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THE EQUATIONS OF THE SECOND ORDER LINEAR MODEL OF SURFACE GRIDS

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1. Introduction

The paper deals with the method of finding the governing equations for a surface structure having a form of a dense and regular grid made of bars. The lateral deformation of elements of the system are taken into account. It is assumed that the material of the structure is elastic, homogeneous and isotropic. The problem of statics is analysed within the linear theory.

The numerical methods employed to solve the problems related to the considered systems were based on discret representation of the structure (see, among others [1, 2, 3]) and lead to a system of algebraic equations with a large number of unknowns. The dimensions of nodes, their deformability and the lateral deformability of structure's bars were not taken into account.

The application of a continuum model of a structure consists in an approximation of the multi-connected geometry of the system by a certain simply-connected and continuous model (see, among others [4, 5, 6]). The advantage of the discussed approach over the previous one lies in the fact that the analytical methods can be employed. The negative aspects are: a) considerable inaccuracy of results for not sufficiently dense grids, b) the required geometrical symmetry of the structure. An interesting idea of a continuum model of such structures based on the concept of a continuum with internal microstructure and higher order internal reactions is presented in [6]. In the present paper Cz. Woźniak's model will be applied to obtain equations of the second order theory. An energetic approach, different from the previously considered one, which will be employed makes it possible to describe in the explicit from all properties of the continuum model. As a special case (in which the higher order effects are neglected) equations of the first approximation will be obtained.

2. Basic assumptions

It is assumed that the structure consist of (homogeneously) deformable cubicoid nodes connected by means of the prismatic links of rectangular cross-sections (and subject to homogeneous deformation in their plane) and constitute the regular and orthogonal surface grid made of bars (Fig. 1). The lengths of the elements of structure are small as compared with the lengths of the surface and its curvature radii.

A system of x^1, x^2 coordinates on the π surface on which the structure is shaped and a z coordinate in the direction normal to surface π were chosen in such a way that x^1, x^2, z axes represent a right-hand system of coordinates. It was assumed that the geometric centers of the nodes lie at intersections of parametric lines x^1 = const, x^2 = const, $z = 0$ and that the axes of the links coincide with directions of these parametric lines. A typical segment of such a structure is shown in Fig. 1.

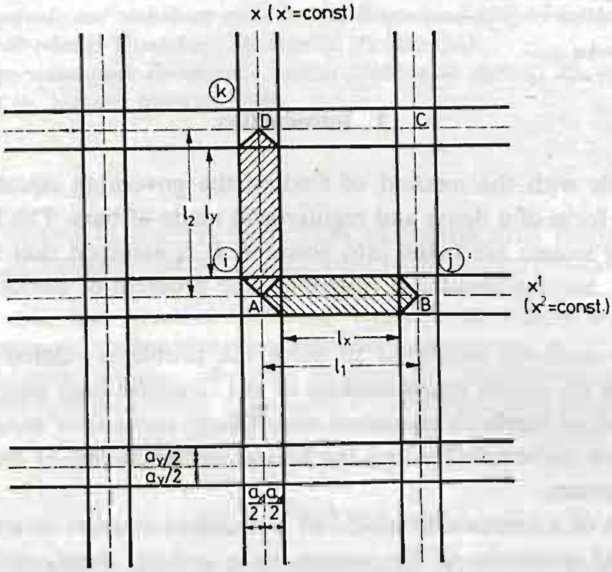

Fig. 1

We shall introduce twelve continuous, sufficiently smooth functions defined on the surface π of the structure. These functions represent translations, rotations, deformations along the coordinate axes and the shape deformations. The forementioned functions constitute unknown quantities of the model and have a physical sense only at the node centres. In every net mesh they can be treated as linear nature.

3. The analysis of the structure components

Node. When a structure is loaded a typical node is subjected to a homogeneous deformation having 12 degrees of freedom. Let (u_x, u_y, u_z) be the displacements, $(\vartheta_x, \vartheta_y, \vartheta_z)$ — the components of an independent vector of rotation, $(\omega_x, \omega_y, \omega_z)$ and $(\omega_{xy}, \omega_{xz}, \omega_{yz})$ — the linear and deviatoric components of a homogeneous deformation, respectively. Denoting by $\overline{w}_x$, $\overline{w}_y$, $\overline{w}_z$ displacements within the node area in directions x, y, z, respoctively the following formula hold

$$\begin{aligned} \overline{w}_x(x, y, z) &= u_x+\omega_x x+(\omega_{xy}-\vartheta_z)y+(\vartheta_y+\omega_{xz})z, \\ \overline{w}_y(x, y, z) &= u_y+(\vartheta_z+\omega_{xy})x+\omega_y y+(\omega_{yz}-\vartheta_x)z, \\ \overline{w}_z(x, y, z) &= u_z+(\omega_{xz}-\vartheta_y)x+(\vartheta_x+\omega_{yz})y+\omega_z z. \end{aligned} \tag{3.1}$$

