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MATHEMATICAL MODELLING OF A DISC WEAKENED BY AN ECCENTRIC CIRCULAR HOLE

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The task of this paper is identification of stresses in a homogeneous isotropic disc weakened with an eccentric circular hole which is loaded by pressure in the internal contour of the hole. By application of a complex variable method, the mathematical model that allows complete analytical solution of stresses of the disc is formed. The methodology can be applied for the solution of any disc weakened with an eccentric circular hole. The comparative analysis has shown a high accuracy of analytically obtained results with FEM results obtained by calculations in ANSYS 12 software package. The application of the results of this paper is of great importance for quality design and optimization of thin-walled structures of disc type weakened by a circular hole.

Key words: modelling, disc, eccentric hole

1. Introduction

The papers by Timoshenko and Goodier (1951), Timoshenko and Woinowsky-Kreiger (1959) provide methods for solving some typical problems of the theory of elasticity. These methods are applied broadly to some practical engineering problems, particularly those with lower approximation, and give reasonably accurate results. However, when there is a case where the problem solving significantly differs from the given theoretical model, it is necessary to make quite a rough approximation, which is the reason why the obtained results are not sufficiently accurate. A particular problem is the calculation of machine elements and structures with sudden change of geometry, the existence of openings, sharp corners, etc. At these places, there is the phenomenon of stress concentration which occupies a significant place in the study of problems of the theory of elasticity. The data from technical practice show that frequent breakdowns and accidents happen on such machines or structural elements and that the consequences are often tragic human victims and enormous material damage. In many cases, the cause is an incomplete and inaccurate identification of stresses and strains.

In studies of this phenomenon, in most cases, the thin plate weakened by a circular hole loaded with a certain type of loading is analysed (Bakhshandeh *et al.*, 2008; Bizic and Petrovic, 2011; Bojic *et al.*, 2010; Chandrashekhara and Muthanna, 1978; Chen and Archer, 1989; Mizushima and Hamada, 1983; Troyani *et al.*, 2002; Wang, 2004; Yang *et al.*, 2010; Zhang and Shen, 2011). Also, the analysis may include determination of stresses in a thin plate in the case when there are two adjacent circular holes or more holes (Arshadnejad *et al.*, 2009; Chen *et al.*, 2000;

Kratochvil and Becker, 2011; Wu and Markenscoff, 1996). Such cases are very frequently present in construction of railway and road vehicles, vessels, aircrafts, civil engineering machinery, mining and transportation machinery, cranes, tooling machines, steel structures and many others. The holes exist on them for various reasons such as construction requirements, optimization of the structure, reduction of self-weight, esthetic reasons, and so on. In all cases, the problem is the accurate determination of stresses in some points of the element in loaded plates weakened by circular holes, in which the stress concentration is present. One of directions for solving this problem, whose theoretical basis are defined in Bower (2010), Lu (1995), Muskhelishvili (1963), Savin (1961), is based on the use of complex functions and complex analysis (Mitrinovic, 1981). The basic equations of the theory of elasticity and stress functions are expressed in a complex form, and for their solution the most widely used method is the method of conformal mapping. By applying the complex variable method, it is possible to determine theoretically the exact stress state of the observed element in a plate weakened by a hole (Huan-chun *et al.*, 1987; Simha and Mohapatra, 1998). In the design and calculation phase of the mentioned machinery and structures, this allows very accurate theoretical determination of stresses, which are impossible to be found by conventional procedures. The papers by Batra and Nie (2010). Radi and Strozzi (2009) deal with the analysis of mechanical elements similar to the disk weakened with an eccentric circular hole. In line with these researches, a very interesting problem is mathematical modelling and identification of stresses in a disc weakened with an eccentric circular hole when it is loaded by the pressure in the internal contour of the hole. This was motivation for the research published in this paper, where the mathematical model that allows analytical solution of stresses is obtained by using the complex variable method (CVM). In order to verify analytically the obtained results, the stresses are also determined by the finite element method (FEM).

2. Theoretical formulation

For obtaining an analytical solution of the problem of stresses in a disc weakened with an eccentric circular hole, it is necessary to start with the theoretical formulation of the plane stress condition. If the thin plate is loaded by forces that are evenly spaced along its thickness and act in parallel to its base, there is a plane stress condition. The plane stress condition is defined by the following four groups of equations (Timoshenko and Goodier, 1951).

The first group of equations link the stresses and volume forces

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \tag{2.1}$$

where: σ_x, σ_y – components of normal stresses, τ_{xy} – shear stress, X, Y – components of volume forces.

