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EQUATIONS OF THE SPHERICAL SHELL WITH AXIALLY SYMMETRIC, STOCHASTIC IMPERFECTIONS

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1. Introduction

Realization of the shell construction often yields some deformations and since the changes in the geometry of the middle surface are unpredictable, it is convenient to consider the problem from the probabilistic point of view. There have been in the literature up to now a few approaches to the description of the stochastic shell. C. BRANICKI and M. SKOWRONEK [1] analized stochastically nonlinear static of a shallow spherical shell, which middle surface was a random function of a rather simple form. E. FILIPOW, J. WEKEZER and P. WILDE [2] proposed a stochastic model for the dislocations of the surface of a cylindrical container based on the discretization of the problem. Random fields theory applicable to thin elastic shells was discussed in the expository paper of E. BIELEWICZ and P. WILDE [3].

The subject of this note is a statical analysis of the spherical shell loaded uniformly by its weight taking into consideration geometrical nonlinearity and axially symmetric random displacement. It is proposed to describe stochastic displacement by auxiliary six dimensional two parameter random field and the corresponding Meissner-type equations are derived.

2. Description of the random shell

Let the undeformed middle surface of the shell be given by the equations in the vector form

$$\vec{r} = \vec{r}(\Theta, \varphi, \omega) = \vec{r}_0(\Theta, \varphi)+\vec{r}_s(\Theta, \varphi, \omega), \tag{1}$$

where $\vec{r}_0$ describes the points of the middle surface of the deterministic shell designed and $\vec{r}_s$ is the stochastic initial deformation.

In order to describe the stochasticaly displaced shell by the equations close to the deterministic case, we rewrite equations (1) in the following equivalent form

$$\begin{aligned}\frac{\partial \vec{r}}{\partial \Theta} &= \frac{\partial \vec{r}_0}{\partial \Theta}+\hat{t}_2(\Theta, \varphi, \omega),\\ \frac{\partial r}{\partial \varphi} &= \frac{\partial r_0}{\partial \varphi}+\hat{t}_1(\Theta, \varphi, \omega),\end{aligned} \tag{2}$$

with the boundary condition

$$\vec{r}(\Theta_0, \varphi, \omega) = \vec{r}_0(\Theta_0, \varphi),$$
$$\vec{r}(\Theta, 0, \omega) = \vec{r}(\Theta, 2\pi, \omega). \tag{3}$$

Here $\hat{t}_1, \hat{t}_2$ is a 6-dimensional 2-parameter random field satisfying the consistency condition

$$\frac{\partial \hat{t}_2}{\partial \varphi} = \frac{\partial \hat{t}_1}{\partial \Theta}. \tag{4}$$

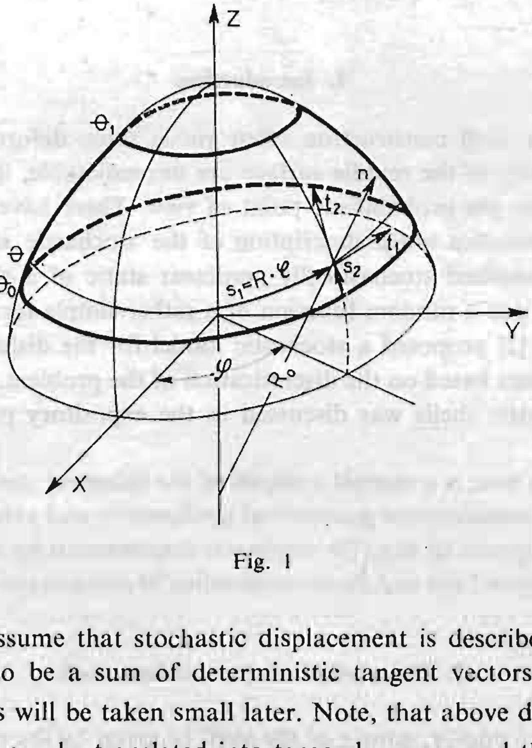

Fig. 1

In other worlds we assume that stochastic displacement is described by tangent vectors which are supposed to be a sum of deterministic tangent vectors and random vectors $\hat{t}_1, \hat{t}_2$, the second ones will be taken small later. Note, that above description making use of random field $\hat{t}_1, \hat{t}_2$ can be translated into tensor language and that our assumption is slightly different from the on the first sight natural representation of the first metric form of the middle surface of the stochastic shell as a sum of a purely deterministic and stochastic part. Although for smaller random field $\hat{t}_1, \hat{t}_2$ both mentioned above approaches become closer, we found our approach as given by equations (2) more consistent with the intuition and therefore we will not deal in the sequel with the equations in the tensor form. Morever we will restrict ourselves to axially symmetric random fields $\hat{t}_1, \hat{t}_2$ and we will look for the equations of the stochasticaly displaced spherical shell as given on fig. 1 loaded uniformly by its weight.

