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FINITE ELEMENT REPRESENTATION FOR STIFFENED THIN CYLINDRICAL SHELL INSTABILITY ANALYSIS

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1. Introduction

Problems of the elastic instability analysis for thin shell structures are highly complex due to the nonlinear character of the actual buckling mechanism. In general, the instability investigation of such structures may include the solution of the problem of equilibrium bifurcation, and a nonlinear analysis based on tracing the nonlinear load-displacement path and determining singular points of load-displacement behavior see ([1], [2], [3]). The basic problem in the instability investigation of thin shells, therefore, lies in a determination of critical loads related to such points (bifurcation, limit point or the other points of decrease in stiffness). An approach to the resolution of the above-cited problems can be based on the finite element method.

The different levels of a nonlinearity can be considered for thin shell instability analysis. It leads to different numerical problems. A hierarchy of nonlinearity was made clear by MALLETT and MARCAL [4]. The objective of this paper, therefore, is not an extension of the finite element approach to the analysis of all the above-cited problems, but the attention is focused on the formulation of matrices appropriate to instability investigation of stiffened cylindrical shell, using for this purpose a ribbed curved element. Fundamental governing relations to be derived are in the class of geometrically nonlinear formulation. Then, the element is verified by comparing numerical results for the linear, stable analysis to the alternative solution for the same problem. The relevant matrices of the finite element model appropriate to the linear stability analysis are given by the explicit definition [5]. Herein, computational procedures are not developed for the nonlinear analysis. The explicit numerical procedures to be outlined are in the class of "linear bifurcational stability" formulation. The fundamental concept of the element model is, that a set of discrete stiffenings (stringers and rings) is considered in the element stiffness connections. The approach based upon the introduction of the stiffenings from within the finite element model is restricted to the thin and flexible ribs referred to as the second order stiffenings. Assuming that the real structure will be stiffened with large number of such ribs the local buckling is not taken into account. The number of the element ribs may be chosen arbitrarily. The ribs eccentricity is taken into account. Attention is restricted to linearly elastic material behavior.

2. Element geometry and displacement functions

A description of the element geometry is given in Fig. 1. The element consists of thin cylindrical panel and a set of thin flexible ribs. The radius of curvature (R) and thickness (t) are constant. The element nodes are corner points numered from 1 to 4 as in Fig. 1 Let ξ, η represent a set of orthogonal curvilinear coordinates for the mid-surface and ζ the normal coordinate. The coordinates are defined as follows:

$$\xi = \frac{x}{R}, \quad \eta = \frac{s}{R}, \quad \zeta = z, \tag{2.1}$$

where

x — length along the axial direction,
s — arc length along the parametric line η,
z — length along the normal direction.

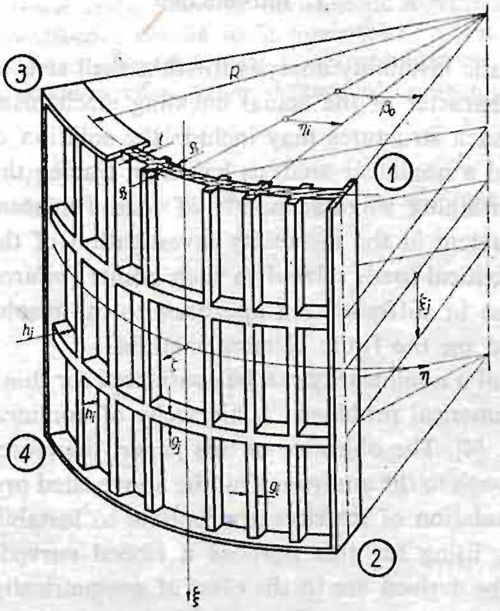

Fig. 1. Element Geometry

As the chosen nodal displacements we take the mid-surface translations u, v, w in the ξ, η, ζ directions and parameters $\frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}, \frac{\partial^2 w}{\partial \xi \partial \eta}$. Therefore, the total number of element degrees of freedom is 24. The displacement functions representing the element behavior are assumed in the form:

