbf{B}(i)^\top \mathbf{H}\mathbf{B}(i) < 0 \qquad i = 1, \dots, N \tag{5.2}$$

then $V(x) = \boldsymbol{x}^\top \mathbf{H}\boldsymbol{x}$ is called the common quadratic Lyapunov function for all subsystems.

Theorem 4. If there exists a common quadratic Lyapunov function for all subsystems, then hybrid system (5.1) is exponentially mean-square stable for any switching $\sigma(t)$.

Proof: For $V(\boldsymbol{x}) = \boldsymbol{x}^\top \mathbf{H}\boldsymbol{x}$, we have

$$\lambda_{min}(\mathbf{H})|\boldsymbol{x}|^2 \leqslant V(\boldsymbol{x}) \leqslant \lambda_{max}(\mathbf{H})|\boldsymbol{x}|^2 \tag{5.3}$$

To show the exponential stability of hybrid system (5.1), we first find positive scalars κ_i such that

$$\mathbf{A}(i)^\top \mathbf{H} + \mathbf{H}\mathbf{A}(i) + \mathbf{B}(i)^\top \mathbf{H}\mathbf{B}(i) < -\kappa_i \mathbf{H} \tag{5.4}$$

hold for all i. Then, from (5.4), we find that

$$\mathcal{L}_i V(\boldsymbol{x}) \leqslant -\kappa_i V(\boldsymbol{x}) \tag{5.5}$$

where

$$\mathcal{L}_i V(\boldsymbol{x}) = \frac{\partial V}{\partial t} + (\mathbf{A}(i)\boldsymbol{x}(t))^\top \frac{\partial V}{\partial \boldsymbol{x}} + \frac{1}{2}\Big\langle \mathbf{B}(i)\boldsymbol{x}(t), \frac{\partial V}{\partial \boldsymbol{x}}\Big\rangle^2 \tag{5.6}$$

Hence, we obtain

$$\frac{d}{dt}E[V(\boldsymbol{x})] \leqslant -\kappa E[V(\boldsymbol{x})] \tag{5.7}$$

where $\kappa = \min_i\{\kappa_i\}$ and

$$E[V(\boldsymbol{x})] \leqslant V(\boldsymbol{x}_0)\exp\{-\kappa(t - t_0)\} \tag{5.8}$$

From (5.3), we have

$$E\left[|\boldsymbol{x}(t)|^2\right] \leqslant E\left[|\boldsymbol{x}(0)|^2\right] \frac{\lambda_{max}(\mathbf{H})}{\lambda_{min}(\mathbf{H})} \exp\{-\kappa(t-t_0)\} \tag{5.9}$$

Theorem 4 with use of the common quadratic Lyapunov function establishes sufficient conditions for the mean-square exponentially stability for linear hybrid systems with parametric excitations for any switching.

Another sufficient condition of the mean-square exponentially stability for hybrid systems can be proposed for linear systems with a special structure of matrices defined by the Lie algebra.

Theorem 5. If the Lie algebra $\mathbb{L}(\mathbf{A}(i), \mathbf{B}(i), i = 1, \dots, N)$ is solvable and furthermore

$$2\mathrm{Re}(\lambda_j(\mathbf{A}(i))) + |\lambda_j(\mathbf{B}(i))|^2 < 0 \qquad \begin{array}{l} j = 1, \dots, n \\ i = 1, \dots, N \end{array} \tag{5.10}$$

then hybrid system (5.1) is mean-square exponentially stable for any switching $\sigma(t)$.

Proof: According to Theorem 4, the proof is reduced to finding a common definite matrix $\mathbf{H}$ for all subsystems of (5.1). If the Lie algebra $\mathbb{L}(\mathbf{A}(i), \mathbf{B}(i), i = 1, \dots, N)$ is solvable, we can find a nonsingular complex matrix $\mathbf{P}$ such that for every $i \in \mathbb{S}$ the matrices $\mathbf{A}(i) = [a_{jl}(i)]_{j,l=1,\dots,n}$ and $\mathbf{B}(i) = [b_{jl}(i)]_{j,l=1,\dots,n}$, $i = 1, 2, \dots, N$ have the following form

