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THE ANALYSIS OF A VISCOELASTIC SHELL OF REVOLUTION PERIODICALLY LOADED

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1. Introduction

The shell of revolution periodically loaded by mechanical forces or by the periodically forced displacement is analysed in the paper. The material of a shell is assumed to be linear. The problem will be solved on the basis of the technical theory of thin shells [1]. Under assumption that the displacements and deformations are small the problem is linear. It is also assumed that the loadings of the shell (mechanical and nonmechanical) are described by the harmonic functions with the circular frequency ω and are applied in the duration of the sufficient length. It allows to assume that the motion of the construction is stationary, i.e. that the displacements and internal forces vary periodically with the frequency ω. It enables searching of the solution without explicit time function.

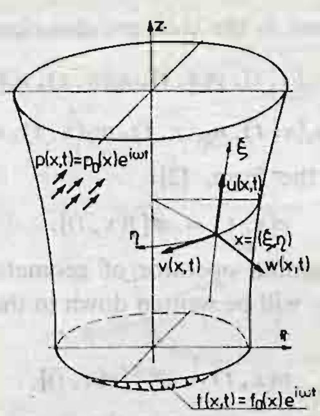

Fig. 1

The set of curvilinear coordinates $x = (\xi, \eta)$ (Fig. 1) is chosen in the middle surface of the shell. For simplification it is assumed that all loads are in the same phase and that the origin of the time axis is chosen so, that the phase angle is equal to zero. It does not restrict the generality of considerations because the problem is linear and the principle of superposition holds. For $\omega = 0$ the statical problem has been received.

The solution of the problem has been obtained numerically by Finite Element Method [2, 3]. The construction of the FEM algorithm for eight parameter conical elements are given in the paper. The solution formulated in this way has been applied to the analysis of the rubber construction of the seal of rotational shafts.

2. Basic definitions and linear viscoelasticity equations

Let the system of cylindrical coordinates $X = (r, z, \eta)$ be introduced in the Euclidean space. The middle surface of the shell immersed in this space occupies the region ϑ. The curvilinear coordinates $x = (\xi, \eta)$, interrelated with (r, z, η) by means of $X = [r(\xi), z(\xi), \eta]$, parametrize the surface. The material of the shell is isotropic with mass density ϱ_0 and thickness distribution $h(\xi)$.

All functions which describe the motion, deformations and stresses in the shell will be described in terms of the physical coordinates of the two dimensional tensor fields on the middle surface of the shell. Keeping in mind the applied method of the solution we have assumed it in the form of vectors in the local set of coordinates (x, ζ).

The motion of the middle surface is described by the displacement vector

$$f(x, t) = [u(x, t), v(x, t), w(x, t)]^T. \tag{1}$$

Moreover, the concept of the generalized vector of displacement is introduced

$$\tilde{f}(x, t) = [u(x, t), v(x, t), w(x, t), \varphi_\xi(x, t), \varphi_\eta(x, t)]^T, \tag{2}$$

where $\varphi_\xi(x, t)$, $\varphi_\eta(x, t)$ are angles of rotations of the material fibre normal to the mid-surface.

The stresses and deformations in the shell are described by vectors

$$\varepsilon(x, t) = [\varepsilon_\xi(x, t), \varepsilon_\eta(x, t), \gamma(x, t), \varkappa_\xi(x, t), \varkappa_\eta(x, t), \chi(x, t)]^T, \tag{3}$$

$$\sigma(x, t) = [n_\xi(x, t), n_\eta(x, t), n_{\xi\eta}(x, t), m_\xi(x, t), m_\eta(x, t), m_{\xi\eta}(x, t)]^T. \tag{4}$$

Geometrical relations have the form, [2]

$$\varepsilon(x, t) = \mathscr{B}[f(x, t)], \tag{5}$$

where $\mathscr{B}[...]$ is the linear differential operator of geometrical relations for thin shells of revolution. Stress-strain relations will be written down in the form of the Voltera equation, [3]

$$\sigma(x, t) = \mathscr{F}^*[\varepsilon(x, t)]. \tag{6}$$

For ν = const Eq. (6) can be rewritten in the form

$$\sigma(x, t) = D(\xi)\left[\varepsilon(x, t) - \int_0^t \Gamma(t-\tau)\, \varepsilon(x, \tau) d\tau\right], \tag{7}$$

where $\Gamma(t, \tau) \equiv \Gamma(t-\tau)$ is the relaxation speed function, while $D(\xi)$ is the elasticity matrix for the isotropic linear elastic body.

