MECHANIKA
TEORETYCZNA
I STOSOWANA
2/3, 21 (1983)

ON CERTAIN INEQUALITIES IN THE LINEAR SHELL PROBLEMS

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1. In the paper the weak formulations of the mixed boundary value problems of the linear elastostatics with constraints are presented. By constraints we mean here the known restrictions, imposed on the displacement and stress fields, represented by the closed convex sets in suitable linear spaces. It has been shown that the forementioned problems are equivalent to the saddle-point problems for the Reissner functional. Interrelations between different solutions to the problems under considerations are obtained. For the formulations of the shell boundary value problems, some special cases of constraints are proposed and detailed.

2. Now we are to state the three following problems:
A — the mixed boundary value problem of the linear elasticity,
B — the problem with constraints for deformations only,
C — the problem with constraints for both deformation and stresses.

Let Ω be the open domain in R^3 with boundary $\partial\Omega = \bar{\Gamma}_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let $H^1(\Omega)$ be the Sobolev space of square integrable functions with square integrable first derivatives on Ω. Define

$$V = (H^1(\Omega))^3, \quad U = \{u \in V / u = 0 \quad \text{on} \quad \Gamma_0\}$$

$$W = \{\varepsilon / \varepsilon = (\varepsilon_{\alpha\beta}), \quad \varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha}, \quad \varepsilon_{\alpha\beta} \in L^2(\Omega), \quad \alpha, \beta = 1, 2, 3\},$$

Moreover, let U^*, W^* be the duals of U and W, respectively. Let us define the "deformation operator" $L: U \to W$ putting

$$(Lu)_{\alpha\beta} = \frac{1}{2}(p^k_{,\alpha}, u_{k,\beta} + p^k_{,\beta} u_{k,\alpha}),$$

where $p: \Omega \to R^3$ is the known smooth invertible mapping. Let the mapping $C: W \to W^*$ represents the strain-stress relation, being determined by the tensor of elastic moduli $C_{\alpha\beta\gamma\delta}$ satisfying the known assumptions

$$C_{\alpha\beta\gamma\delta} \in L^\infty(\Omega), \quad C_{\alpha\beta\gamma\delta} = C_{\alpha\beta\delta\gamma} = C_{\gamma\delta\alpha\beta},$$

$$\exists c_0 > 0, \quad C_{\alpha\beta\gamma\delta}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta} > c_0\,\varepsilon_{\alpha\beta}\,\varepsilon_{\alpha\beta}, \quad \forall \varepsilon \in W.$$

The duality pairing on $W^* \times W$ and on $U^* \times U$ will be denoted by $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$, respectively. The inner product on W^* is denoted by $[\cdot, \cdot]$. Let

$$(\sigma, \varepsilon) = \int_{\Omega} \sigma_{\alpha\beta} \varepsilon_{\alpha\beta} dv,$$

$$[\sigma, \tau] = \int_{\Omega} \sigma_{\alpha\beta} D_{\alpha\beta\gamma\delta} \tau_{\gamma\delta} dv$$

where $D_{\alpha\beta\gamma\delta}$ represent the operator which is inverse to C. The adjoint L^* of the operator L is defined via Gauss-Green's formula by

$$(\sigma, Lu) = \langle L^*\sigma, u\rangle = -\int_{\Omega} (p^k_{,\alpha}, \sigma_{\alpha\beta})_{,\beta} u_k dv + \int_{\Gamma_1} p^k_{,\alpha} \sigma_{\alpha\beta} n_\beta u_k ds.$$

The functional $F \in U^*$ is assumed to be given by body forces $f_k \in L^2(\Omega)$ and surface tractions $p_k \in L^2(\Gamma_1)$ and has the form

$$\langle F, u\rangle = \int_{\Omega} f_k u_k dv + \int_{\Gamma_1} p_k u_k ds.$$

The mixed boundary value problem of the linear elastostatics will be stated as follows:

Problem A. Find (u, σ) in $U \times W^*$ such that

$$\langle L^*\sigma - F, v - u\rangle \geqslant 0, \quad \forall v \in U, \tag{2.1}$$

$$[\sigma - CLu, \tau - \sigma] \geqslant 0, \quad \forall \tau \in W. \tag{2.2}$$

