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ON LINEAR THEORY OF ANISOTROPIC SHELLS OF MODERATE THICKNESS

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1. Introduction

The classical theory of thin elastic shells often referred as the Kirchhoff-Love (KL) theory may be unsatisfactory in some problems of practical importance such as moderately thick shells, shells with short wave length of the deformation pattern of the middle surface, shells with a high degree of anisotropy, etc. In this paper, limited to the statics of shells, such problems are considered within the framework of a theory engaging six unknown kinematical parameters defined on the midsurface. The theory called in the sequel the six parameter (SP) theory is based on the assumption of linear distribution of the displacement vector across the thickness, previously used in [1 - 4]. To a similar theory lead the supposition that the deformation in a vicinity of the middle surface is homogenous [5] and the concept of a Cosserat surface [6]. In the present paper the basic equations of SP are derived from the equations of three-dimensional elasticity via variational approach. Then the range of applicability and the accuracy of SP are investigated by evaluation of the strain energy density. Since SP proves not to be generally consistent with respect to the strain energy approximation it may only be useful in specific problems or in a limited region of a shell. As an illustration to this conclusion a numerical example is given concerning the rotationally-symmetric bending of an isotropic circular cylindrical shell loaded by an abruptly changing normal pressure.

2. Basic equations

Let us consider a shell of constant thickness h parametrized by usual normal coordinate system $\{x^k\} = \{x^\alpha, x^3 = z\}$ with the z axis perpendicular to the middle surface coordinate lines $\{x^\alpha\} = \{x^1, x^2\}$. In above and in the sequel the Latin and the Greek indices range over the integers $\{1, 2, 3\}$ and $\{1, 2\}$, respectively. Components of tensors related to the local basis on the middle surface ($z = 0$) and on an arbitrary surface ($z =$ const.) are accordingly distinguished by the indices $\{i, j, k, l, p, q; \alpha, \beta, \lambda, \eta\}$ and $\{a, b, c, d; \varphi, \psi\}$. Indices preceded by a comma and by a vertical stroke denote partial and surface covariant derivatives in the middle surface metrix. The Kronecker symbols are denoted by δ^i_a, δ^β_α, etc., b^β_α stands for the mixed components of the second metric tensor of the midsurface, H and K are the mean and Gaussian curvatures of that surface. The translators

μ^i_a, μ^a_i we define as composed of the above listed midsurface tensors

$$\mu^\alpha_\varphi = \delta^\beta_\varphi(\delta^\alpha_\beta - z b^\alpha_\beta), \qquad \mu^\varphi_\alpha = (1/\mu)\,\delta^\varphi_\beta[\delta^\beta_\alpha + z(b^\beta_\alpha - 2H\delta^\beta_\alpha)],$$
$$\mu^3_3 = 1, \qquad \mu^\alpha_3 = \mu^3_\alpha = 0, \qquad \mu = 1 - 2zH + z^2 K. \tag{2.1}$$

The basic equations of SP can be easily derived from the three-dimensional equations of elasticity. Starting from the HU-WASHIZU [7] variational theorem one only has to assume a distribution of the displacement $U_i(x^k)$ and the deformation $e_{ab}(x^k)$ across the shell thickness. Let us adopt for these quantities the following power series expansions

$$U_i(x^k) = w_i + z\beta_i + \underline{z^2\delta_i} + \ldots,$$
$$2e_{\varphi\psi}(x^k) = (\mu^\alpha_\varphi\,\delta^\beta_\psi + \mu^\alpha_\psi\,\delta^\beta_\varphi)(\gamma_{\alpha\beta} + z\varkappa_{\alpha\beta} + \underline{z^2\mu_{\alpha\beta}} + \ldots),$$
$$2e_{\varphi 3}(x^k) = \delta^\alpha_\varphi(\gamma_{3\alpha} + z\varkappa_{3\alpha} + \underline{z^2\mu_{3\alpha}} + \ldots), \quad e_{33}(x^k) = \gamma_{33} + \underline{z\varkappa_{33}} + \ldots, \tag{2.2}$$

involving six generalized middle surface displacements w_i and β_i and thirteen middle surface and its vicinity strains $\gamma_{\alpha\beta}$, $\varkappa_{\alpha\beta}$, $\gamma_{3\alpha}$, $\varkappa_{3\alpha}$ and γ_{33}; the underlined terms in (2.2) should be omitted throughout as far as SP is concerned.