The equation of motion of the shell will be derived from the law of conservation of the energy. For isothermic problems we have

(8) $$\dot{K}+\dot{U} = \dot{L},$$

where: $\dot{K}$ is the power of the kinetic energy

(9) $$\dot{K} = \int_{\vartheta} [\dot{f}(x, t)]^T \varrho(\xi)\ddot{f}(x, t)d\vartheta,$$

(10) $$\varrho(\xi) = \mathrm{diag}[\varrho_0, \varrho_0, \varrho_0, h^2\varrho_0/12, h^2\varrho_0/12],$$

$\dot{U}$ is the power of the internal energy

(11) $$\dot{U} = \int_{\vartheta} [\dot{\varepsilon}(x, t)]^T\sigma(x, t)dv,$$

and $\dot{L}$ is the power of the external forces

(12) $$\dot{L} = \int_{\vartheta} [\dot{f}(x, t)]^T p(x, t)d\vartheta.$$

3. The algorithm of FEM

Let the shell be divided into a number of the conical finite elements and the lines of nodes coincide with the chosen parallels [2]. The nodal parameters in the node $\alpha(\alpha = 1, 2)$ are represented by the vector (Fig. 2)

(13) $$\delta_\alpha^{(e)}(\eta, t) = [u_\alpha^{(e)}(\eta, t), v_\alpha^{(e)}(\eta, t), w_\alpha^{(e)}(\eta, t), \varphi_\alpha^{(e)}(\eta, t)]^T.$$

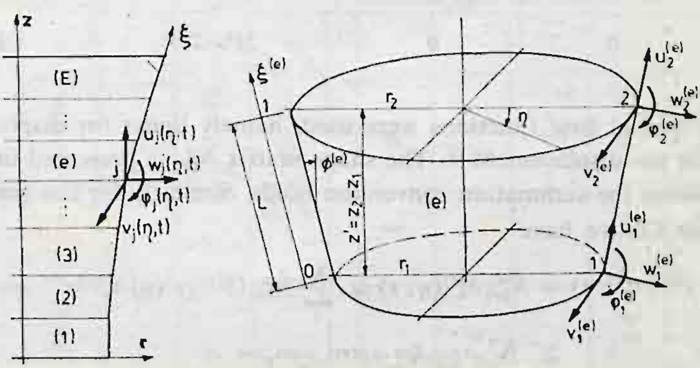

Fig. 2

Taking into account the axial symmetry of the structure, we are to expand the nodal parameters into the Fourier series along the η variable. If the motion is an effect of the periodical excitation with the frequency ω, then

(14) $$\delta_\alpha^{(e)}(\eta, t) = \sum_{l=0}^{\infty} y^l(\eta)\,\delta_\alpha^{l(e)} e^{i\omega t} = \sum_l y^l(\eta)(\bar{\delta}_\alpha^{l(e)} + i\bar{\bar{\delta}}_\alpha^{l(e)})e^{i\omega t},$$

where $y^l(\eta) = \mathrm{diag}\ [c_l, s_l, c_l, c_l]$,

$$c_l = \begin{cases} \cos l\eta \\ \sin l\eta \end{cases}, \quad s_l = \begin{cases} \sin l\eta & \text{for even loads} \\ \cos l\eta & \text{for odd loads} \end{cases}$$

$\bar{\delta}^l_\alpha$, $\bar{\bar{\delta}}^l_\alpha$ — real part and imaginary part (respectively) of the displacement vector of the α node.

Displacements of e-element will be written down in the form

$$(15) \qquad f^{(e)}(x^{(e)}, t) = N^\alpha_{(e)}(\xi^{(e)})\,\delta^{(e)}_\alpha(\eta, t) = \sum_l N^\alpha_{(e)}(\xi^{(e)})\,y^l(\eta)\,\delta^{l(e)}_\alpha e^{i\omega t} =$$

$$= \sum_l z^l(\eta) N^\alpha_{(e)}(\xi^{(e)})\,\delta^{l(e)}_\alpha e^{i\omega t},$$

where: $x^{(e)} = (\xi^{(e)}, \eta) \in \vartheta^{(e)}$,
$z^l(\eta) = \mathrm{diag}\ [c_l, s_l, c_l]$.