After applying the principle of ideal constraints we can arrive at 12 equations describing the node equilibrium with 6 generalized internal forces and 12 generalized external forces. From the equations of the linear theory of elasticity the general constitutive relations can be obtained together with a formula for the strain energy of a node.

Link. Let us take into account a typical element connecting the i-th and the j-th nodes situated on the x^2 = const parametric line (see Fig. 1). Let w_x, w_y, w_z represent displacements of the link area in directions of a local coordinates x, y, z (see Fig. 2). It is assumed that the lateral cross-sections of a link are subjected to homogeneous deformations in their planes as well as to the rigid displacements (9 degrees of freedom). Hence:

$$\begin{aligned} w_x(x, y, z) &= v_x(x)-y\varphi_z(x)+z\varphi_y(x), \\ w_y(x, y, z) &= v_y(x)+y\gamma_y(x)+z\left[\frac{1}{2}\gamma_{yz}(x)-\varphi_x(x)\right], \\ w_z(x, y, z) &= v_z(x)+y\left[\frac{1}{2}\gamma_{yz}(x)+\varphi_x(x)\right]+z\gamma_z(x), \end{aligned} \tag{3.2}$$

where v_x, v_y, v_z are dislocations, $\varphi_x, \varphi_y, \varphi_z$ — rotations, γ_y, γ_z, γ_{yz} — deformations of the cross-section of an element along the x coordinate.

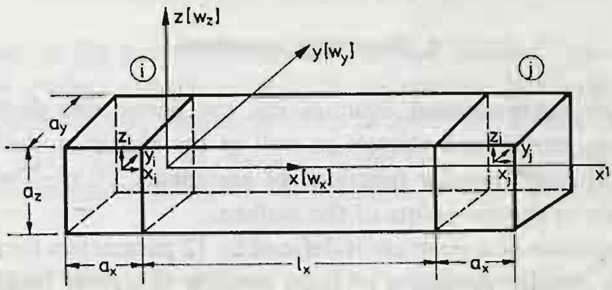

Fig. 2

The state of link area displacements is described by 9 functions of the variable x being the Lagrange's generalized displacements. The assumption (3.2) can be called the hypothesis of a flat, homogeneously deformable cross-section with independent rotations. This is a generalization of the well-known hypothesis of Bernoulli and Timoshenko for the classical model of a bar.

The constraints for stresses are assumed in the form

$$\sigma_y = 0, \quad \sigma_z = 0. \tag{3.3}$$

This assumption simplifies considerably the formulae given below. After applying the principle of ideal constraints of the static and kinematic types we shall obtain 9 equations describing the equilibrium of the link with 10 generalized internal forces and 9 generalized external forces. Taking into account the known equations of the linear elasticity, the generalized constitutive equations and the formulae defining the strain energy of a link can be found.

Node-link-node system. A system consisting of the i-th node, the j-th node and the $(i-j)$-th link connecting these two nodes is presented below (see Fig. 2).
From the equilibrium and constitutive equations describing the link, the differential equations for the generalized displacements can be obtained. The kinematic boundary conditions result from the assumption that the displacements of the appriopriate boundaries of the i-th and j-th nodes have to be competible with the displacements of the suitable boundaries of the link situated between them. In this way we obtain, functions $v_x, v_y, v_z, \varphi_x, \varphi_y, \varphi_z, \gamma_y, \gamma_z, \gamma_{yz}$, which are expressed as the functions of the i-th node and the Δ operator defined as follows:

$$\Delta(\ ) = (\ )^j - (\ )^i. \tag{3.4}$$

These functions can be understood as certain shape functions of the bar treated as a three-dimensional body. The total elastic energy of the $(i-j)$-th link is a function of parameters attributed to the i-th node and the Δ operator defined above.