Variational inequalities in this problem are equivalent to the equations

$$L^*\sigma = F, \quad \sigma = CLu, \tag{2.3}$$

which constitute the stationary conditions for the Reissner functional

$$\mathscr{F}(v, \tau) = -\frac{1}{2}||\tau||^2 + \langle L^*\tau - F, v\rangle. \tag{2.4}$$

Let $K \subset U$ and $\Sigma \subset W^*$ be nonempty closed convex sets. Problem with constraints for deformations will be formulated as

Problem B. Find (u, σ) in $K \times W^*$ such that

$$\langle L^*\sigma - F, v - u\rangle \geqslant 0, \quad \forall \tau \in K, \tag{2.5}$$

$$[\sigma - CLu, \tau - \sigma] \geqslant 0, \quad \forall \tau \in W^*. \tag{2.6}$$

Inequality (2.6) is equivalent to the equation

$$\sigma = CLu. \tag{2.7}$$

By problem with constraints for deformation and stresses we shall mean

Problem C. Find (u, σ) in $K \times \Sigma$ such that

$$\langle L^*\sigma - F, v - u\rangle \geqslant 0, \quad \forall v \in K, \tag{2.8}$$

$$[\sigma - CLu, \tau - \sigma] \geqslant 0, \quad \forall \tau \in \Sigma. \tag{2.9}$$

3. Now we are to observe that the problems A, B, C are equivalent to the suitable saddle points problems for the Reissner functional (2.4). It means that the pairs $(u_A, \sigma_A) \in U \times W^*$, $(u_B, \sigma_B) \in K \times W^*$, $(u_C, \sigma_C) \in K \times \Sigma$ are solutions of problems A, B, C, respectively if they are the saddle points of the Reissner functional defined on the sets

$U\times W^*$, $K\times W^*$, $K\times \Sigma$, respectively. If follows that the following inequalities have to hold

$$\mathscr{F}(u_A, \tau) \leqslant \mathscr{F}(u_A, \sigma_A) \leqslant \mathscr{F}(v, \sigma_A), \quad \forall (v, \tau) \in U\times W^*, \tag{3.1}$$

$$\mathscr{F}(u_B, \tau) \leqslant \mathscr{F}(u_B, \sigma_B) \leqslant \mathscr{F}(v, \sigma_B), \quad \forall (v, \tau) \in K\times W^*, \tag{3.2}$$

$$\mathscr{F}(u_C, \tau) \leqslant \mathscr{F}(u_C, \sigma_C) \leqslant \mathscr{F}(v, \sigma_C), \quad \forall (v, \tau) \in K\times \Sigma. \tag{3.3}$$

To observe this fact it can be easily seen that the left hand sides of (3.1), (3.2), (3.3) are derived from the constitutive relations (2.2), (2.6) and (2.9), while the right hand sides are obtained from equilibrium conditions (2.1), (2.5) (2.8), respectively. The sufficient condition follows directly from the definition of the Reissner functional (2.4).

Since

$$\mathscr{F}(v, CLv)-\mathscr{F}(v, \tau) = \frac{1}{2}\, ||\tau-CLv||^2,$$

for $v\in U$ and $\tau\in W^*$, then $\mathscr{F}(v, CLv) \geqslant \mathscr{F}(v, \tau)$ for every $\tau \in W^*$ holds. Then from the right hand sides of (3.1), (3.2), (3.3) and after taking into acount (2.3), (2.7) we conclude that

$$\mathscr{F}(u_A, \sigma_A) = \min_{v\in U} \mathscr{F}(v, CLv),$$

$$\mathscr{F}(u_B, \sigma_B) = \min_{v\in K} \mathscr{F}(v, CLv),$$

$$\mathscr{F}(u_C, \sigma_C) \leqslant \min_{v\in K} \mathscr{F}(v, CLv).$$

4. Some relations between solutions of problems A, B, C will be now derived. Let us denote by P_Σ the orthogonal projection from W^* on Σ. For any $\sigma\in W^*$, projection $P_\Sigma\sigma$ is the best approximation to σ in the closed set Σ given by

$$||P_\Sigma\sigma-\sigma|| \leqslant \tau-\sigma||, \quad \forall\tau\in\Sigma.$$