The second group of equations is the relation between stress and strain

$$\sigma_x = \lambda\overline{\varepsilon} + 2\mu\frac{\partial u}{\partial x} \qquad \sigma_y = \lambda\overline{\varepsilon} + 2\mu\frac{\partial v}{\partial y} \qquad \tau_{xy} = \mu\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) \tag{2.2}$$

where: $\overline{\varepsilon}$ – surface deformation, λ, μ – Lame's constants, that are

$$\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

and E – modulus of elasticity, ν – Poisson's ratio, u, v – displacements in the direction of the coordinate axes x and y, respectively.

The third group of equations are compatibility conditions, where solutions have a unique ID only when they satisfy the conditions of compatibility

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{2.3}$$

where: ε_x, ε_y – deformational components, $\gamma_{xy} = (\partial u/\partial y) + (\partial v)/(\partial x)$ – slip component.

Finally, the fourth group of equation are boundary conditions

$$X_n = l\sigma_x + m\tau_{xy} \qquad\qquad Y_n = l\tau_{xy} + m\sigma_y \tag{2.4}$$

where: X_n, Y_n – components of the vector of external forces, $l = \cos(n, x)$, $m = \cos(n, y)$ – direction cosine, n – vector of the external normal to the contour.

Therefore, the plane stress condition is completely defined by equations (2.1)-(2.4). In one point with the stresses σ_x, σ_y and τ_{xy}, it is always possible to find a coordinate system in which the normal stresses have extreme values and in which the shear stresses equal zero. These stresses are named the principal stresses and are defined by the following equations

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\Big(\frac{\sigma_x - \sigma_y}{2}\Big)^2 + \tau_{xy}^2} \tag{2.5}$$

2.1. Plane stress condition expressed through complex potentials

The problem solving in the plane stress condition is reduced to finding a stress function that will uniquely determine stresses and deformations, while equations (2.1)-(2.4) will match with it. Determination of stresses in the disc weakened by an eccentric circular hole is this type of problem in which the volume forces can be neglected. As a consequence, there is always a function $U(x, y)$ through which it is possible to express the stresses by the following expressions

$$\sigma_x = \frac{\partial^2 U(x, y)}{\partial^2 y} \qquad\qquad \sigma_y = \frac{\partial^2 U(x, y)}{\partial^2 x} \qquad\qquad \tau_{xy} = -\frac{\partial^2 U(x, y)}{\partial x \partial y} \tag{2.6}$$

The function $U(x, y)$ is called the stress function and it has to match with the following biharmonic equation

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0 \tag{2.7}$$

Solving equation (2.7) and determining the stress function in real form is often very complex, and for a number of problems is practically impossible. One method of solving this problem is the transition into the complex area and solution of the problem in a complex form. The stress function, which is a function of two independent variables x and y, is expressed through two functions of one complex variable. In this way, the problem of solving a single function of two independent variables reduces the problem of two complex functions of one independent variable. These complex functions are $\phi(z)$ and $\psi(z)$, and they are called complex potentials.

Equations of the plane stress condition expressed trough complex potentials are (Muskhelishvili, 1963)

$$\begin{aligned} &\sigma_x + \sigma_y = 2[\phi(z) + \overline{\phi}(z)] = 4\,\mathrm{Re}\,\phi(z) \\ &\sigma_y - \sigma_x + 2\mathrm{i}\tau_{xy} = 2[\overline{z}\phi'(z) + \psi(z)] \end{aligned} \tag{2.8}$$

where: Re – real part, $\overline{z}$ – conjugated complex number.

The complex potentials are determined through the conditions made at the contour. In solving the problem of determining the stresses of the disc weakened by an eccentric circular hole, several methods for determining the complex potentials can be applied. In this paper, the method of power series is applied, but before that the conformal mapping is defined.

2.2. Conformal mapping

If $z = x + \mathrm{i}y$ and $\zeta = \zeta + \mathrm{i}\eta$ are two complex variables that are linked with the relation $z = \omega(\zeta)$, where $\omega(\zeta)$ is an unambiguous analytical function in the area of Σ in the plane area of change ζ, then each mapping (Fig. 1) that is applied by using these functions in which values of the angles are preserved, is called a conformal mapping (Muskhelishvili, 1963).