The analogous procedure can be applied to link situated on the x^1 = const. parametric line and connecting the i-th and the k-th nodes. Instead of the Δ operator we deal now with the $\tilde{\Delta}$ operator defined as follows:

$$\tilde{\Delta}(\ ) = (\ )^k - (\ )^i \tag{3.5}$$

4. Boverning equations

According to the forementioned assumptions, the parameters describing the displacement, deformation, strains and stresses as well as the elastic moduli are described by the continuous, sufficiently regular functions of arguments x^1, x^2. These functions have a physical sense only in certain points of the surface.

The displacement state of a structure is defined by 12 parameters for each node. A continuous, sufficiently regular extension of these discrete functions leads to the relations:

$$\begin{aligned}
&u_1(x^1, x^2) = u_x, && u_2(x^1, x^2) = u_y, && u(x^1, x^2) = u_z,\\
&\vartheta_1(x^1, x^2) = \vartheta_x, && \vartheta_2(x^1, x^2) = \vartheta_y, && \vartheta(x^1, x^2) = \vartheta_z,\\
&\omega_{11}(x^1, x^2) = \omega_x, && \omega_{22}(x^1, x^2) = \omega_y, && \omega(x^1, x^2) = \omega_z,\\
&\omega_{12}(x^1, x^2) = \omega_{21}(x^1, x^2) = \omega_{xy}, && \omega_1(x^1, x^2) = \omega_{xz}, && \omega_2(x^1, x^2) = \omega_{yz}.
\end{aligned} \tag{4.1}$$

The strain energy of a typical structure segment (i.e. the energy of the $(i-j)$-th and the $(i-k)$-th links, $\frac{1}{2}$ energy of the i-th node, $\frac{1}{4}$ of energy of the j-th node, $\frac{1}{4}$ of energy of the k-th node) is related to ABCD surface segment with $l_1 \cdot l_2$ dimensions (see Fig. 1). This energy is a function of parameters assigned to the i-th node and involve Δ and $\tilde{\Delta}$ operators.

Assuming that:

$$\frac{1}{l_1}\Delta(\,) = \frac{\partial}{\partial x^1} = (\,)_{,1}, \quad \frac{1}{l_2}\tilde{\Delta}(\,) = \frac{\partial}{\partial x^2} = (\,)_{,2} \tag{4.2}$$

and that the density of elastic energy σ° is equal to the density of energy $\sigma^{(i)}$ in the i-th node, the basic relation of the continuum model of the considered structure in its explicit form was found:

$$\begin{aligned}\sigma^0 = {} & \frac{1}{2} C^{KLMN}\varkappa_{KL}\varkappa_{MN} + \frac{1}{2} A^{KL}\gamma_K\gamma_L + G^{KLM}\varkappa_{KL}\gamma_M + F^{KLMN}\varkappa_{KL}\tau_{MN} + \\ & + D^{KL}\gamma_K\omega_L + H^{KLM}\varkappa_{KL}\omega_M + R^{KLM}\gamma_K\tau_{LM} + \frac{1}{2} G^{KLMN}\tau_{KL}\tau_{MN} + \frac{1}{2} G^{KL}\tau_K\tau_L + \\ & + \frac{1}{2} F^{KL}\omega_K\omega_L + \frac{1}{2} A\omega^2 + C^{KLM}\omega_K\tau_{LM} + A^K\tau_K\omega + \frac{1}{2} A^{KLMN}\gamma_{KL}\gamma_{MN} + \\ & + \frac{1}{2} C^{KL}\varkappa_K\varkappa_L + E^{KLM}\varkappa_K\gamma_{LM} + B^{KLMN}\gamma_{KL}\omega_{MN} + D^{KLMN}\varkappa_K\tau_{LMN} + \\ & + F^{KLM}\varkappa_K\omega_{LM} + A^{KLMNP}\gamma_{KL}\tau_{MNP} + \frac{1}{2} E^{KLMN}\omega_{KL}\omega_{MN} + {}'E^{KLMN}\omega_{KL}\omega_{MN} + \\ & + \frac{1}{2} A^{KLMPRS}\tau_{KLM}\tau_{PRS} + {}'A^{KLMPRS}\tau_{KLM}\tau_{PRS} + C^{KLMPR}\omega_{KL}\tau_{MPR} + \\ & + H^{KL}\omega_{KL}\omega + D^{KLM}\omega_{KL}\tau_M + B^{KLM}\tau_{KLM}\omega + H^{KLMN}\tau_{KLM}\tau_N. \\ & K, L, M, N, P, R, S = 1, 2.\end{aligned} \tag{4.3}$$