$P_\Sigma\sigma$ satisfies the variational inequality

$$[P_\Sigma\sigma-\sigma, \tau-P_\Sigma\sigma] \geqslant 0, \quad \forall\tau\in\Sigma.$$

Therefore from (2.9) we have

$$\sigma_C = P_\Sigma CLu_C. \tag{4.1}$$

By virtue of the right hand side of (3.2) and taking into account (2.3) and (2.7) we conclude that

$$\sigma_B = P_{CLK}\sigma_A, \tag{4.2}$$

where CLK is the image of K under mapping CL. If $\sigma_A\in CLK$, then $\sigma_B=\sigma_A$, If $K\cap Ker\; CL \neq \emptyset$ then from inequalities (2.5) and (2.8) we have

$$||\sigma_B||^2 \leqslant \langle F, u_B\rangle = [\sigma_A, \sigma_B], \tag{4.3}$$

$$[\sigma_C, CLu_C] \leqslant \langle F, u_C\rangle = [\sigma_A, CLu_C]. \tag{4.4}$$

By virtue of (4.3), $||\sigma_B|| \leqslant ||\sigma_A||$ holds.

From the inequality on the left hand side of (3.2) we obtain

$$\mathscr{F}(u_B, \sigma_A) \leqslant \mathscr{F}(u_B, \sigma_B),$$

hence

$$\|\sigma_A\|^2 \geqslant 2\langle F, u_B\rangle - \|\sigma_B\|^2. \tag{4.5}$$

Putting $v = u_C$ in (2.5) and $v = u_B$ in (2.8) and combining the obtained inequalities, we arrive at

$$\langle L^*\sigma_C - L^*\sigma_B, u_B - u_C\rangle \geqslant 0,$$

which can be transformed to the form

$$\left\|\sigma_B - \frac{1}{2}(\sigma_C + CLu_C)\right\| \leqslant \|\sigma_C - CLu_C\|. \tag{4.6}$$

The direct consequence of inequality (4.6) and equation (4.1) leads to the corollary: If $CLK \subset \Sigma$ then $\sigma_B = \sigma_C$.

5. Now we are to give the example of constraints for deformation and stresses leading to the boundary value shell problems. To this aid assume that: $\Omega = \Pi \times (-h, h)$, $\Gamma_1 = \partial_1\Pi \times (-h, h) \cup \Pi \times \{-h\} \cup \Pi \times \{h\}$, $\Theta = (\Theta^1, \Theta^2) \in \Pi$, $\zeta \in (-h, h)$. Let us introduce linear spaces $\bar{V}$, $\bar{Y}$, $\mathring{Y}$ by means of:

$$\bar{V} = \{\bar{v}/\bar{v} = (\bar{v}^n), \quad \bar{v}^n \in H^1(\Pi), \quad \bar{v}^n = 0 \quad \text{on} \quad \partial_1\Pi, \quad n = 1, \ldots, N\},$$

$$\bar{Y} = \{\bar{\varepsilon}/\bar{\varepsilon} = (\bar{\varepsilon}_a), \quad \bar{\varepsilon}_a \in L^2(\Pi), \quad a = 1, \ldots, A\},$$

$$\mathring{Y} = \{\mathring{\varepsilon}/\mathring{\varepsilon} = (\mathring{\varepsilon}_i), \quad \mathring{\varepsilon}_i \in L^2(\Omega), \quad i = 1, \ldots, I\}.$$

Let $\bar{V}^*$. $\bar{Y}^*$, $\mathring{Y}^*$ stand for dual spaces of $\bar{V}$, $\bar{Y}$, $\mathring{Y}$, respectively. The elements of $\bar{Y}^*$ and $\mathring{Y}^*$ can be identified with the elements of $\bar{Y}$ and $\mathring{Y}$. Let us introduce the linear mapping $P: \bar{V} \to V$, putting

$$P(\bar{v}) = \gamma_n(X)\bar{v}^n, \tag{5.1}$$

where functions $\gamma_n: \Omega \to R^3$, $n = 1, \ldots, N$, are known, $\gamma_n \in (L^\infty(\Omega))^3$ The adjoint P^* of P is defined by the relation

$$\langle P^*(F), \bar{v}\rangle = \langle F, P(\bar{v})\rangle, \quad \forall \bar{v} \in \bar{V}.$$