Table 1

	$1-\xi$	0	0	0
$N^1_{(e)}(\xi) =$	0	$1-\xi$	0	0
	0	0	$1-3\xi^2+2\xi^3$	$L(\xi-2\xi^2+\xi^3)$
	ξ	0	0	0
$N^2_{(e)}(\xi) =$	0	ξ	0	0
	0	0	$3\xi^2-2\xi^3$	$L(-\xi^2+\xi^3)$

In Eq. (15) typical base functions were used, namely linear for displacements u and v while cubic for the displacement w. The shape matrix $N^\alpha_{(e)}$ is presented in Table 1. Here and in what follows the summation convention holds. Similarly for the generalized vector of displacements (2) we have

$$(16) \qquad \tilde{f}^{(e)}(x^{(e)}, t) = \tilde{N}^\alpha_{(e)}\,\delta^{(e)}_\alpha(\eta, t) = \sum_l \tilde{N}^\alpha_{(e)}(\xi^{(e)})\,y^l(\eta)\,\delta^{l(e)}_\alpha e^{i\omega t} =$$

$$= \sum_l z^l(\eta)\,\tilde{N}^\alpha_{(e)}(\xi^{(e)})\,\delta^{l(e)}_\alpha e^{i\omega t}.$$

Geometrical relations in accordance with (5) have the form

$$(17) \qquad \varepsilon^{(e)}(x^{(e)}, t) = \mathscr{B}[f^{(e)}(x^{(e)}, t)] = \mathscr{B}\Big[\sum_l z^l(\eta) N^\alpha_{(e)}(\xi^{(e)})\,\delta^{l(e)} e^{i\omega t}\Big] =$$

$$= \sum_l z^l(\eta)\,B^{l\alpha}_{(e)}(\xi^{(e)})\,\delta^{l(e)}_\alpha e^{i\omega t},$$

where: $z(\eta) = \mathrm{diag}\ [c_l, c_l, s_l, c_l, c_l, s_l]$.

Table 2

$$
B^{11}_{(e)}(\xi) = \left[\begin{array}{c|c|c|c}
-\dfrac{1}{L} & 0 & 0 & 0 \\
\dfrac{r'}{rL}(1-\xi) & \dfrac{\psi l}{r}(1-\xi) & \dfrac{z'}{rL}(1-3\xi^2+2\xi^3) & \dfrac{z'}{r}(\xi-2\xi^2+\xi^3) \\
-\dfrac{\psi l}{r}(1-\xi) & -\dfrac{r'}{rL}(1-\xi)-\dfrac{1}{L} & 0 & 0 \\
0 & 0 & -\dfrac{1}{L^2}(-6+12\xi) & -\dfrac{1}{L}(-4+6\xi) \\
0 & \dfrac{\psi z' l}{r^2 L}(1-\xi) & -\dfrac{r'}{rL^2}(-6\xi+6\xi^2)+\dfrac{l^2}{r^2}(1-3\xi^2+2\xi^3) & -\dfrac{r'}{rL}(1-4\xi+3\xi^2)+\dfrac{l^2 L}{r^2}(\xi-2\xi^2+\xi^3) \\
0 & -\dfrac{2r'z'}{r^2L^2}(1-\xi)-\dfrac{2z'}{rL^2} & -\dfrac{2r'\psi l}{r^2 L}(1-3\xi^2+2\xi^3)+\dfrac{2\psi l}{rL}(-6\xi+6\xi^2) & -\dfrac{2r'\psi l}{r^2}(\xi-2\xi^2+\xi^3)+\dfrac{2\psi l}{r}(1-4\xi+3\xi^2)
\end{array}\right]
$$

$$
B^{12}_{(e)}(\xi) = \left[\begin{array}{c|c|c|c}
\dfrac{1}{L} & 0 & 0 & 0 \\
\dfrac{r'}{rL}\xi & \dfrac{\psi l}{r}\xi & \dfrac{z'}{rL}(3\xi^2-2\xi^3) & \dfrac{z'}{r}(-\xi^2+\xi^3) \\
0 & -\dfrac{r'}{Lr}\xi+\dfrac{1}{L} & 0 & 0 \\
-\dfrac{rz'}{L^4}\xi & 0 & -\dfrac{1}{L^2}(6-12\xi) & -\dfrac{1}{L}(-2+6\xi) \\
0 & \dfrac{\psi z' l}{r^2 L}\xi & -\dfrac{r'}{rL^2}(6\xi-6\xi^2)+\dfrac{l^2}{r^2}(3\xi^2-2\xi^3) & -\dfrac{r'}{rL}(-2\xi+3\xi^2)+\dfrac{l^2 L}{r^2}(-\xi^2+\xi^3) \\
0 & -\dfrac{2r'z'}{r^2L^2}\xi+\dfrac{z'}{rL^2} & -\dfrac{2r'\psi l}{r^2 L}(3\xi^2-2\xi^3)+\dfrac{2\psi l}{rL}(6\xi-6\xi^2) & -\dfrac{2r'\psi l}{r^2}(-\xi^2+\xi^3)+\dfrac{2\psi l}{r}(-2\xi+3\xi^2)
\end{array}\right]
$$