$$\widetilde{\mathbf{A}}(i) = \mathbf{P}\mathbf{A}(i)\mathbf{P}^{-1} \qquad \widetilde{\mathbf{B}}(i) = \mathbf{P}\mathbf{B}(i)\mathbf{P}^{-1} \tag{5.11}$$

where the complex matrices $\widetilde{\mathbf{A}}(i) = [\widetilde{a}_{jl}(i)]_{j,l=1,\dots,n}$, $\widetilde{\mathbf{B}}(i) = [\widetilde{b}_{jl}(i)]_{j,l=1,\dots,n}$, $i = 1, \dots, N$ are upper triangular. First, we show that there exists a real positive definite matrix $\widetilde{\mathbf{H}}$ such that $\forall i = 1, \dots, N$

$$\widetilde{\mathbf{A}}(i)^*\widetilde{\mathbf{H}} + \widetilde{\mathbf{H}}\widetilde{\mathbf{A}}(i) + \widetilde{\mathbf{B}}(i)^*\widetilde{\mathbf{H}}\widetilde{\mathbf{B}}(i) < 0 \qquad \widetilde{\mathbf{H}} = \mathrm{diag}\,\{\widetilde{h}_j\}_{j=1,\dots,n} \tag{5.12}$$

The matrices $\widetilde{\mathbf{A}}(i)^*\widetilde{\mathbf{H}} + \widetilde{\mathbf{H}}\widetilde{\mathbf{A}}(i) + \widetilde{\mathbf{B}}(i)^*\widetilde{\mathbf{H}}\widetilde{\mathbf{B}}(i)$ are Hermitian for every $i \in \mathbb{S}$, for instance, for $n = 3$ have the following form

$$\widetilde{\mathbf{A}}(i)^*\widetilde{\mathbf{H}} + \widetilde{\mathbf{H}}\widetilde{\mathbf{A}}(i) + \widetilde{\mathbf{B}}(i)^*\widetilde{\mathbf{H}}\widetilde{\mathbf{B}}(i) = [q^i_{jl}] \tag{5.13}$$

where

$$
\begin{aligned}
q^i_{11} &= \widetilde{h}_1\Big(2\mathrm{Re}(\widetilde{a}_{11}(i)) + |\widetilde{b}_{11}(i)|^2\Big)\\
q^i_{12} &= \widetilde{h}_1\Big(\widetilde{a}_{12}(i) + \overline{\widetilde{b}}_{11}(i)\widetilde{b}_{12}(i)\Big)\\
q^i_{13} &= \widetilde{h}_1\Big(\widetilde{a}_{13}(i) + \overline{\widetilde{b}}_{11}(i)\widetilde{b}_{13}(i)\Big)\\
q^i_{22} &= \widetilde{h}_2\Big(2\mathrm{Re}(\widetilde{a}_{22}(i)) + |\widetilde{b}_{22}(i)|^2\Big) + \widetilde{h}_1(|\widetilde{b}_{12}(i)|^2)\\
q^i_{23} &= \widetilde{h}_2\Big(\widetilde{a}_{23}(i) + \overline{\widetilde{b}}_{22}(i)\widetilde{b}_{23}(i)\Big) + \widetilde{h}_1(\overline{\widetilde{b}}_{12}(i)\widetilde{b}_{13}(i))\\
q^i_{33} &= \widetilde{h}_3\Big(2\mathrm{Re}(\widetilde{a}_{33}(i)) + |\widetilde{b}_{33}(i)|^2\Big) + \widetilde{h}_1(|\widetilde{b}_{13}(i)|^2) + \widetilde{h}_2(|\widetilde{b}_{23}(i)|^2)
\end{aligned}
\tag{5.14}
$$

From the assumption of Theorem 5, we have

$$
\Big(2\mathrm{Re}(\widetilde{a}_{jj}(i)) + |\widetilde{b}_{jj}(i)|^2\Big) < 0 \qquad \begin{array}{l} j = 1, \dots, n \\ i = 1, \dots, N \end{array} \tag{5.15}
$$

Hence we can find sufficiently large positive scalars $\widetilde{h}_j$, $j = 1, 2, 3$, such that matrix (5.13) is negative definite.