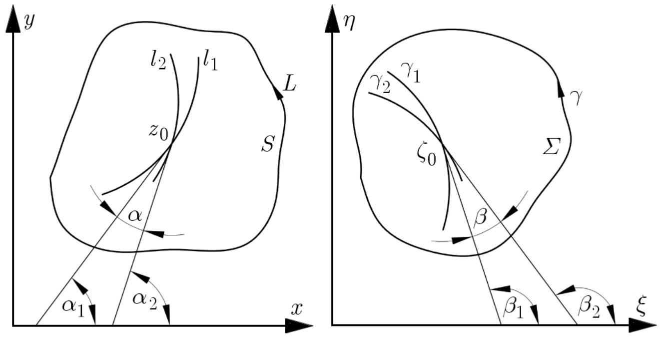

Fig. 1. Conformal mapping

In other words, the task is to find a function for the mapping in which the angle between the two curves in the plane z will be copied without changes in the angle between the corresponding curves in the plane ζ ($\alpha = \beta$). There are advanced methods for formation of the conformal mapping function $\omega(\zeta)$. In this paper, the ready-made function of conformal mapping is used.

2.3. Stresses in a mirrored area

Some area S in the plane z is mapped into a circular area Σ in the plane ζ (Fig. 1). In a new field Σ in the plane ζ, the polar coordinates θ and ρ are introduced by using the following relation

$$\zeta = \rho \mathrm{e}^{\mathrm{i}\theta} \tag{2.9}$$

The circles $\rho =$ const and $\theta =$ const of the field Σ in the plane ζ match curves which are also marked with $\rho =$ const and $\theta =$ const. The mapping is done by using the following analytical function

$$x + \mathrm{i}y = \omega(\rho \mathrm{e}^{\mathrm{i}\theta}) \tag{2.10}$$

So, the lines $\rho =$ const and $\theta =$ const are coordinate lines in the mirrored area Σ, and in the plane z they intersect at right angles. If through the point on the plane z pass the curves $\rho =$ const and $\theta =$ const (Fig. 2), which are mutually orthogonal, then the angle between the tangent to the curve $\theta =$ const in the direction of increasing ρ and x axis can be marked with α.

The link between the stress ratio in Cartesian and polar coordinates is as follows:

$$\begin{aligned} &\sigma_\rho + \sigma_\theta = \sigma_x + \sigma_y \\ &\sigma_\theta - \sigma_\rho + 2\mathrm{i}\tau_{\rho\theta} = \mathrm{e}^{2\mathrm{i}\alpha}(\sigma_y - \sigma_x + 2\mathrm{i}\tau_{xy}) \end{aligned} \tag{2.11}$$

where: σ_ρ – normal component of stress on the curve $\rho =$ const, σ_θ – normal component of stress on the curve $\theta =$ const, $\tau_{\rho\theta}$ – tangential component of stress on both curves.

The stresses that are expressed in the polar coordinate system through the complex potential in the new mirrored area are defined by the expressions

$$\begin{aligned} &\sigma_\rho + \sigma_\theta = 4\,\mathrm{Re}\,\phi(z) \\ &\sigma_\theta - \sigma_\rho + 2\mathrm{i}\tau_{\rho\theta} = \frac{2\zeta^2}{\rho^2\overline{\omega}'(\zeta)}[\overline{\omega}(\zeta)\phi'(\zeta) + \omega'(\zeta)\psi(\zeta)] \end{aligned} \tag{2.12}$$

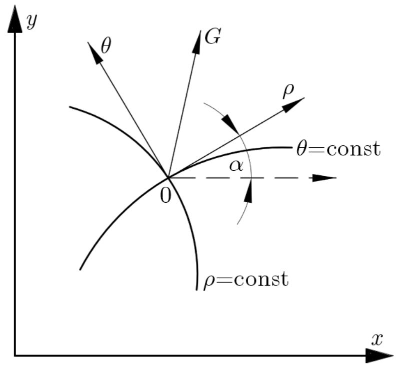

Fig. 2. The link between stress in Cartesian and polar coordinates in the mirrored area

3. Determination of complex potentials for a disc weakened with an eccentric circular hole

The disc is limited by two circles whose centers are shifted by the eccentricity e, as shown in Fig. 3.

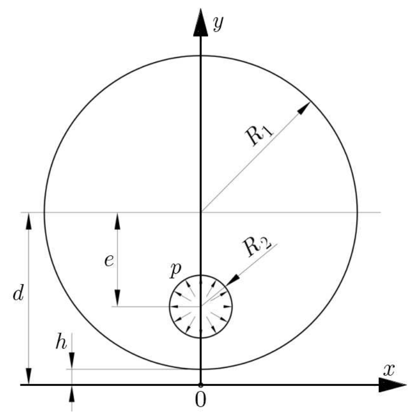

Fig. 3. The disc weakened by an eccentric circular hole

The condition is that the main moment and the main vector on the contours equal zero. The function of conformal mapping by which the disc mirrors on a circular ring is

$$z = \omega(\zeta) = c\frac{\zeta + 1}{\zeta - 1} \tag{3.1}$$

where: c – real constant determined from the expression

$$c = \frac{\sqrt{\mathrm{e}^4 + R_1^4 + R_2^4 - 2(R_1R_2)^2 - 2\mathrm{e}^2(R_1^2 + R_2^2)}}{2e} \tag{3.2}$$