$r = r(\xi)$; $r' = r_2 - r_1$; $z' = z_2 - z_1$; $L = \sqrt{r'^2 + z'^2}$; $\psi = \begin{cases} 1 & \text{for even loads} \\ -1 & \text{for odd loads.} \end{cases}$

$B^{l\alpha}_{(e)}(\xi^{(e)})$ — geometrical matrix is presented in Table 2.

After the conversion of Eq. (7) and substitution into Eq. (17) we have

$$\text{(18)} \qquad \begin{aligned} \sigma^{(e)}(x, t) &= D(\xi^{(e)})\Big[\varepsilon^{(e)}(x^{(e)}, t) - \int_0^t \Gamma(t-\tau)\,\varepsilon^{(e)}(x^{(e)}, \tau)\,d\tau\Big] = \\ &= D(\xi^{(e)})\,\varepsilon^{(e)}(x^{(e)}, t)(\Gamma_c + i\Gamma_s) = \\ &= (\Gamma_c + i\Gamma_s) D(\xi^{(e)}) \sum_l z^l(\eta) B^{l\alpha}_{(e)}(\xi^{(e)})\,\delta^{l(e)}_\alpha e^{i\omega t}, \end{aligned}$$

where $\Gamma_c = 1 - \int_0^\infty \Gamma(z)\cos\omega z\,dz$, $\Gamma_s = \int_0^\infty \Gamma(z)\sin\omega z\,dz$.

The shell as an assemblage of elements $e = 1, 2, \ldots, E$ occupies the region $\bar{\vartheta}$ which is different from the region ϑ. The total nodal parameters of the structure δ^l_j are related to the local parameters of the element by the transformation

$$\text{(19)} \qquad \delta^{l(e)}_\alpha = \Omega^{j(e)}_\alpha \delta^l_j, \qquad \delta^l_j = \overline{\delta^l_j} + i\overline{\overline{\delta^l_j}},$$

where

$$\text{(20)} \qquad \Omega^{j(e)}_\alpha = \begin{cases} \lambda^{(e)} & \text{for} \quad e + \alpha - 1 = j, \\ 0 & \text{in the all remaining cases,} \end{cases}$$

$$\text{(21)} \qquad \lambda^{(e)} = \begin{bmatrix} \cos\phi^{(e)} & 0 & \sin\phi^{(e)} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi^{(e)} & 0 & \cos\phi^{(e)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Applying the transformation (19) to Eqs. (15), (16), (17) and (18) we arrive at the expressions describing the state of displacement, deformations and stresses at each point of the region v in the form

$$\text{(22)} \qquad \begin{aligned} f(\bar{x}, t) &= \sum_{e=1}^{E} \sum_l z^l(\eta) N^\alpha_{(e)} \Omega^{j(e)}_\alpha \delta^l_j e^{i\omega t} = \sum_l z^l(\eta) N^j(\bar{\xi})\,\delta^l_j e^{i\omega t}, \\ \tilde{f}(\bar{x}, t) &= \sum_e \sum_l z^l(\eta) \tilde{N}^\alpha_{(e)} \Omega^{j(e)}_\alpha \delta^l_j e^{i\omega t} = \sum_l z^l(\eta) \tilde{N}^j(\bar{\xi})\,\delta^l_j e^{i\omega t}, \\ \varepsilon(\bar{x}, t) &= \sum_e \sum_l \bar{z}^l(\eta) B^{l\alpha}_{(e)}(\xi^{(e)}) \Omega^{j(e)}_\alpha \delta^l_j e^{i\omega t} = \sum_l \bar{z}^l(\eta) B^{lj}(\bar{\xi})\,\delta^l_j e^{i\omega t}, \\ \sigma(\bar{x}, t) &= (\Gamma_c + i\Gamma_s) \sum_e D(\xi^{(e)}) \sum_l \bar{z}^l(\eta) B^{l\alpha}_{(e)}(\xi^{(e)}) \Omega^{j(e)}_\alpha \delta^l_j e^{i\omega t} = \\ &= (\Gamma_c + i\Gamma_s) \sum_l D(\bar{\xi}) \bar{z}^l(\eta) B^{lj}(\bar{\xi})\,\delta^l_j e^{i\omega t}, \end{aligned}$$

where $\bar{x} = (\bar{\xi}, \eta)$ and $\bar{\xi}$ — is the coordinate which coincides with the generator of the element.