$$\begin{aligned} u &= \alpha_1+\alpha_2\xi+\alpha_3\eta+\alpha_4\xi\eta+\alpha_6\sin\eta-\alpha_{10}(\cos\eta-\cos\beta_0), \\ v &= -\alpha_5(1-\cos\eta\cos\beta_0)+\alpha_6\xi\cos\eta+\alpha_7\eta+\alpha_8\xi\eta-\alpha_{10}\xi\sin\eta+\alpha_{11}\cos\eta+\alpha_9\sin\eta, \\ w &= \alpha_9\cos\eta+\alpha_{10}\xi\cos\eta+\alpha_{11}\sin\eta+\alpha_{12}\xi^2+\alpha_{13}\xi\eta+\alpha_{14}\eta^2+\alpha_{15}\xi^3 \\ &\quad+\alpha_{16}\xi^2\eta+\alpha_{17}\xi\eta^2+\alpha_{18}\eta^3+\alpha_{19}\xi^3\eta+\alpha_{20}\xi^2\eta^2+\alpha_{21}\xi\eta^3+\alpha_{22}\xi^3\eta^2 \\ &\quad+\alpha_{23}\xi^2\eta^3+\alpha_{24}\xi^3\eta^3+\alpha_5\sin\eta\cos\beta_0+\alpha_6\xi\sin\eta. \end{aligned} \tag{2.2}$$

Similar functions was previously used by CANTIN and CLOUGH [6] to the linear static analysis of a thin cylindrical shell. The element is nonconforming since the employed displacement functions (2.2) do not satisfy required convergence conditions. ZIENKIEWICZ and CHEUNG [7] summarized conditions to be met by displacement function chosen in the representation of element behavior for the purpose of matrix displacement analyses. Reference [6] examines the violation of the above-mentioned conditions with the reference to a thin cylindrical shell element and, through numerical evaluation, conclude that such functions exhibit convergence for the linear case.

References [8], [9] conduct a similar study. GALLAGHER [1] concludes, that the finite element method, when based on variational principles, requires interelement continuity of derivatives up to one order lower than it appears in the associated functional, or energy. With a reference to the strain displacement relationships the highest derivative to appear in the nonlinear terms is the first. Thus, since the linear terms consist of the second order derivatives, this opens up the possibility of using the same or simplified field for the nonlinear terms as compared to the field used for linear terms. Since functions (2.2) exhibit convergence for the linear case, thus, there is no objection to application such field in nonlinear case and stability analysis.

3. Governing nonlinear equations

Define u and p as the mid-surface translation and external force intensity matrices, ε, ε_i, ε_j as the strain matrices referred to cylindrical panel, stringers and rings, respectively, and σ, σ_i, σ_j as the stress matrices corrpesponding to ε, ε_i, ε_j, expressed by

$$
\begin{gathered}
u = [uvw]^T, \quad p = [p_\xi p_\eta p_\zeta]^T, \\
\varepsilon = [\varepsilon_1^\zeta \varepsilon_2^\zeta \varepsilon_{12}^\zeta]^T, \quad \varepsilon_i = [\varepsilon_1^\zeta 0 \varepsilon_{12}^\zeta]^T, \quad \varepsilon_j = [0 \varepsilon_2^\zeta \varepsilon_{12}^\zeta]^T, \\
\sigma = [\sigma_1^\zeta \sigma_2^\zeta \sigma_{12}^\zeta]^T, \quad \sigma_i = [\sigma_1^\zeta 0 \sigma_{12}^\zeta]^T, \quad \sigma_j = [0 \sigma_2^\zeta \sigma_{12}^\zeta]^T, \\
i = 1, 2, \ldots, n. \quad j = 1, 2, \ldots, m.
\end{gathered} \tag{3.1}
$$

where

n — total number of element stringers,
m — total number of element rings,
ε_1^ζ, ε_2^ζ — extensional strains in ξ, η directions,
ε_{12}^ζ — shear strain.

The stress components are shown in Fig. 2.

Let v be the strain energy per unit area of the middle surface, expressed by

$$
v = V_0 + \sum_{i=1}^{n} V_i + \sum_{i=1}^{m} V_j \tag{3.2}
$$

where

V_0 — cylindrical panel strain energy density,
V_i — stringer strain energy density,
V_j — ring strain energy density.