Then for the functional $F \in V^*$ given by

$$\langle F, v\rangle = \int_\Omega fv\,dv + \int_{\Gamma_1} pv\,ds, \quad f \in (L^2(\Omega))^3, \quad p \in (L^2(\Gamma_1))^3,$$

we obtain

$$\langle P^*(F), \bar{v}\rangle = \int_\Pi \bar{f}_n\bar{v}\,ds + \int_{\partial_1\Pi} \bar{p}_n\bar{v}^n\,dl, \tag{5.2}$$

where

$$\bar{f}_n = \int_{-h}^{h} \gamma_n f\,d\zeta + [\gamma_n p]_{-h,h},$$

$$\bar{p}_n = \int_{-h}^{h} \gamma_n p\,d\zeta.$$

Let $\bar{K}$ be the closed, nonempty convex set in $\bar{V}$. If K in problem B is the image of $\bar{K}$ under mapping P, $K = P(\bar{K})$, then problem B is to be reduced to the following shell problem:

Find $\bar{u} \in \bar{V}$ such that

(5.3) $$\langle \bar{A}\bar{u} - \bar{F}, \bar{v} - \bar{u} \rangle \geqslant 0 \quad \forall \bar{v} \in \bar{K},$$

where

$$\bar{A} = P^*L^*CLP, \qquad \bar{F} = P^*F.$$

Let the constraints for stresses be introduced by the mapping $Q^*: \bar{Y}^* \times \mathring{Y}^* \to Y^*$

(5.4) $$Q^*(\bar{\sigma}, \mathring{\sigma}) = \mu\bar{\sigma} + \nu\mathring{\sigma},$$

where $\mu: \bar{Y}^* \to Y^*$, $\nu: \mathring{Y}^* \to Y^*$ are uniquely determined by the representations $\mu_a, \nu_i \in \in (L^\infty(\Omega))^6$.

The conjugate mapping $Q: Y \to \bar{Y} \times \mathring{Y}$ is given by the relation

$$\big(Q(\varepsilon), (\bar{\sigma}, \mathring{\sigma})\big) = \big(\varepsilon, Q^*(\bar{\sigma}, \mathring{\sigma})\big), \quad \forall (\bar{\sigma}, \mathring{\sigma}) \in \bar{Y}^* \times \mathring{Y}^*;$$

hence

$$\big(Q(\varepsilon), (\bar{\sigma}, \mathring{\sigma})\big) = (\mu^*\varepsilon, \bar{\sigma}) + (\nu^*\varepsilon, \mathring{\sigma}).$$

Putting now $K = P(\bar{K})$ and $\Sigma = Q^*(\bar{\Sigma} \times \mathring{Y}^*)$, where $\bar{\Sigma}$ is nonempty, closed, convex set in $\bar{Y}^*$, the problem C will be reduced to the following problem

Find $\bar{u} \in \bar{V}$ and $(\bar{\sigma}, \mathring{\sigma}) \in \bar{\Sigma} \times \mathring{Y}^*$ such that

(5.5) $$\langle P^*L^*Q^*(\bar{\sigma}, \mathring{\sigma}) - \bar{F}, \bar{v} - \bar{u} \rangle \geqslant 0, \quad \forall \bar{v} \in \bar{K},$$

(5.6) $$\big(QDQ^*(\bar{\sigma}, \mathring{\sigma}) - QLP\bar{u}, (\bar{\tau} - \bar{\sigma}, \mathring{\tau} - \mathring{\sigma})\big) \geqslant 0, \quad (\bar{\tau}, \mathring{\tau}) \in \bar{\Sigma} \times \mathring{Y}^*.$$

The inequality (5.6) yields

(5.7) $$(\mu^*DQ^*(\bar{\sigma}, \mathring{\sigma}) - \mu^*LP\bar{u}, \bar{\tau} - \bar{\sigma}) \geqslant 0 \quad \forall \bar{\tau} \in \bar{\Sigma},$$