Substituting RHS of Eqs. (22) into Eq. (8) and the performing simple conversions, we obtain the equation of equilibrium of the structure in the form

$$(\Gamma_c + i\Gamma_s)K^{ljn}\delta_n^l - \omega^2 M^{ljn}\delta_n^l = F^{lj}, \tag{23}$$
$$\text{for } l = 0, 1, 2, \ldots; \quad j = 1, 2, \ldots, E+1;$$

where

$$K^{ljn} = \int_{\vartheta} [\bar{z}^l(\eta) B^{lj}(\bar{\xi})^T] D(\bar{\xi}) \bar{z}^l(\eta) B^{ln}(\bar{\xi}) d\vartheta,$$

$$M^{ljn} = \int_{\vartheta} [z^l(\eta)\tilde{N}^j(\bar{\xi})]^T \varrho(\xi) z^l(\eta)\tilde{N}^n(\bar{\xi}) d\vartheta, \tag{24}$$

$$F^{lj} = \int_{\vartheta} [z^l(\eta) N^j(\bar{\xi})]^T p_0(\bar{\xi}, \eta) d\vartheta.$$

Comparing real and imaginary parts of Eq. (23) we obtain the conjugate system of equations with real roots

$$\begin{aligned} &(\Gamma_c K^{ljn} - \omega^2 M^{ljn})\bar{\delta}_n^l - \Gamma_s K^{ljn}\bar{\bar{\delta}}_n^l = \bar{F}^{lj}, \\ &-\Gamma_s K^{ljn}\bar{\delta}_n^l - (\Gamma_c K^{ljn} - \omega^2 M^{ljn})\bar{\bar{\delta}}_n^l = \bar{\bar{F}}^{lj}, \end{aligned} \tag{25}$$

for $l = 0, 1, 2, \ldots; j = 1, 2, \ldots, E+1$.

Together with Eqs. (25) the suitable displacement boundary conditions have to be taken into account.

4. The numerical example

Forced vibrations of the shell of revolution being the model of the rubber construction of the seal of rotational shafts (Fig. 3) is analysed. The linear viscoelastic body, for which the velocity of relaxation function takes the form

$$\Gamma(t) = Ce^{-\beta t}t^{\alpha-1}, \quad \text{for} \quad t > 0. \tag{26}$$

is assumed as the model of the material.

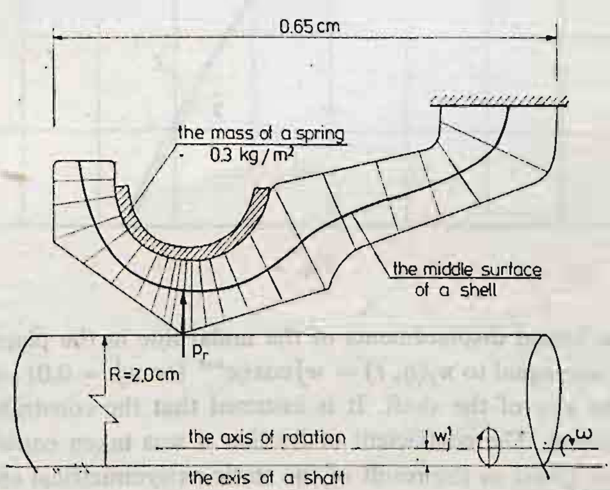

Fig. 3

The material constants were found on the basis of the laboratory tests: $C = 0{,}236$; $\beta = 0{,}01$; $\alpha = 0{,}1$; Youngs modulus of elasticity $E = 9{,}4$ MPa. The mass density of material is equal to $\varrho_0 = 1250$ kg/m³.