The outer contour of the disc of radius R_1 is mapped to the inner contour of the circular ring of radius ρ_0, according to the equation

$$\rho_0 = \frac{c + \sqrt{c^2 + R_1^2}}{R_1} \tag{3.3}$$

The inner contour of the disc of radius R_2 is mapped to the outer contour of the circular ring of radius R_1, according to the equation

$$\rho_1 = \frac{c + \sqrt{c^2 + R_2^2}}{R_2} \tag{3.4}$$

The conformal mapping is defined by the following relation

$$x^2 + \Big(y - c\frac{\zeta\overline{\zeta}+1}{\zeta\overline{\zeta}-1}\Big)^2 = \Big|\frac{2c\sqrt{\zeta\overline{\zeta}}}{\overline{\zeta}\zeta-1}\Big|^2 \tag{3.5}$$

From relation (3.5), it is noticed that the concentric circles $\zeta\overline{\zeta} = \rho^2$ correspond to the circles in S area with the center on the y axis shifted for the value d of radius r. The two parameters (d and r) are concluded from relation (3.5) as follows

$$d = c\frac{\rho^2+1}{\rho^2-1} \qquad r = \frac{2c\rho}{\rho^2-1} \tag{3.6}$$

The expressions for the stresses in polar coordinates are obtained by solving the system of equations (2.12)

$$\begin{aligned}
&\sigma_\theta = 2\,\mathrm{Re}\,\phi(\zeta) + \frac{1}{2}\,\mathrm{Re}\,\Omega(\zeta) \qquad \sigma_\rho = 2\,\mathrm{Re}\,\phi(\zeta) - \frac{1}{2}\,\mathrm{Re}\,\Omega(\zeta)\\
&\tau_{\rho\theta} = \frac{1}{2}\,\mathrm{Im}\,\Omega(\zeta)
\end{aligned} \tag{3.7}$$

In equations (3.7), $\Omega(\zeta)$ is a complex function that is defined by the following equation

$$\Omega(\zeta) = \frac{2\zeta^2}{\rho^2\overline{\omega}'(\zeta)}[\overline{\omega}(\zeta)\phi'(\zeta) + \omega'(\zeta)\psi(\zeta)] \tag{3.8}$$

The complex potentials are

$$\begin{aligned}
&\phi(\zeta) = C_0 + (\zeta-1)^2\Big(C_1 + \frac{C_2}{\zeta^2}\Big)\\
&\psi(\zeta) = \frac{\rho_0^2(\zeta-1)^2}{(\rho_0^2-\zeta)^2}\Big[\phi(\zeta) + C_0 + \Big(\frac{\rho_0^2}{\zeta}-1\Big)^2\Big(C_1 + \frac{C_2\zeta^2}{\rho_0^4}\Big)\Big] + \frac{(\rho_0^2+\zeta)(\zeta-1)^2}{2(\rho_0^2-\zeta)}\phi'(\zeta)
\end{aligned} \tag{3.9}$$

The real constants are determined in the following way

$$\begin{aligned}
&C_0 = \frac{1}{2}C_1\Big[2(\rho_0\rho_1)^2 + 2(\rho_0^2+\rho_1^2) - 2(\rho_0^2+\rho_1^2)^2 - \frac{p\rho_1^2}{2(\rho_1^2-\rho_0^2)}\Big]\\
&C_1 = -\frac{p\rho_1^2\rho_0^2}{(\rho_1^2-\rho_0^2)(\rho_1^2\rho_0^2)(\rho_0^2+\rho_1^2) - 4(\rho_0\rho_1)^2 + \rho_0^2\rho_1^2}\\
&C_2 = -C_1(\rho_0\rho_1)^2
\end{aligned} \tag{3.10}$$

In relations (3.10), p is the pressure that operates in the inner part of the contour of the eccentric hole.