Further we assume that $\hat{t}_2$ is a vector tangent to the meridian of the middle surface

$$\hat{t}_2(\Theta, \varphi, \omega) = [-B_\Theta(\omega)\sin\Theta\cos\varphi, \; -B_\Theta(\omega)\sin\Theta\sin\varphi, \; B_\Theta(\omega)\cos\Theta]$$

where $B_\Theta(\omega)$ is a scalar 1-parameter random field. In this case $\hat{t}_1$ is uniquelly determined by (4) and the axial symetricity

$$\hat{t}_1(\Theta, \varphi, \omega) = \left[\sin\varphi \int_{\Theta}^{\Theta} B_\Theta \sin\Theta d\Theta, \quad -\cos\varphi \int_{\Theta_0}^{\Theta} B_\Theta \sin\Theta d\Theta, 0\right]$$

Since we are in the axially symmetric case, we can use the well known Meissner-type equations in the form taking into account geometric nonlinearity, (c.f. [4]). Regarding an additional assumption, that $\hat{t}_1$ and $\hat{t}_2$ are small when compared with the radius of the shell (i.e. $B_\Theta \ll R_0$) we get the following system of nonlinear second order differential equations

$$\begin{aligned} (D_+ + L)\psi + R_1\psi\vartheta + N\psi\vartheta' + P_1\vartheta + Q\vartheta' &= F_1(\Theta, \omega) \\ D_-\vartheta + R_2\vartheta\psi + P_2\psi &= F_2(\Theta, \omega) \end{aligned} \tag{5}$$

where $D_\pm$ are the second order deterministic differential operators of the form

$$D_\pm = \cos\Theta \frac{d^2}{d\Theta^2} - \sin\Theta \frac{d}{d\Theta} - \left(\frac{\sin^2\Theta}{\cos\Theta} \pm \nu\cos\Theta\right). \tag{6}$$

L is the random differential operator

$$L = (-S^\Theta - 2\delta_\Theta\cos\Theta)\frac{d^2}{d\Theta^2} + (\delta_\Theta\sin\Theta - \delta'_\Theta\cos\Theta)\frac{d}{d\Theta} - \left(\frac{\sin^2\Theta}{\cos^2\Theta} S^\Theta + \nu\delta_\Theta\cos\Theta\right). \tag{7}$$

R_i, P_i, N are some deterministic functions and F_i are random functions.

Also we denoted by R_0 radius of the spherical shell and $\delta_\Theta = \dfrac{B_\Theta}{R_0}$, $\delta'_\Theta = \dfrac{d}{d\Theta}\delta_\Theta$, $S^\Theta = \int_{\Theta_0}^{\Theta} \delta_\Theta \sin\Theta d\Theta$, ν is the Poisson coeficient.

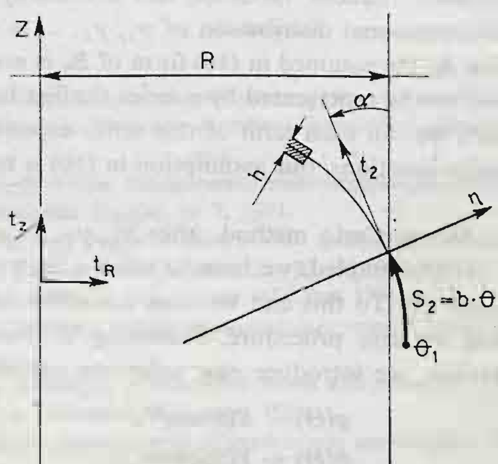

Fig. 2

Equations (5) contain the following unknown quantities, ϑ — is an increment of an angle α after deformation $\alpha^* = \alpha+\vartheta$ (fig. 2)

$$\psi = V\sqrt{\frac{12(1-\nu^2)}{Eh^2}}, \quad V' = (1+\delta_\Theta) R_0 T_1 \text{ (fig. 3)}$$

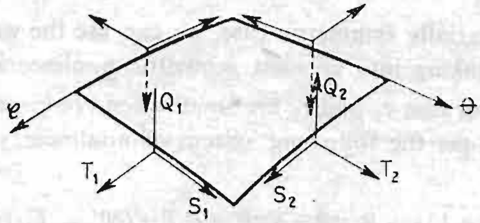

Fig. 3

3. Numerical solution

For the solution of the equations (5) we have to consider the boundary conditions e.g. we consider the shell with fixed lower boundary

$$\Theta = \Theta_0, \quad \varepsilon_1 = 0, \quad \vartheta = 0, \tag{8}$$

and with free upper boundary

$$\Theta = \Theta_1, \quad T_r = -T_1 \sin\alpha^* + Q_2 \cos\alpha^* \text{ (fig. 3).} \tag{9}$$