The density of work of the external forces can be olso defined. The relation (4.3) was originally expressed in the Cartesian coordinate system and then generalized to a curvilinear orthogonal system of coordinates in terms of which the surface system is described. The parameters: $\gamma_{KL}, \gamma_K, \varkappa_{KL}, \varkappa_K, \tau_{KLM} = \tau_{KML}, \tau_{KL}, \tau_K, \omega_{KL} = \omega_{LK}, \omega_K, \omega$ constitute generalized components of the state of deformation, with the geometric relations taking the form

$$\begin{aligned} \gamma_{KL} &= u_L|_K - b_{LK}u + e_{LK}\vartheta, \quad \gamma_K = u|_K + b^L_K u_L + e_{KL}\vartheta^L, \\ \varkappa_{KL} &= \vartheta_L|_K - b_{LK}\vartheta, \quad \varkappa_K = \vartheta|_K + b^L_K\vartheta_L, \\ \tau_{KLM} &= \omega_{LM}|_K - b_{MK}\omega_L - b_{LK}\omega_M, \\ \tau_{KL} &= \omega_L|_K + b^N_K\omega_{NL} - b_{LK}\omega, \quad \tau_K = \omega|_K + 2b^N_K\omega_N, \end{aligned} \tag{4.4}$$

where $b_{KL}, e_{KL}, (\,)|_K$ represent the components of the second metric tensor of the surface, Ricci's bivector, and the symbol of covariant differentiation on the surface, respectively. The functions $A^{KLMPRS}, \dots C^{KLM}, \dots, A^K, A$ stand for the tensor of elastic moduli of the structure and describe its geometric and physical properties.

The components of the stress state of the structure are given by the formulae

$$p^{KL} = \frac{\partial\sigma^0}{\partial\gamma_{KL}}, \quad p^K = \frac{\partial\sigma^0}{\partial\gamma_K}, \quad m^{KL} = \frac{\partial\sigma^0}{\partial\varkappa_{KL}}, \quad m^K = \frac{\partial\sigma^0}{\partial\varkappa_K}, \tag{4.5}$$

(4.5) [cont.]
$$s^{KLM} = \frac{\partial\sigma^0}{\partial\tau_{KLM}}, \quad s^{KL} = \frac{\partial\sigma^0}{\partial\tau_{KL}}, \quad s^K = \frac{\partial\sigma^0}{\partial\tau_K},$$
$$r^{KL} = \frac{\partial\sigma^0}{\partial\omega_{KL}}, \quad r^K = \frac{\partial\sigma^0}{\partial\omega_K}, \quad r = \frac{\partial\sigma^0}{\partial\omega}.$$

We see that 12 from 30 introduced aqove components of the stress state is of the force type $(p^{KL}, p^K, r^{KL}, r^K, r)$, and the remaining 18 is of the couple type $(m^{KL}, m^K, s^{KLM}, s^{KL}, s^K)$.

With the aid of the principle of virtual work the equilibrium equations and the boundary conditions for the continuum model of the structure can be obtained in the form

(4.6)
$$p^{KL}|_K - b^L_K p^K + q^L = 0, \quad p^K|_K + b_{LK}p^{KL} + q = 0,$$
$$m^{KL}|_K - b^L_K m^K + e^L_K p^K + h^L = 0, \quad m^K|_K + e_{KL}p^{KL} + b_{LK}m^{KL} + h = 0,$$
$$s^{KLM}|_K - \frac{1}{2}(b^L_K s^{KM} + b^M_K s^{KL}) - r^{LM} + f^{LM} = 0,$$
$$s^{KL}|_K + 2b_{MK}s^{KLM} - 2b^L_K s^K - r^L + f^L = 0, \quad s^K|_K + b_{LK}s^{KL} - r + f = 0.$$