4. Determination of stress using CVM

The previously defined theoretical equations were applied to the concrete example of the disc of thickness $h = 5\,\mathrm{mm}$, radius $R_1 = 50\,\mathrm{mm}$, weakened by an eccentric circular hole of eccentricity $e = 30\,\mathrm{mm}$, radius $R_2 = 10\,\mathrm{mm}$, loaded on the inside contour with the pressure $p = 1.0\,\mathrm{kN/cm^2}$. By variation of the polar coordinates ρ and θ in the mirrored area, it is possible to determine stress at any point of the disc. In this paper using the CVM, the specific numerical values of stresses σ_ρ and σ_θ and principal stresses σ_1 and σ_2 on the outer contour of the disc and at the contour of the eccentric hole, as well as the stresses σ_x and σ_y at the intersection along to the vertical axis of symmetry of the disc are determined. The numerical values of stresses σ_ρ and σ_θ are determined in function of the coordinates ρ_1 and ρ_0 in the outer contour of the disc and the contour of the eccentric hole in the mapped field calculated according to equations (3.3) and (3.4), for 36 points that are defined by the angle θ that ranges from 0°-360° with a step of 10°. The numerical values of stresses σ_x and σ_y are calculated in 22 points on the interval from ρ_0 to ρ_1 ($y = 0.5\,\mathrm{cm}$ to $y = 10.5\,\mathrm{cm}$) with step

$$\Delta = \frac{\rho_1 - \rho_0}{10} \tag{4.1}$$

Therefore, the stresses σ_x and σ_y are determined for values of the coordinate ρ with the step Δ according to equation (4.1) and for the angles $\theta = 0°$ and $\theta = 180°$. This means that the numerical values of stresses σ_x and σ_y are determined for the intersection that matches the y axis.

Based on the previously defined equations, an algorithm for numerical solution of the problem is created. The fact that is worth mentioning is that the programming is done with complex sizes. Based on such obtained numerical values of stresses, appropriate diagrams of stresses change at the outer contour of the disc, and at the contour of the eccentric hole are created as given in Section 5. Analysis of the data for the stress value σ_ρ at the contour of the eccentric hole obtained by using theoretical models has shown that the stresses are constant at any point of the eccentric hole, which is logical because the first boundary condition was made on that contour which implied that it was loaded with a constant pressure $p = 1.0\,\mathrm{kN/cm^2}$. Also, at the outer contour of the disc, the values of stress σ_ρ equal zero. The values of principal stresses σ_1 and σ_2 at any point of the disc with the eccentric circular hole are obtained when the values of stresses σ_ρ, σ_θ and $\tau_{\rho\theta}$ are replaced in the following equations

$$\sigma_1 = \frac{\sigma_\rho + \sigma_\theta}{2} + \sqrt{\Big(\frac{\sigma_\rho - \sigma_\theta}{2}\Big)^2 + \tau_{\rho\theta}^2} \qquad \sigma_2 = \frac{\sigma_\rho + \sigma_\theta}{2} - \sqrt{\Big(\frac{\sigma_\rho - \sigma_\theta}{2}\Big)^2 + \tau_{\rho\theta}^2} \tag{4.2}$$

At the outer contour of the disc, the values of stress σ_2 equal zero at any point. The stresses values σ_x, σ_y and τ_{xy} at any point of the disc with the eccentric hole can be obtained when σ_ρ, σ_θ and $\tau_{\rho\theta}$ are replaced in the following equations

$$\begin{aligned}
\sigma_x &= \frac{\sigma_\rho + \sigma_\theta}{2} + \frac{\sigma_\rho - \sigma_\theta}{2}\cos 2\theta - \tau_{\rho\theta}\sin 2\theta \\
\sigma_y &= \frac{\sigma_\rho + \sigma_\theta}{2} - \frac{\sigma_\rho - \sigma_\theta}{2}\cos 2\theta + \tau_{\rho\theta}\sin 2\theta \\
\tau_{xy} &= \frac{\sigma_\rho - \sigma_\theta}{2}\sin 2\theta + \tau_{\rho\theta}\cos 2\theta
\end{aligned} \tag{4.3}$$

5. Determination of stress using FEM

The FEM is based on physical discretization of the considered continuum with elements of finite dimensions and simple shape. In the study, a numerical model was created. It was a steel disc weakened by an eccentric circular hole whose dimensions and load are identical to the dimensions and load used in the previous analysis by the CVM. Also, a homogeneous isotropic disc has been considered, while the material of the disc is steel with modulus of elasticity $E = 21000\,\mathrm{kN/cm^2}$, and Poisson's ratio $\nu = 0.33$. The calculation was carried out by using ANSYS 12 software package and the finite elements such as thin plates were applied. The FEM model consists of 10492 nodes and 1816 finite elements. It is important to note that the input data for the spatial discretization and mesh generation were not previously adjusted, but a mesh that is generated automatically by the program ANSYS 12 was used. The disc was loaded with internal