The forced displacements of the structure are given by its loading as a result of the eccentric location of the axis of rotational shaft in the relation to the axis of rotation. The motion of the shaft can be decomposed into two simple harmonic motions in two perpendicular directions, displaced in relation to each other by phase angle equal $\pi/2$. Dynamical thrusts of the seal lip on the shift are the most interesting values from the point of view of applications. On Fig. 4 the plot of amplitudes of the unitary thrusts of the seal lip as the function of angular velocity ω is given. The motion of the structure in which different influences are taken into account has been analysed:

a) the motion of the structure in which the mass of the spring and the friction are neglected, (line 1 on Fig. 4),
b) the motion of the structure with the influence of the mass of the spring and without the friction (line 2 on Fig. 4),
c) the motion of the structure with the influence of the mass of the spring and the friction (line 3 on Fig. 4).

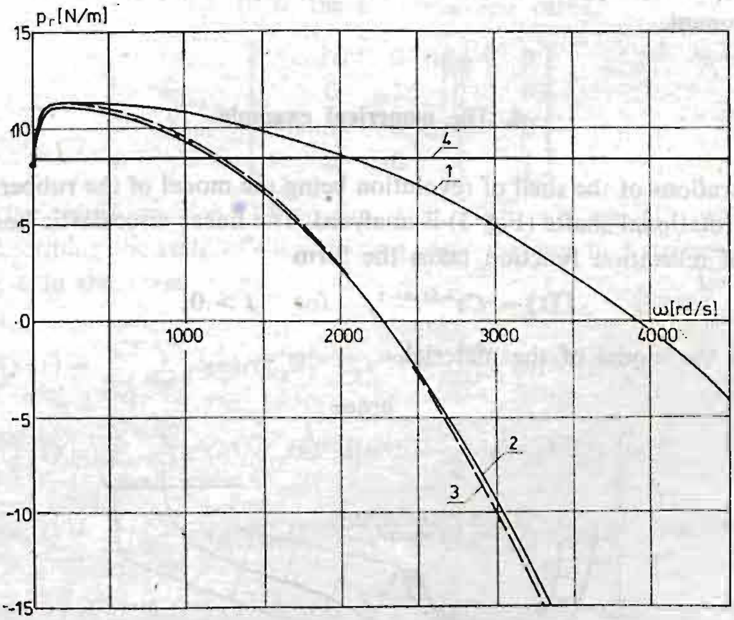

Fig. 4

In each case the forced displacements of the nodal line in the place of contact of the seal with the shaft are equal to $w_j(\eta, t) = w_j^1 \cos \eta e^{i\omega t}$ for $w_j^1 = 0{,}01$ cm, in the direction perpendicular to the axis of the shaft. It is assumed that the constraints in the place of the contact are bilateral. The coefficient of friction μ was taken equal to 0,1. Line 4 on Fig. 4 shows unitary thrust as the result of the static axisymmetrical extension of the seal lip by value $w_j^0 = 0{,}01$ cm.

Literature

1. W. Fluge, *Powłoki, obliczenia statyczne*, Arkady, Warszawa 1972.
2. O. C. Zienkiewicz, *Metoda elementów skończonych*, Arkady, Warszawa, 1972.
3. J. T. Oden, *Finite elements of nonlinear continua*, McGraw-Hill Book Company, New York, 1972.
4. J. N. Rabotnow, *Elementy nasljedstwiennoj mechaniki twierdych tjel*, Nauka, Moskwa, 1977.

Р е з ю м е

АНАЛИЗ ВЯЗКО-УПРУГОЙ ОБОЛОЧКИ ВРАЩЕНИЯ ПЕРИОДИЧЕСКИ НАГРУЖЕННОЙ

В работе анализируются оболочки вращения циклически нагруженные механической нагрузкой или циклически вынужденным перемещением. Модель материала это тело линейно вязко-упругое. Проблемы решаются исходя из уравнений технической теории тонких оболочек пользуясь методом конечных элементов. Для иллюстрации анализируются вынужденные колебания оболочки вращения составляющей модель конструкции резинового уплотнения вращающихся валов.

S t r e s z c z e n i e

ANALIZA LEPKOSPRĘŻYSTEJ POWŁOKI OBROTOWEJ OBCIĄŻONEJ PERIODYCZNIE

W pracy analizowano powłokę obrotową obciążoną cyklicznie obciążeniem mechanicznym lub cyklicznie wymuszonym przemieszczeniem. Modelem materiału jest ciało liniowo lepkosprężyste. Zagadnienie rozwiązano w oparciu o związki technicznej teorii powłok cienkich, posługując się metodą elementów skończonych. Jako ilustrację analizowano drgania wymuszone powłoki obrotowej będącej modelem konstrukcji gumowego uszczelnienia wirujących wałów.

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