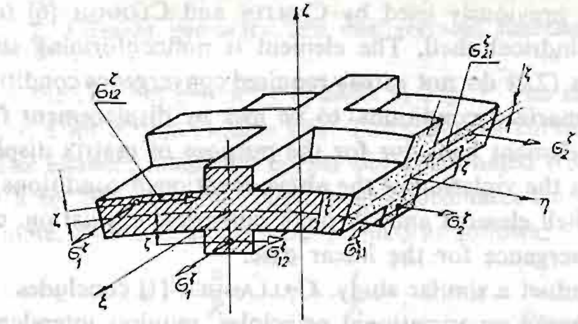

Fig. 2. Stress Components

The governing eguilibrium equations are obtained by applying the Principle of Virtua Displacements which requires

$$(3.3)\qquad \int_A \delta V_0 dA + \sum_{i=1}^{n} \int_{A_i} \delta V_i dA_i + \sum_{j=1}^{m} \int_{A_j} \delta V_j dA_j = \int_A \delta u^T p dA,$$

and has to be satisfied for an arbitrary admissible variation δu. The formulae for the strain energy density can be written as follows

$$(3.4)\qquad V_0 = \frac{1}{2}\int_{\zeta} \varepsilon^T \sigma\left(1+\frac{\zeta}{R}\right) d\zeta, \qquad V_{\substack{i\\j}} = \frac{1}{2}\int_{\zeta} \varepsilon_{\substack{i\\j}}^T \sigma_{\substack{i\\j}}\left(1+\frac{\zeta}{R}\right) d\zeta.$$

The derived nonlinear strain — displacement relationships constitute a generalization of those due to NOVOZHILOV [10]:

$$(3.5)\qquad \begin{aligned} \varepsilon_1^{\zeta} &= \frac{1}{R}\frac{\partial u}{\partial \xi} + \frac{1}{2R^2}\left(\frac{\partial w}{\partial \xi}\right)^2 - \zeta\frac{1}{R^2}\frac{\partial^2 w}{\partial \xi^2}, \\ \varepsilon_2^{\zeta} &= \frac{1}{R}\frac{\partial v}{\partial \eta} + \frac{w}{R} + \frac{1}{2R^2}\left(\frac{\partial w}{\partial \eta}\right)^2 - \zeta\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \eta^2} - \frac{\partial v}{\partial \eta}\right), \\ \varepsilon_{12}^{\zeta} &= \frac{1}{R}\frac{\partial v}{\partial \xi} + \frac{1}{R}\frac{\partial u}{\partial \eta} + \frac{1}{R^2}\frac{\partial w}{\partial \xi}\frac{\partial w}{\partial \eta} - \zeta\frac{2}{R^2}\left(\frac{\partial^2 w}{\partial \xi \partial \eta} - \frac{\partial v}{\partial \xi}\right). \end{aligned}$$

After restricting our attention to the linearly elastic material behavior, the dependence of stress upon strain is assumed to be governed by

$$(3.6)\qquad \sigma = D(\varepsilon - \varepsilon^0), \qquad \sigma_i = E_i(\varepsilon_i - \varepsilon_i^0), \qquad \sigma_j = E_j(\varepsilon_j - \varepsilon_j^0),$$

in which ε^0, ε_i^0, ε_j^0 contain the initial strains. The rigidity matrices are defined as follows

$$(3.7)\qquad D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \qquad E_{\substack{i\\j}} = E_{\substack{i\\j}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2(1+\nu_{\substack{i\\j}})} \end{bmatrix}$$

where

E, E_i, E_j — elastic modulus of cylindrical panel and ribs, respectively

ν, ν_i, ν_j — Poisson's ratios of panel and ribs.

It is convenient to segregate the nonlinear terms in the expression for strains, thus, the strain matrices ε, ε_i, ε_j can be written as follows

$$\varepsilon = e+\epsilon+\zeta k, \quad \varepsilon_{\substack{i\\j}} = e_{\substack{i\\j}}+\epsilon_{\substack{i\\j}}+\zeta k_{\substack{i\\j}}, \tag{3.8}$$

in which e, e_i, e_j contain the linear terms of stretching deformations, $\epsilon, \epsilon_i, \epsilon_j$ contain the nonlinear terms and k, k_i, k_j contain the curvature changes. Substituting for u we obtain from Eq. (2.2)

$$u = A\alpha, \tag{3.9}$$

where A contains prescribed functions of the local coordinates and α contains unknown parameters α_i. Substituting this relation into Eq. (3.8) we are able to express the relevant matrices in the following form

$$\begin{aligned} e &= B\alpha, & e_{\substack{i\\j}} &= B_{\substack{i\\j}}\alpha,\\ \epsilon &= H\alpha, & \epsilon_{\substack{i\\j}} &= H_{\substack{i\\j}}\alpha,\\ k &= G\alpha, & k_{\substack{i\\j}} &= G_{\substack{i\\j}}\alpha. \end{aligned} \tag{3.10}$$