(4.7)
$$p^{KL}n_K - \overset{*}{p}{}^L = 0 \quad \text{or} \quad u_L = \overset{*}{u}_L; \quad p^K n_K - \overset{*}{p} = 0 \quad \text{or} \quad u = \overset{*}{u},$$
$$m^{KL}n_K - \overset{*}{m}{}^L = 0 \quad \text{or} \quad \vartheta_L = \overset{*}{\vartheta}_L; \quad m^K n_K - \overset{*}{m} = 0 \quad \text{or} \quad \vartheta = \overset{*}{\vartheta},$$
$$s^{KLM}n_K - \overset{*}{s}{}^{LM} = 0 \quad \text{or} \quad \omega_{LM} = \overset{*}{\omega}_{LM}; \quad s^{KL}n_K - \overset{*}{s}{}^L = 0 \quad \text{or} \quad \omega_L = \overset{*}{\omega}_L,$$
$$s^K n_K - \overset{*}{s} = 0 \quad \text{or} \quad \omega = \overset{*}{\omega}$$

where n_K represents the components of a unit normal vector to the boundary $\partial\Omega$ of the structure, $q^K, q, h^K, h, f^{KL}, f^K, f$ — are the densities of the surface-type external stress; $\overset{*}{p}{}^K, \overset{*}{p}, \overset{*}{m}{}^K, \overset{*}{m}, \overset{*}{s}{}^{KL}, \overset{*}{s}{}^K, \overset{*}{s}$ — are the densities of boundary stresses, $\overset{*}{u}{}^L, \overset{*}{u}, \overset{*}{\vartheta}{}^L, \overset{*}{\vartheta}, \overset{*}{\omega}_{LM}, \overset{*}{\omega}_L, \overset{*}{\omega}$ are the given values of generalized displacements within the $\partial\Omega$.

The equilibrium equations (4.6) and the boundary conditions (4.7) together with the constitutive (4.5) and geometric (4.4) relations form the basic system of equations describing the continuous model of the structure. This system enables us to calculate the displacement distribution in the link and node areas as well as the stress distribution.

It must be stressed that parameters $\gamma_K, \varkappa_{KL}, \tau_{KL,K}, \omega_L$, [see (4.4)] define the components of the plate-like deformation state, while $\gamma_{KL}, \omega_{KL}, \varkappa_K, \tau_{KLM}$ — the components of the plane-like deformation state. The components $\gamma_{KL}, \omega_{kl}, \varkappa_K, \tau_{KLm}, \tau_k, \omega$ are defined exactly as in the rst order model (see [6]), however the parameters which do not appear in that model, i.e. $\tau_{KLM}, \tau_{KL}, \tau_K, \omega_{KL}, \omega_K, \omega$ result from the deformability of a node and the deformability of link's lateral cross-sections.

The analysis of the influence of the second order parameters on the internal forces together with the suitable numerical calculation will be the subject of separate papers.

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Р е з ю м е

УРАВНЕНИЯ ЛИНЕЙНОЙ ТЕОРИИ ВТОРОГО РЯДА УПРУГИХ ПОВЕРХНОСТНЫХ РОСТВОРОК

В работе представлено линейные уравнения статики упругих поверхностных растворок имеющих плотную и регулярную сетку элементов, которых деформированные, прямоугольные узлы соединенные между собой рпи помощи призматических стержней имеющих прямоугольное сечение.

Принимая исходные данные: уравнения линейной теории упругости, а также подходящие кинематические гипотезы получено вариационным методом уравнения сплошного моделя прогона.

Работа содержит обобщение теории Возняка, выходящие за пределы теории I-го ряда, позволяющие учитывать эффекты ,,высших рядов'' (размеры узлов, их деформирование, деформирование поперечного сечения стержней соединяющих узлы).

S t r e s z c z e n i e

RÓWNANIA LINIOWEJ TEORII DRUGIEGO RZĘDU SPRĘŻYSTYCH RUSZTÓW POWIERZCHNIOWYCH

W pracy wyprowadzono liniowe równania statyki sprężystych rusztów powierzchniowych o gęstej i regularnej siatce elementów, których odkształcalne, prostopadłościenne węzły połączone są za pomocą pryzmatycznych prętów o przekroju prostokątnym.

Przyjmując za punkt wyjścia równania liniowej teorii sprężystości oraz zakładając odpowiednie hipotezy kinematyczne otrzymano na drodze wariacyjnej równania ciągłego modelu dźwigara.

Praca zawiera uogólnienie teorii Woźniaka, wykraczające poza teorię I-go rzędu, zezwalające na uwzględnienie efektów ,,wyższych rzędów'' (wymiary węzłów, ich odkształcalność, odkształcalność przekrojów poprzecznych prętów łączących węzły).

Praca zostala zlożona w Redakcji dnia 3 marca 1983 roku