Using the obtained $\widetilde{\mathbf{H}}$, and substituting (5.11) into (5.12), we obtain

$$
(\mathbf{P}^{-1})^*\mathbf{A}(i)^\top\mathbf{P}^*\widetilde{\mathbf{H}} + \widetilde{\mathbf{H}}\mathbf{P}\mathbf{A}(i)\mathbf{P}^{-1} + (\mathbf{P}^{-1})^*\mathbf{B}(i)^\top\mathbf{P}^*\widetilde{\mathbf{H}}\mathbf{P}\mathbf{B}(i)\mathbf{P}^{-1} < 0 \tag{5.16}
$$

Multiplying equation (5.16) from the left-hand side by the matrix $\mathbf{P}^*$ and from the right-hand side by the matrix $\mathbf{P}$, we have

$$
\mathbf{A}(i)^\top\mathbf{P}^*\widetilde{\mathbf{H}}\mathbf{P} + \mathbf{P}^*\widetilde{\mathbf{H}}\mathbf{P}\mathbf{A}(i) + \mathbf{B}(i)^\top\mathbf{P}^*\widetilde{\mathbf{H}}\mathbf{P}\mathbf{B}(i) < 0 \tag{5.17}
$$

Hence, we obtain

$$
\mathbf{A}(i)^\top\mathbf{H} + \mathbf{H}\mathbf{A}(i) + \mathbf{B}(i)^\top\mathbf{H}\mathbf{B}(i) < 0 \qquad \text{for} \qquad \mathbf{H} = \mathbf{P}^*\widetilde{\mathbf{H}}\mathbf{P} \tag{5.18}
$$

We write the complex matrix $\mathbf{H}$ as $\mathbf{H} = \mathrm{Re}(\mathbf{H}) + i\mathrm{Im}(\mathbf{H})$. Since $\mathbf{H}$ is Hermitian, $\mathrm{Im}(\mathbf{H})$ is skew-symmetric and $\boldsymbol{x}^\top\mathbf{H}\boldsymbol{x} = \boldsymbol{x}^\top\mathrm{Re}(\mathbf{H})\boldsymbol{x} > 0$, $\boldsymbol{x} \neq \mathbf{0}$. Thus, $\mathrm{Re}(\mathbf{H})$ is a real positive definite matrix and

$$
\mathbf{A}(i)^\top\mathrm{Re}(\mathbf{H}) + \mathrm{Re}(\mathbf{H})\mathbf{A}(i) + \mathbf{B}(i)^\top\mathrm{Re}(\mathbf{H})\mathbf{B}(i) < 0 \tag{5.19}
$$

which implies that $\mathrm{Re}(\mathbf{H})$ is the common Lyapunov matrix we want to compute.

Remark 3. If all eigenvalues of matrices $\mathbf{A}(i)$ and $\mathbf{B}(i)$ are real, condition (5.10) is equivalent to the Hurwitz character of matrices $2\mathbf{A}(i) + \mathbf{B}^2(i)$, $i = 1, \dots, N$.

Example 2. Let us consider hybrid system (5.1) with two exponentially mean-square stable structures, where

$$\mathbf{A}(1) = \begin{bmatrix} -2 & -1 \\ 0.5 & -0.5 \end{bmatrix} \qquad \mathbf{B}(1) = \begin{bmatrix} -0.01 & 0.05 \\ 0.05 & -0.01 \end{bmatrix} \tag{5.20}$$

and

$$\mathbf{A}(2) = \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \qquad \mathbf{B}(2) = \begin{bmatrix} 0.05 & 0.01 \\ 0.01 & 0.05 \end{bmatrix} \tag{5.21}$$

The Lie algebra $\mathbb{L}(\mathbf{A}(1), \mathbf{A}(2), \mathbf{B}(1), \mathbf{B}(2))$ is solvable with the matrix $\mathbf{P}$ given by

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{5.22}$$

The matrices $\mathbf{A}(i) + \frac{1}{2}\mathbf{B}^2(i)$, $i = 1, 2$ are Hurwitz, and from Theorem 5, hybrid system (5.1) with matrices given by (5.20) and (5.21) is mean-square exponentially stable for any switching (Fig. 2). Theorem 5 cannot be applied to the system considered in Example 1 because we cannot use the random switching rule in it, and to make this system mean-square exponentially stable we have to apply limitations concerning the switching rule of this system.