Note that H, H_i, H_j depend on α whereas B, B_i, B_j, G, G_i, G_j are independent of α. The relevant variations are

$$\begin{aligned} \delta e &= B\delta\alpha & \delta e_{\substack{i\\j}} &= B_{\substack{i\\j}}\delta\alpha\\ \delta\epsilon &= 2H\delta\alpha & \delta\epsilon_{\substack{i\\j}} &= 2H_{\substack{i\\j}}\delta\alpha\\ \delta k &= G\delta\alpha & \delta k_{\substack{i\\j}} &= G_{\substack{i\\j}}\delta\alpha\\ \delta u &= A\delta\alpha \end{aligned} \tag{3.11}$$

Substituting Eqs. (3.10) and (3.11) into Eq. (3.3) and integrating over the element we obtain

$$\delta\alpha^T(\bar{k}_n\alpha+\bar{P}_n^0) = \delta\alpha^T\bar{P}_n. \tag{3.12}$$

Eq. (3.12) must be satisfied for an arbitrary admissible $\delta\alpha$, the equality of coefficients, therefore, leads to the equilibrium equations

$$\bar{k}_n\alpha+\bar{P}_n^0 = \bar{P}_n, \tag{3.13}$$

where k_n is the nonlinear stiffness matrix. The matrices $\bar{P}_n^0$, and $\bar{P}_n$ depend on the initial and external loadings, respectively. The above matrices are given explicit definitions in dissertation [5]. It remains to express the element parameters, α, in terms of the nodal displacement. Define u_l as the displacement matrix for node „l":

$$u_l = \left[u\ v\ w\ \frac{\partial w}{\partial\xi}\ \frac{\partial w}{\partial\eta}\ \frac{\partial^2 w}{\partial\xi\partial\eta}\right]^T, \quad l = 1,2,3,4. \tag{3.14}$$

Let U denote the matrix containing the nodal displacement matrices u_l listed in a prescribed order

$$U = [u_1\, u_2\, u_3\, u_4]^T. \tag{3.15}$$

Substituting Eq. (3.9) into the foregoing expression leads to the relation defining transformation matrix C

(3.16) $$U = C\alpha,$$

Performing this transformation on Eq. (3.12) we obtain

(3.17) $$k_n U + P_n^0 = P_n,$$

where

$$k_n = (C^{-1})^T k_n C^{-1}, \quad P_n^0 = (C^{-1})^T P_n^0, \quad P_n = (C^{-1})^T P_n.$$

The system of equations for the complete structure can be obtained in the known manner (see [11]). Defining r as the system nodal displacement matrix while R_n^0, R_n as the initial and external loading matrices, respectively, we can express the total system equations

(3.18) $$K_n r + R_n^0 = R_n,$$

where K_n is the total nonlinear stiffness matrix.

4. Generation of tangent stiffness matrix

Herein, the definition of tangent stiffness matrix is obtained by applying the Trefftz criterion ([12], [13]). A necessary and sufficient condition for the stability of the prebuckled state is the existence of some nonvanishing but infinitesimally close perturbed configuration in which the energy increment is always non-negative. The critical point may be characterized by a positive, semidefinite, second variation of the potential energy. Thus, at the critical load there exist nonzero virtual displacements for which the second variation in the total potential energy vanishes. The total potential energy may be expressed as follows

(4.1) $$\Pi = V + \Omega$$

where V is the strain energy and Ω is the potential energy of the external load. If the external loading is considered to be independent of the displacements, $\delta^2\Pi$ reduced to $\delta^2 V$ and Trefftz criterion may be written as $\delta^2 V \geqslant 0$. Thus, the attention is turned here to the formulation of the second variation of this portion of the potential energy. According to Eqs. (3.2) and (3.3) the second variation of „V" can be expressed as