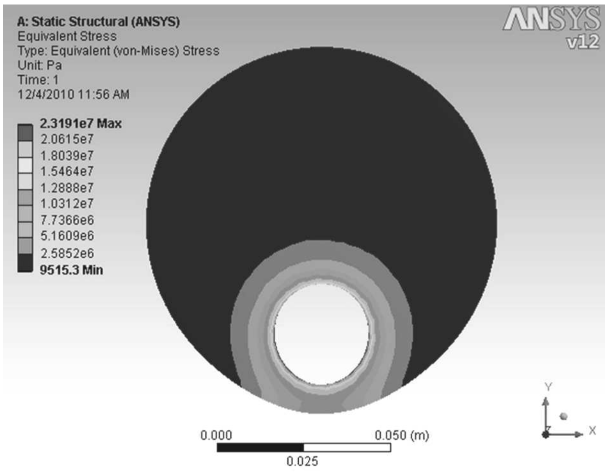

Fig. 4. Equivalent stress

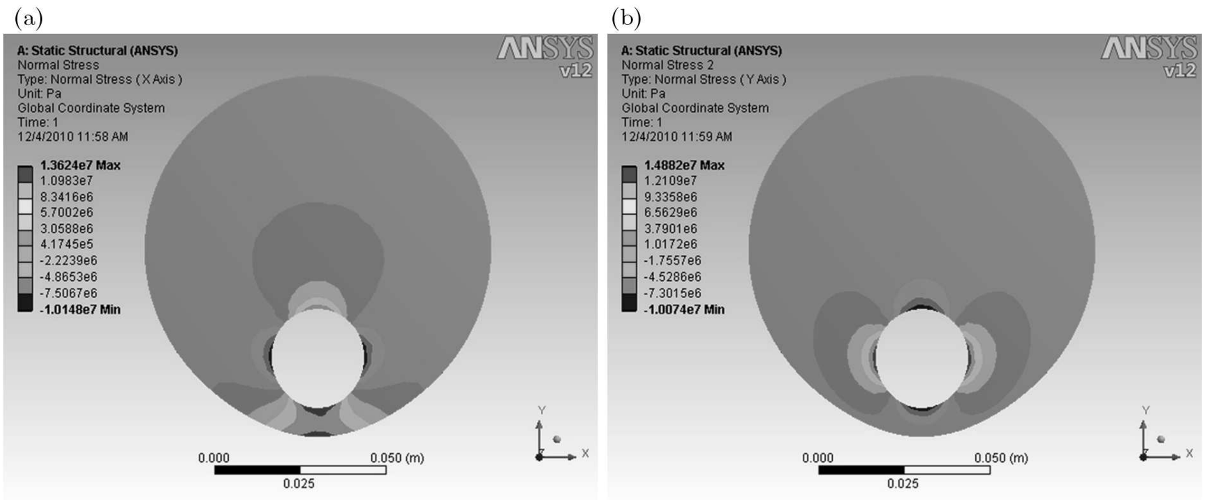

Fig. 5. Normal stress σ_x (a) and σ_y (b)

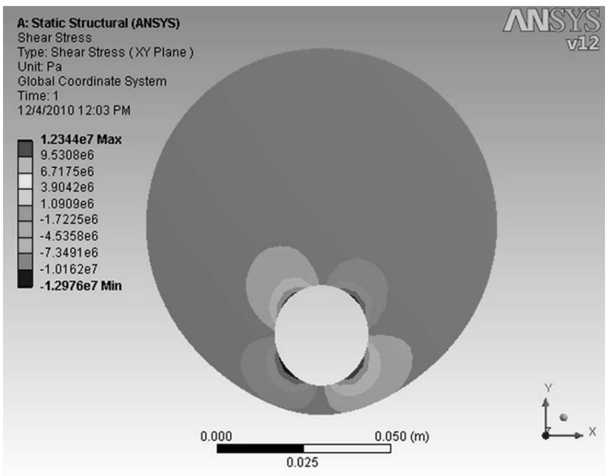

Fig. 6. Shear stress τ_{xy}

pressure equal to $p = 1.0\,\mathrm{kN/cm^2}$, acting within the internal contours of the eccentric hole. As a consequence of the given discretization and load, the corresponding stresses in the disc were obtained as shown in Figs. 4-6.

In order to compare the results, the numerical stresses obtained by the FEM were read in specific points of the disc. Locations of these points at the outer contour of the disc and at the contour of the eccentric circular hole were the same as 36 points that were defined by the angle θ ranging from 0°-360°, with a step of 10°, defined in Section 3. Locations of points along the y axis for reading the stresses values were also the same as those in Section 3.

The principal stresses σ_1 and σ_2 were determined according to equation (2.5), and also the diagrams of principal stresses at the outer contour of the disc and at the contour of the eccentric hole were formed. Also, the diagrams of the stresses σ_x and σ_y for the intersection that matches the y axis were formed.