(5.8) $$\nu^*DQ^*(\bar{\sigma}, \mathring{\sigma}) - \nu^*LP\bar{u} = 0.$$

By virtue of the definition of mapping Q^*, from Eq. (5.8) we see that

$$\nu^*D\nu\mathring{\sigma} = \nu^*LP\bar{u} - \nu^*D\mu\bar{\sigma}$$

Hence, provided that det $(\nu^*D\nu) \neq 0$ we obtain

(5.9) $$\mathring{\sigma} = \mathring{C}\nu^*LP\bar{u} - \mathring{C}\nu^*D\mu\bar{\sigma}$$

where $\mathring{C} = (\nu^*D\nu)^{-1}$. Substituting the right hand side of Eq. (5.9) to (5.7) and taking into account (5.4) we arrive at

(5.10) $$(\mu^*D(I - \nu\mathring{C}\nu^*D)\mu\bar{\sigma} - \mu^*(I - D\nu\mathring{C}\nu^*)LP\bar{u}, \bar{\tau} - \bar{\sigma}) \geqslant 0 \quad \forall \bar{\tau} \in \bar{\Sigma}.$$

Furthermore, substituting the right hand side of (5.9) to (5.5) we obtain ..

(5.11) $$\langle P^*L^*(I - \nu\mathring{C}\nu^*D)\mu\bar{\sigma} + P^*L^*\nu\mathring{C}\nu^*LP\bar{u} - \bar{F}, \bar{v} - \bar{u} \rangle \geqslant 0, \quad \forall \bar{v} \in \bar{K}.$$

The resulting inequalities (5.10) and (5.11) of the shell problem can be written down in the final form:

(5.12) $$\langle \bar{L}^*\bar{\sigma} + \bar{R}\bar{u} - \bar{F}, \bar{v} - \bar{u} \rangle \geqslant 0 \quad \forall \bar{v} \in \bar{K},$$
$$(\bar{D}\bar{\sigma} - \bar{L}\bar{u}, \bar{\tau} - \bar{\sigma}) \geqslant 0 \quad \forall \bar{\tau} \in \bar{\Sigma},$$

where

$$L = \mu^*(I - D\nu\mathring{C}\nu^*)LP,$$
$$L^* = P^*L^*(I - \nu\mathring{C}\nu^*D)\mu,$$
$$R = P^*L^*\nu\mathring{C}\nu^*LP,$$
$$D = \mu^*D(I - \nu\mathring{C}\nu^*D)\mu.$$

Inequalities (5.12) were obtained by imposing the constraints for deformations and stresses, on the three-dimensional equilibrium problem of the linear elasticity. The constraints for deformations modify the equilibrium equations, while the constraints for stresses modify the constitutive relations. The constraints under consideration, (cf. [5]. [1]), are different from those used in papers [2], [3], [4], which modify the constitutive relations only. From the abstract inequalities (5.12), by the specification of constraints, different boundary value shell problems can be obtained. The inequalities analogous to these given by Eqs. (5.12) can also be derived from a tolerance interpretation of the boundary-value problems of the classical elasticity. [6].

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Р е з ю м е

НЕКОТОРЫЕ НЕРАВЕНСТВА В ЛИНЕЙНОЙ ТЕОРИИ ОБОЛОЧЕК

В работе дана слабая формулировка смешанных граничных задач линейной теории упругости со связями. Доказано, что она равносильна задачи седловой точки для функционала Рейсснера. Выведено несколько соотношений между решениями сформулированных задач. Рассмотрено класс связи ведущих к проблемам теории оболочек.

S t r e s z c z e n i e

O PEWNYCH NIERÓWNOŚCIACH W LINIOWYCH ZAGADNIENIACH POWŁOK

W pracy podano slabe sformułowanie mieszanych zagadnień brzegowych w liniowej teorii sprężystości z więzami. Wykazano ich równoważność z zagadnieniem punktu siodłowego dla funkcjonału Reissnera. Wyprowadzono kilka relacji między rozwiązaniami sformułowanych zagadnień. Rozpatrzono klasę więzów prowadzącą do zagadnień teorii powłok.

Praca zostala złożona w Redakcji dnia 27 marca 1983 roku