(4.2) $$\delta^2 V = \int_A \delta^2 V_0 dA + \sum_{i=1}^{n} \int_{A_i} \delta^2 V_i dA_i + \sum_{j=1}^{m} \int_{A_j} \delta^2 V_j dA_j,$$

Expanding the right-hand side od Eq. (4.2) we obtain

(4.3) $$\delta^2 V = \delta\alpha^T \bar{k}_T \delta\alpha,$$

where $\bar{k}_T$ is the element tangent stiffness matrix. The expression for $\bar{k}_T$ takes the form

(4.4) $$\bar{k}_T = \bar{k} + \bar{k}_\sigma + \bar{k}_l,$$

where $\bar{k}$ is the linear stiffness matrix, $\bar{k}_\sigma$ is refered to as the initial stress matrix and $\bar{k}_l$ is the matrix of large deformations. The matrices are given explicit definitions in dissertation [5]. Applying the transformation (3.16) to Eq. (4.3) we obtain

$$\delta^2 V = \delta U^T k_T \delta U. \tag{4.5}$$

Therefore, the element tangent stiffness matrix k_T takes the form

$$k_T = (C^{-1})^T \bar{k}_T C^{-1}, \tag{4.6}$$

the total tangent stiffness matrix for complete structure can be obtained in a known manner

$$K_T = \sum_g a_g^T k_T a_g, \tag{4.7}$$

where a_g is a Boolean matrix and „g" element number. According to Eq. (4.4) the total tangent stiffness matrix takes the form

$$K_T = K + K_\sigma + K_l \tag{4.8}$$

5. Calculation of bifurcation state

Herein, the previously outlined finite element formulation is applied to the linearized analysis of a bifurcation state of the shell. Since the external loading is considered to be independent of the displacement, the application of the Trefftz criterion to the complete structure leads to the relation $\delta^2 V_C \geq 0$ where „V_C" is the total strain energy of the system. The second variation of the total strain energy may be expressed as follows

$$\delta^2 V_C = \delta r^T K_T \delta r. \tag{5.1}$$

In the linearized stability analysis, the prebuckled geometry corresponds to the undeformed (initial) geometry. This assumption is introduced by disregarding the second order nonlinear terms, therefore, the matrix K_l is neglected. The total tangent stiffness matrix reduces to

$$K_T = K + K_\sigma \tag{5.2}$$

where K is the linear stiffness matrix and K_σ depends on the applied loading. If the distribution of internal forces in the structure does not change along the fundamental path the matrix K_σ may be assumed to vary linearly with the load level. If the loading R_0 is arbitrary, first we obtain the linear solution r_0 and then we generate K_σ. When the loading can be specified in terms of a single parameter, say λ, then K_σ can be written as

$$K_\sigma(r) = \lambda K_\sigma(r_0), \tag{5.3}$$

moreover

$$K_T = K + \lambda K_\sigma(r_0). \tag{5.4}$$

According to Eqs. (5.1) and (5.4) the bifurcation problem can be interpreted as a generalized eigenvalue problem, expressed by

$$K\delta r - \lambda K_\sigma \delta r = 0 \tag{5.5}$$

Many research efforts have been devoted to solve the foregoing problem (see [14], [15], [16], [17]). Therefore, two independent computational procedures have been developed for this purpose. The first procedure represents application of the Householder-Cholesky method, the second one, based upon Peters-Wilkinson approach may be reffered to as the trial method. The basis of the trial method may be stated as follows:

> "If the matrix K is positive definite then the number of eigenvalues of the problem (5.5) smaller than the chosen trial value of parameter λ, say λ_0, equals to the number of negative diagonal elements of the top triangular matrix obtained from $(K-\lambda_0 K_\sigma)$ by the Gauss elimination"

The number of negative diagonal elements displaying during the trial process enables sensible selection the succeeding values of parameter λ_0 during the calculation. These values can be introduced into computer storage simply via monitor. The trial process advances iteratively until the desirable accuracy of solution is accomplished.