6. Comparison of results obtained by CVM and FEM

The values of the principal stress σ_1 at the outer contour of the disc (Fig. 7a) coincide only at the maximum stress values for the angle $\theta = 180°$ ($y = 0.5\,\mathrm{cm}$). At other points of the outer contour, the stresses differ by a greater extent. The trend of stress distribution in both cases is approximately the same. By the FEM, some very small and negligible values of the principal stress σ_2 at the outer contour of the disc (Fig. 7b) are obtained, while according to theoretically obtained equations (4.2) derived by the CVM, such stress is equal to zero. The maximum values of the principal stress σ_1 at the contour of the eccentric hole (Fig. 8a) calculated by the CVM are obtained for the angles $\theta = 100°$ and $\theta = 260°$, while by using the FEM are obtained for the angles $\theta = 110°$ and $\theta = 240°$. The deviations are very small and, in this case, the trend of stress distribution is also approximately the same. The values of the principal stress σ_2 at the contour of the eccentric hole (Fig. 8b) calculated by the CVM are constant $\sigma_2 = -1.0\,\mathrm{kN/cm^2}$, which is logical because there is constant pressure of the same intensity inside the contour of the eccentric hole. In comparison with the FEM, these values are somewhat different, where the deviations range from -5.9% to 2.1%. As for the values of stress components σ_x and σ_y at the intersection which coincides with the y axis, the comparative analysis has shown that the values largely overlap, and that overlapping ranges below 2% (Figs. 9a and 9b). Therefore, the stresses obtained by the CVM and FEM are very similar in values and trend of distribution.

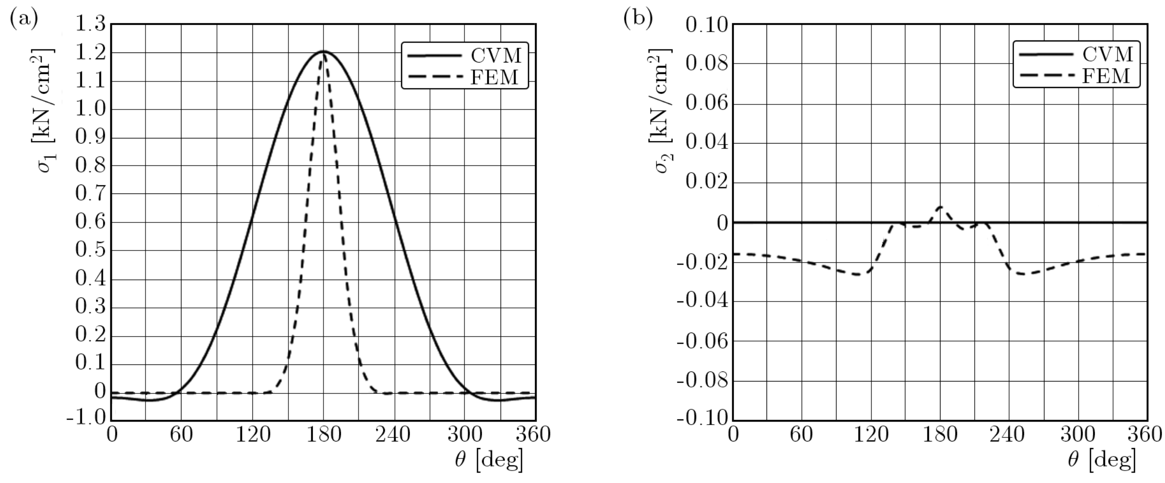

Fig. 7. Comparative diagram of the stress σ_1 (a) and σ_2 (b) at the outer contour of the disc