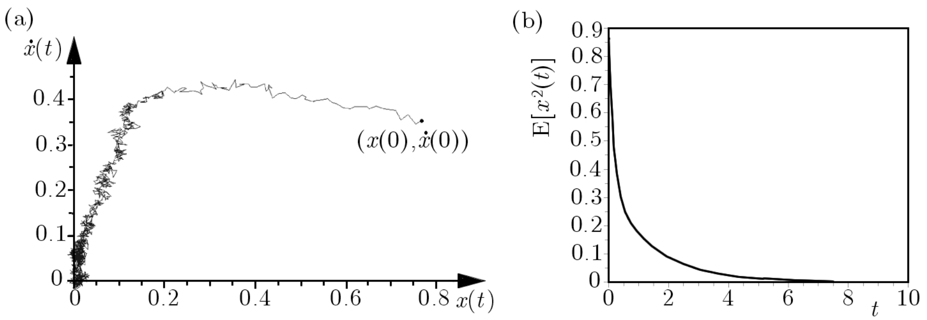

Fig. 2. An exemplary simulation for system (5.1) with matrices (5.20) and (5.21). (a) A stable sample for system (5.1), (b) the mean-square of the solution to system (5.1)

Example 3. Let us consider a special case of hybrid system (5.1) with two unstable structures, where

$$\mathbf{A}(1) = \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & 0.4 \end{bmatrix} \qquad \mathbf{B}(1) = \begin{bmatrix} -0.1 & 0.5 \\ 0.5 & -0.1 \end{bmatrix} \tag{5.23}$$

and

$$\mathbf{A}(2) = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.5 \end{bmatrix} \qquad \mathbf{B}(2) = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix} \tag{5.24}$$

The Lie algebra $\mathbb{L}(\mathbf{A}(1), \mathbf{A}(2), \mathbf{B}(1), \mathbf{B}(2))$ is solvable with the matrix $\mathbf{P}$ given by

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{5.25}$$

If both structures are unstable and matrices (5.23), (5.24) create the solvable Lie algebra, hybrid system (5.1) is always unstable regardless of the switching (Fig. 3).

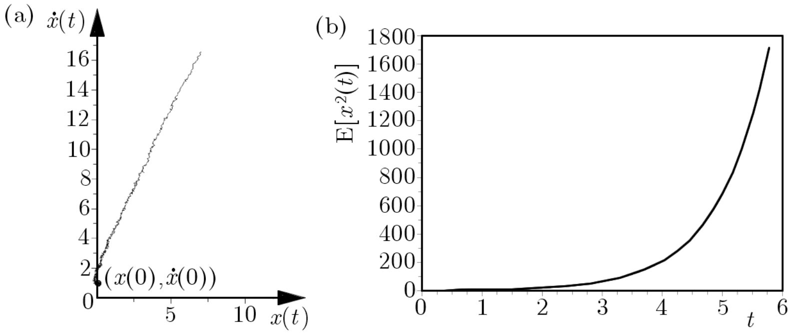

Fig. 3. An exemplary simulation for system (5.1) with matrices (5.23) and (5.24). (a) An unstable sample for system (5.1), (b) the mean-square of the solution to system (5.1)

6. Limitation of the applicability of the Lie algebra approach to mechanical hybrid systems

In what follows, we show that for an oscillator hybrid system the Lie algebra sufficient conditions of stability are not satisfied, and one can consider both unstable a hybrid system consisting of stable subsystems and stable a hybrid system consisting of unstable subsystems.