6. Numerical examples and accuracy comparisons

In this section the previously described element is verified by comparing numerical results for the linear stability analysis to the alternative classical solution for the same problem. The basis for appropriate computations are the computer programs created in accordance with the methods applied to resolution of the eigenvalue problem. These programs were coded in FORTRAN for the computers ODRA 1305 and ODRA 1325. The explicit listings and a concise flow charts of these programs are given in dissertation [5]. The first example considered was a simply supported along the longitudinal edges,

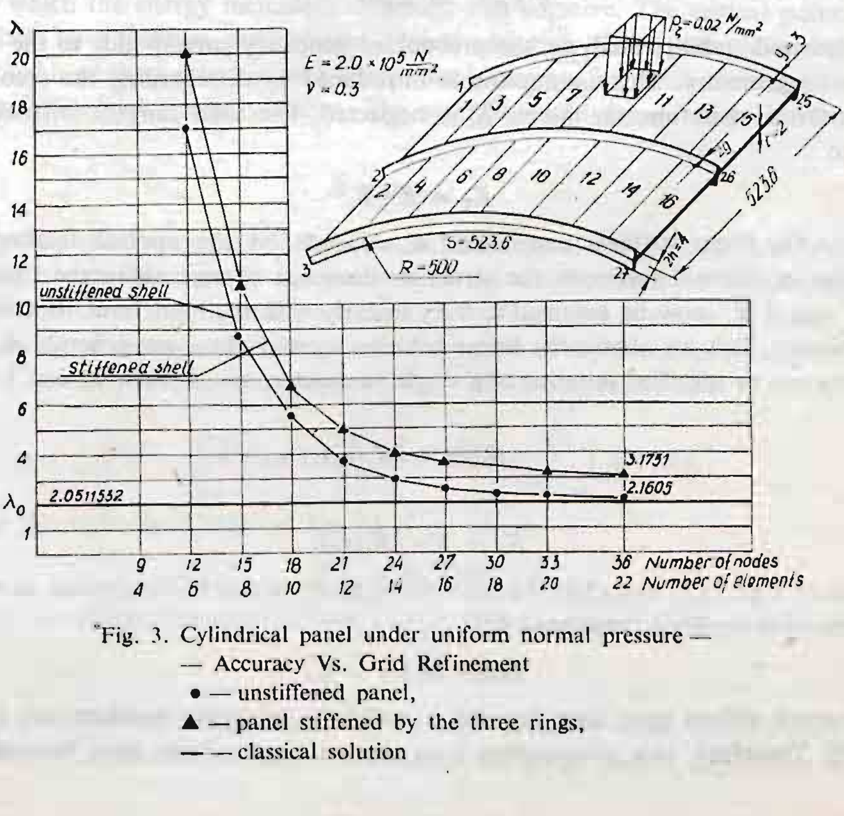

Fig. 3. Cylindrical panel under uniform normal pressure —
— Accuracy Vs. Grid Refinement
• — unstiffened panel,
▲ — panel stiffened by the three rings,
— — classical solution

and free along the circular ends unstiffened cylindrical panel subjected to uniform normal pressure. Fig. 3 shows a comparison of the finite-element solution with the classical solution [18]. The same cylindrical panel stiffened, by the three rings (see Fig. 3) was studied next. Fig. 3 indicates a study of the convergence of the present method. The diagram shows that the non-dimensional buckling load „λ" converges guite rapidly. The difference between the present result and classical solution is approximately 5% for 22 elements.

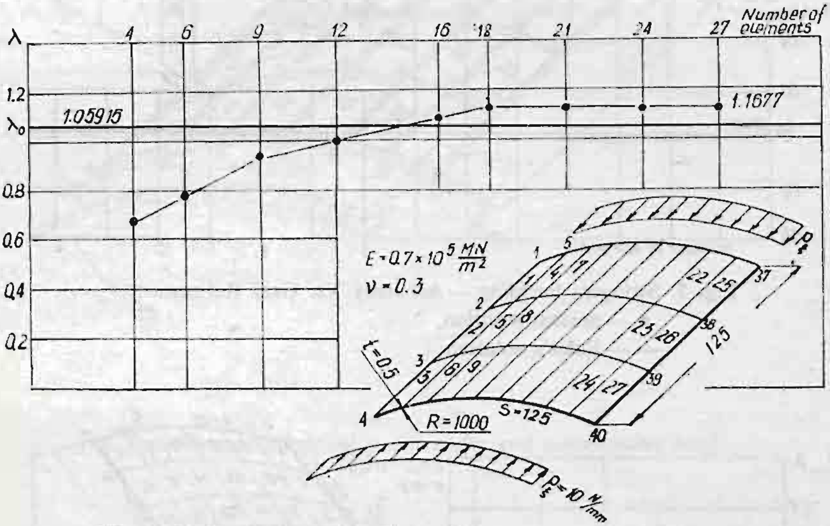

Fig. 4. Cylindrical panel under uniform compressive load — — Accuracy Vs. Grid Refinement
• — present solution,
— — clasical solution