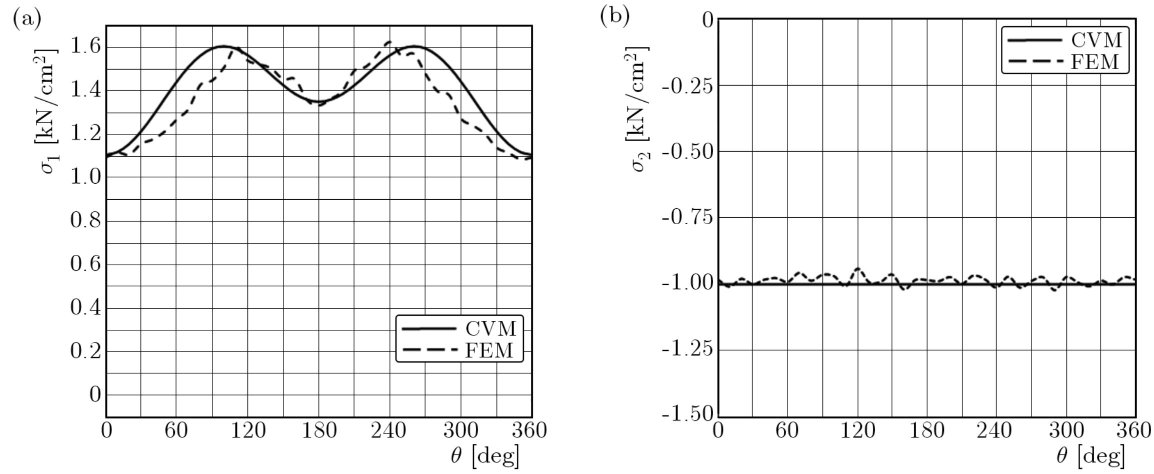

Fig. 8. Comparative diagram of the stress σ_1 (a) and σ_2 (b) at the contour of the eccentric hole

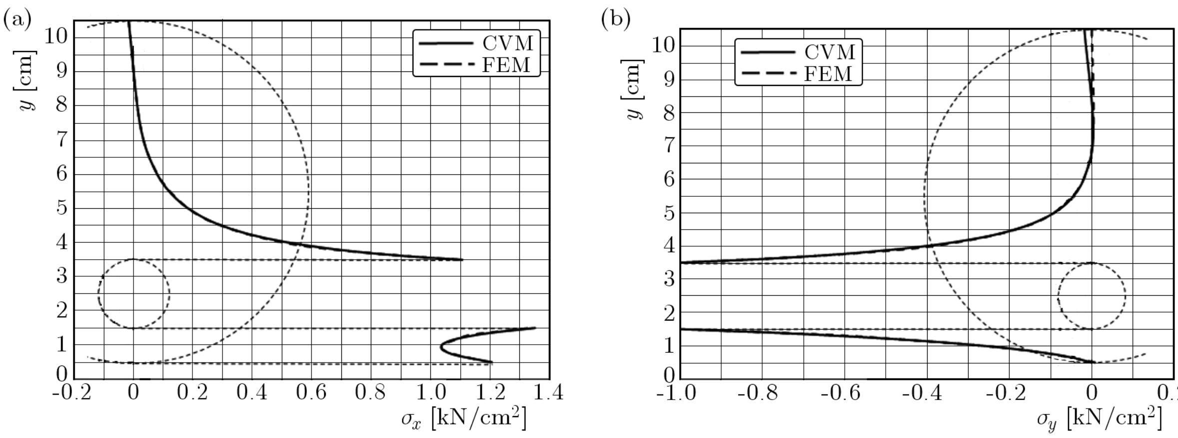

Fig. 9. Comparative diagram of the stress σ_x (a) and σ_y (b) at the intersection of the y axis

7. Conclusion

The task of this paper is stress analysis in a homogeneous isotropic disc weakened with an eccentric circular hole which is loaded by pressure in the internal contour of the hole. The complex variable method (CVM) was applied which is based on the application of Muskhelishvili's complex variable function technique. The formed mathematical model allows complete analytical solution of the stress state of the disc, especially the contour of the hole where the stress concentration is present. The technique was applied to the specific example of the disc, for which the concrete numerical values of stresses were determined. Verification of the obtained results was carried out by the finite element method (FEM) using the software package ANSYS 12. Comparative analysis has shown that stresses obtained by the CVM and FEM are very similar in values and trend of distribution, which confirms the correctness of the established mathematical model. The application of the results of this paper is of great importance for quality design and optimization of thin-walled structures of disc type weakened by a circular hole.

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Modelowanie matematyczne dysku osłabionego mimośrodowo umiejscowionym otworem kołowym

Streszczenie

Zadaniem podjętym w artykule jest identyfikacja stanu naprężeń w jednorodnym, izotropowym dysku osłabionym mimośrodowo umiejscowionym wycięciem kołowym, obciążonym wzdłuż brzegu wewnętrznym naciskiem. Stosując metodę zmiennej zespolonej, w pracy sformułowano matematyczny model układu, który pozwolił na uzyskanie w pełni analitycznego opisu rozkładu naprężeń w dysku. Zastosowana metodologia, zdaniem autorów, może być użyta dla dowolnego dysku z mimośrodowym otworem o kształcie koła. W pracy przeprowadzono ponadto analizę porównawczą z wynikami uzyskanymi numerycznie za pomocą metody elementów skończonych z wykorzystaniem pakietu ANSYS 12. Otrzymane rezultaty badań mogą mieć duże znaczenie praktyczne z punktu widzenia jakości projektowania i optymalizacji cienkościennych konstrukcji zawierających elementy strukturalne z kołowym wycięciem.

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