Example 4. Let us consider a particular case of hybrid system (3.1) with two exponentially mean-square stable structures with matrices

$$\mathbf{A}(1) = \begin{bmatrix} 0 & 1 \\ -0.25 & -0.01 \end{bmatrix} \qquad \mathbf{B}(1) = \begin{bmatrix} 0 & 0 \\ -0.125 & 0 \end{bmatrix} \tag{6.1}$$

and

$$\mathbf{A}(2) = \begin{bmatrix} 0 & 1 \\ -4 & -0.01 \end{bmatrix} \qquad \mathbf{B}(2) = \begin{bmatrix} 0 & 0 \\ -8 & 0 \end{bmatrix} \tag{6.2}$$

Condition (5.10) is satisfied, but a nonsingular matrix which brings matrices (6.1) and (6.2) into an upper triangular form does not exists, and hence matrices (6.1) and (6.2) do not create a solvable Lie algebra. Thus, assumptions of Theorem 5 are not satisfied, and we can find the switching rule that hybrid system (5.1) with this special switching is unstable. An exemplary simulation is shown in Fig. 4.

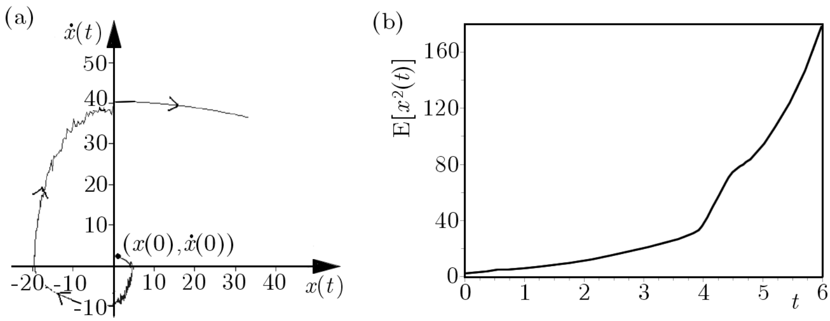

Fig. 4. An exemplary simulation for system (5.1) with matrices (6.1) and (6.2). (a) An unstable sample for system (5.1), (b) the mean-square of the solution to system (5.1)

Example 5. Let us consider a special case of hybrid system (3.1) with two unstable structures with matrices

$$\mathbf{A}(1) = \begin{bmatrix} 0 & 1 \\ -0.25 & -0.01 \end{bmatrix} \qquad \mathbf{B}(1) = \begin{bmatrix} 0 & 0 \\ -0.05 & 0 \end{bmatrix} \tag{6.3}$$

and

$$\mathbf{A}(2) = \begin{bmatrix} 0 & 1 \\ -4 & -0.01 \end{bmatrix} \qquad \mathbf{B}(2) = \begin{bmatrix} 0 & 0 \\ -0.2 & 0 \end{bmatrix} \tag{6.4}$$

Similar to previous Counterexample 4, a nonsingular matrix which brings matrices (6.3) and (6.4) into an upper triangular form does not exists, and hence matrices (6.3) and (6.4) do not create a solvable Lie algebra. Despite of both structures are unstable, hybrid system (5.1) can be stable with the special switching. An exemplary simulation is shown in Fig. 5.

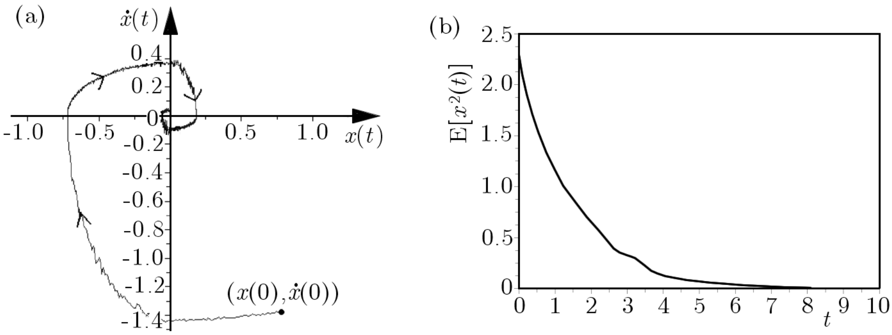

Fig. 5. An exemplary simulation for system (5.1) with matrices (6.3) and (6.4). (a) A stable sample for system (5.1), (b) the mean-square of the solution to system (5.1)

Now we consider the case of stochastic second order simple nonholonomic hybrid system that satisfies Lie algebra conditions but matrices $\mathbf{A}(i)$ have not negative real parts of their eigenvalues.