Figure 4 shows a simply supported on all four sides cylindrical panel under uniform end compressive load p_ξ. The non-dimensional buckling load is plotted against the number of discrete elements. The result for employed class of the finite element mesh division is compared with the classical Timoshenko's solution [18]. In this case the difference between the present result and those of Reference [18] is approximately 10%. The diagram shows that the buckling factor „λ" converges to value higher then those of Timoshenko, however, a finer, in longitudinal direction, type of the grid can be used, giving more accurate answer.

Another problem examined is the buckling of a set of stringers supported and loaded as shown in Fig. 5. The results prove to be in close agreement with the classical Euler's solution.

Fig. 6 indicates a study of the convergence of the present solution for the simply supported on all four sides cylindrical panel in the shear conditions. Volmir [19] has analyzed this problem using Galerkin's method. Volmir's results are compared with results of the present method. Ref. [19] predicts higher loads in comparison with "exact" solution, whereas the present method gives lower buckling loads. The difference between the results of foregoing solutions is approximately 8% for 24 elements.

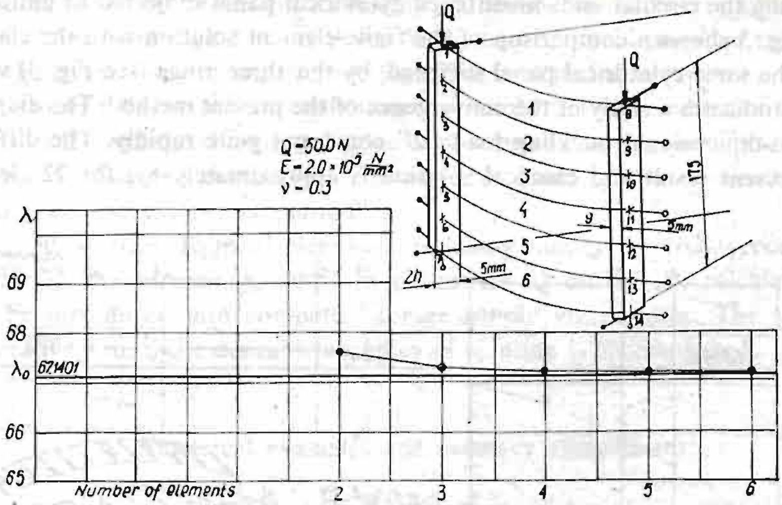

Fig. 5. Stringers buckling — Accuracy Vs. Grid Refinement
● — present solution,
— — Euler's solution

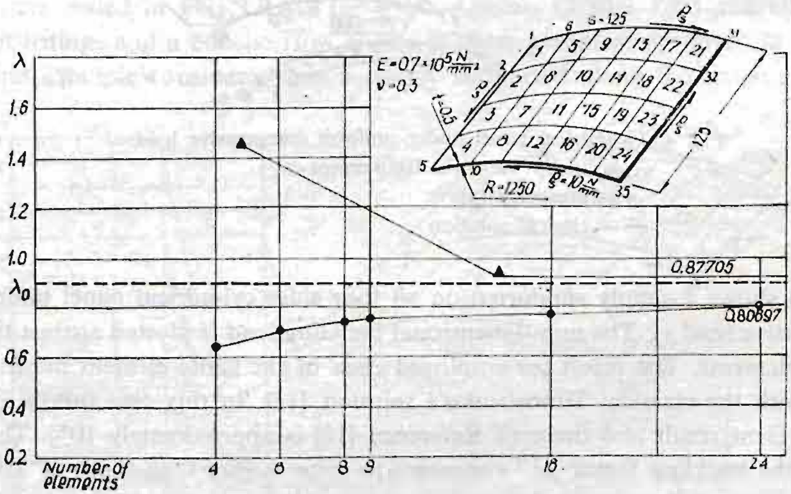

Fig. 6. Cylindrical panel in the shear conditions —
— Accuracy Vs. Grid Refinement
● — present solution,
▲ — Galerkin's method