Determination of the mean value and the standard deviation of the random internal forces was achieved by Monte Carlo method. For this purpose we assumed that random function B_Θ can be represented as a series

$$B_\Theta = \cos\Theta \sum_{n=0}^{\infty} \gamma_n \cos^n 2\Theta \tag{10}$$

where γ_n are one dimensional random variables, not necessarily independent — in the dependent case the multidimensional distribution of $\gamma_1, \gamma_2, \ldots$ is needed. Note, that for Gaussian random function B_Θ the assumed in (10) form of B_Θ is not very much restrictive. Indeed, any Gaussian field can be represented by a series similiar to (10) with independent random variables and then we can each term of this series expand into a Fourier series. Thus, up to the convergence questions, our assumption in (10) is that some of the Fourier coeficients are zero.

As it is usual in the Monte Carlo method, after $\gamma_1, \gamma_2, \ldots$ are sampled, i.e. finite approximation of B_Θ in (10) is sampled, we have to solve a deterministic system of non-linear differential equations (5). To this end we used a combination of the power series method together with the iterative procedure. Following R. NAGÓRSKI [5] with slight changes to avoid singularities, we introduce new unknown variables X, Y defined by

$$\begin{aligned} \psi(\Theta) &= X(x)\cos\Theta, \\ \vartheta(\Theta) &= Y(x)\cos\Theta, \end{aligned} \tag{11}$$

where $x = \cos 2\Theta$.

After the change of variables in (5) we expand the right hand sides of the equations (5) in the power series with respect to x. Then we look for the coeficients in the power series expansion of X and Y by iterative procedure adopted from A. MAHMOUD [6]. The Mahmoud's approach lies on the transposition of the nonlinearities to the right hand sides of the equations (5) treating them as known. This applied to our problem gives the following separated recurrent equations with unknown X_n and Y_n

$$\begin{aligned}(4-4x^2)X_n''+(6-10x)X_n'-(1-\nu)X_n &= G_-(x, X_{n-1}, Y_{n-1}),\\ (4-4x^2)Y_n''+(6-10x)Y_n'-(1+\nu)Y_n &= G_+(x, X_{n-1}, Y_{n-1}).\end{aligned} \tag{12}$$

In (12) the right hand side functions $G_\pm$ can be expanded in the power series of the convergence radius 1, thus the solutions of (12) are given by $\sum_{n=0}^{\infty} a_n^\pm x^n$ (c.f. E. KAMKE [7]). Therefore we get the recurrent linear equations for unknown power series coeficients

$$a_{n+2}^\pm = -\frac{2n+3}{2n+4}a_{n+1}^\pm+\frac{4n^2-(1\pm\nu)}{4(n+1)(n+2)}a_n^\pm+\frac{2(n+1)(2n+1)-(1\pm\nu)}{4(n+1)(n+2)}a_{n-1}^\pm+ \\ +\frac{b_n^\pm}{4(n+1)(n+2)}, \tag{13}$$

where $G_\pm = \sum_{n=0}^{\infty} b_n^\pm x_n$.

From (13) follows also, that the convergence radius of the series expansion of X_n and Y_n is equal 1.

Approximatively strict solutions of the system of nonlinear equations (5) are then determined by $X = \lim X_n$, $Y = \lim Y_n$ together with (11).

The author checked numericaly the above procedure and it appeared, that iterative procedure with $X_0 = Y_0 = 0$ works nicely for small stochastic part (5 - 10 iterations are then sufficient).

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Р е з ю м е

ВРАЩАТЕЛЬНО-СИММЕТРИЧНОЕ СЛУЧАЙНОЕ СОСТОЯНИЕ ПЕРЕМЕЩЕНИЙ СФЕРИЧЕСКОЙ ОБОЛОЧКИ

В работе представлены проблемы связанные со статикой тонкой сферической оболочки постоянной толщины, нагруженной собственным весом с учетом вращательно-симметричных случайных начальных перемещений вместе с геометрической нелинейностью.

Случайная часть проблемы решена методом симмуляции. Численное решение возникнувшей детерминистической задачи проведено методом, который соединяет метод степенных рядов с итерационным методом.

S t r e s z c z e n i e

RÓWNANIA POWŁOKI KULISTEJ W PRZYPADKU OSIOWO SYMETRYCZNYCH LOSOWYCH PRZEMIESZCZEŃ WSTĘPNYCH

W pracy zostały rozpatrzone zagadnienia statyki cienkiej powłoki kulistej o stałej grubości, obciążonej ciężarem własnym z uwzględnieniem obrotowo symetrycznych losowych przemieszczeń wstępnych oraz geometrycznej nieliniowości.

Stochastyczna część zagadnienia została rozwiązana metodą symulacji. Liczbowe rozwiązanie zagadnienia deterministycznego otrzymano metodą łączącą metodę szeregów potęgowych i iteracyjną.

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