Example 6. Let us consider a special case of hybrid system (3.1) with two structures with matrices

$$\mathbf{A}(1) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \qquad \mathbf{B}(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0.05 \end{bmatrix} \tag{6.5}$$

and

$$\mathbf{A}(2) = \begin{bmatrix} 0 & 1 \\ 0 & -0.2 \end{bmatrix} \qquad \mathbf{B}(2) = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{6.6}$$

An exemplary simulation is shown in Fig. 6.

From the structures of matrices (6.5) and (6.6), it follows that the both subsystems are not exponentially stable (one eigenvalue in both structure matrices $\mathbf{A}(i)$, $i = 1, 2$ is equal to zero). Therefore, the sufficient conditions derived in the previous section are not satisfied. However, one can show that the hybrid system is practically mean–square stable in the sense of the following definition (Yin *et al.*, 2008).

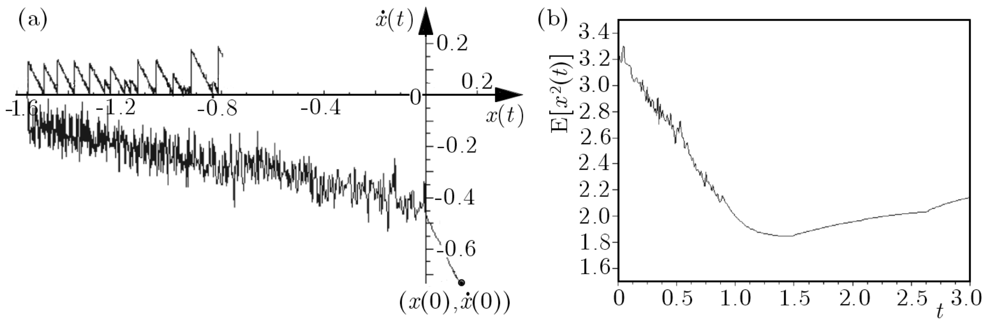

Fig. 6. An exemplary simulation for system (5.1) with matrices (6.5) and (6.6). (a) A stable sample for system (5.1), (b) the mean-square of the solution to system (5.1)

Definition 6. System (2.3) is said to be practically mean-square stable if there exists a pair of positive scalars α, β such that $\alpha < \beta$ that $\forall(\boldsymbol{x}_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$ the condition

$$\mathbb{E}[|\boldsymbol{x}_0|^2] \leqslant \alpha \tag{6.7}$$

implies

$$\mathbb{E}[|\boldsymbol{x}(t, \boldsymbol{x}_0, t_0)|^2]) \leqslant \beta \tag{6.8}$$

7. Conclusions

In this paper, a class of linear hybrid systems have been analyzed from the point of view concerning their stability.

We have considered linear stochastic hybrid systems with a special case of matrices creating the solvable Lie algebra. We have analyzed hybrid systems parametrically excited by a white noise consisted of both stable and unstable structures described by Itô stochastic differential equations. Two cases of the switching rules have been studied: Markovian and any switching. The obtained sufficient conditions of the almost sure and the p-mean stability have been illustrated by examples and simulations. It has been shown that the Lie algebra sufficient conditions of stability can not be found for oscillator hybrid systems, and only in some cases of second order systems sufficient conditions of practical stability are satisfied.

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Metody Lie algebr w badaniu stabilności stochastycznych układów hybrydowych

Streszczenie

W pracy rozważony został problem stabilności klasy liniowych stochastycznych układów hybrydowych ze szczególną strukturą macierzy i multiplikatywnym szumem. Znalezione zostały warunki wystarczające dla eksponencjalnej p-średniej stabilności i eksponencjalnej stabilności prawie na pewno dla klasy stochastycznych liniowych układów hybrydowych z Markowskim przełączaniem. Dodatkowo podane zostały warunki wystarczające dla eksponencjalnej stabilności średnio-kwadratowej dla klasy stochastycznych liniowych układów hybrydowych z dowolnym przełączaniem spełniających warunki Lie algebry. Otrzymane wyniki zilustrowane zostały przykładami i symulacjami.

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