Finally, a stiffened by 7 stringers and 5 rings cylindrical panel under uniform end compressive load is considered (see Fig. 7). The panel is simply supported on all sides. In the discrete element analysis, employing 24 elements the result obtained is $\lambda = 5.46066$. For this case the solution for Householder's method was compared with the employed trial method (see Ref. [20]). The results of the trial method prove to be in close agreement with Householder's solution also in the other cases examined [5].

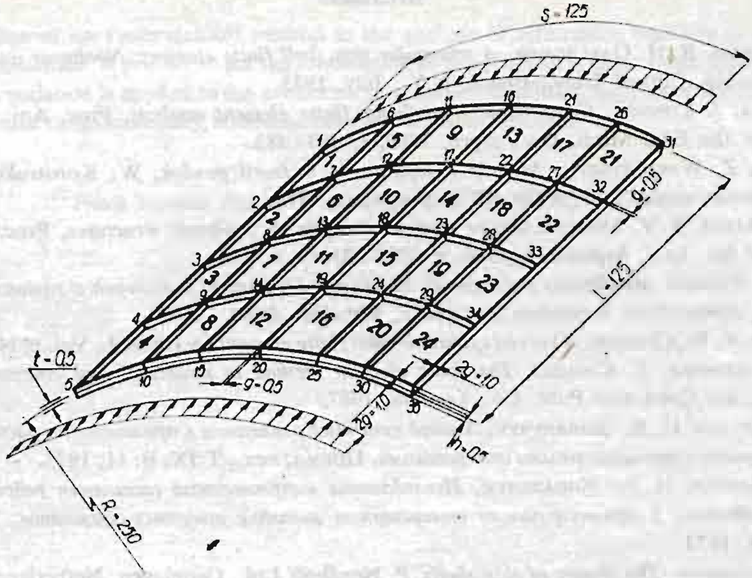

Fig. 7. Stiffened cylindrical panel under end compressive load

7. Conclusions

It is clear from the outlined convergence study, that present method assures convergence for the basic cases of loading. The convergence characteristics depend on these cases of loading. Appropriate curves can converge from opposite directions. It arises from the nature of the assumed displacement field. Namely, displacement function for the normal translation „w" satisfies the continuity condition, whereas the simpler functions for the „u" and „v" components are assumed. It causes that the violations of the continuity conditions for "in plane" translations are present in the element representation. If the normal translation is the dominant component in prebuckled state, the convergence characteristics, therefore, converge from the top direction. On the other hand if the "in plane" translations are in prebuckled state the dominant components the convergence occurs from the bottom direction. Ref. [5] gives more detailed convergence and accuracy analysis. Fairly good agreement was observed between the results of applied methods of solution of the buckling eigenvalue problem in all examined cases. The trial method seems to be very effective in such a class of the eigenvalue problem. Finally, it should be pointed out that discrepancies between the theoretically predicted classical bifurcation buckling loads and test results are always expected for thin-shell structures. The reason is that buckling of thin shells is very sensitive to initial imperfections. More accurate results may be obtained by applying nonlinear stability analysis. The derived nonlinear terms enable an extension of the present formulation for investigation of the nonlinear instability effects of the stiffened cylindrical shells.

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Резюме

КОНЕЧНЫЙ ЭЛЕМЕНТ В РАСЧЕТАХ УСТОЙЧИВОСТИ РЕБРИСТЫХ ЦИЛИНДРИЧЕСКИХ ОБОЛОЧЕК

В работе представлено использование метода конечных элементов в расчетах устойчивости ребристых цилиндрических оболочек. Для исследования устойчивости использован критерий Трефца. Алгоритм применяется в расчете общей потери устойчивости. В работе построены нелинейные матрицы конечного элемента.

Summary

An extension of the finite element method to the analysis of bifurcation buckling of cylindrical stiffened shells is presented. A procedure for the formulation of the problem is based upon the Trefftz criterion. The present formulation is applied to the prediction of general instabilities. Aspects of the element formulation which pertain to geometrically nonlinear behavior are also described.

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