Introduction of the hypotheses (2.2) into the three-dimensional Hu-Washizu functional [7] results in the following two-dimensional Hu-Washizu functional of SP

$$J = \int_\tau \{-N^{ij}\gamma_{ji} - M^{\alpha j}\varkappa_{j\alpha} + (1/2)\,{}_0B^{ijkl}\gamma_{ij}\gamma_{kl} + {}_1B^{ijk\alpha}\gamma_{ij}\varkappa_{k\alpha}$$
$$+ (1/2)\,{}_2B^{\alpha i\beta j}\varkappa_{\alpha i}\varkappa_{\beta j} + N^{\beta\alpha}(w_{\alpha|\beta} - b_{\alpha\beta}w_3) + N^{3\alpha}(\beta_\alpha + w_{3,\alpha} + b^\lambda_\alpha w_\lambda)$$
$$+ N^{33}\beta_3 + M^{\beta\alpha}(\beta_{\alpha|\beta} - b_{\alpha\beta}\beta_3) + M^{\alpha 3}\beta_{3,\alpha} - q^i w_i - m^i\beta_i\}\,d\tau$$
$$- \int_{\partial\tau} (\overset{*}{N}{}^{\alpha i}w_i + \overset{*}{M}{}^{\alpha i}\beta_i)\nu_\alpha\,ds - \int_{\partial\tau} [N^{\alpha i}(w_i - \overset{*}{w}_i) + M^{\alpha i}(\beta_i - \overset{*}{\beta}_i)]\nu_\alpha\,ds, \tag{2.3}$$

defined on the middle surface τ with the edge $\partial\tau$ and ν_α — the outward unit vector normal to $\partial\tau$; the starred quantities are prescribed on $\partial\tau$. Appearing in (2.3) the stress resultants N^{ij} and couples $M^{i\alpha}$, the stiffness tensors ${}_nB^{ijkl}$ and the reduced loadings q^i and m^i are defined as follows

$$\{N^{ij}, M^{ij}\} = \int_{-h/2}^{+h/2} \mu\sigma^{ab}\delta^i_a\mu^j_b\{1, z\}\,dz, \tag{2.4}$$

$${}_nB^{ijkl} = C^{ipkq}\delta^j_a\delta^l_b \int_{-h/2}^{+h/2} \mu\mu^a_p\mu^b_q z^n\,dz, \qquad {}_nB^{\alpha\beta\lambda 3} = {}_nB^{333\alpha} = 0,$$
$${}_nB^{ijkl} = {}_nB^{klij}, \qquad n = 0, 1, 2., \tag{2.5}$$

$$\{q^i, m^i\} = \int_{-h/2}^{+h/2} \mu\mu^i_a F^a\{1, z\}\,dz + [\mu\mu^i_a\sigma^{a3}\{1, z\}]^{+h/2}_{-h/2}, \tag{2.6}$$

where $\sigma^{ab}(x^k)$ is the stress tensor, $C^{ipkq}(x^\alpha)$ the elasticity tensor valid for shells having symmetry of elastic properties relative to the surfaces z = const. (e.g. orthotropic shells), $F^a(x^k)$ denotes the density of the mass forces.

By requiring the functional (2.3) to be stationary under arbitrary variations of w_i, $\beta_i, \gamma_{ij}, \varkappa_{i\alpha}, N^{ij}$ and $M^{i\alpha}$ one obtains the basic equations of SP, to wit:
the geometric eqs.

$$
\begin{aligned}
&\gamma_{\alpha\beta} = w_{\alpha|\beta} - b_{\alpha\beta} w_3, \quad \gamma_{\alpha 3} = \beta_\alpha + w_{3,\alpha} + b^\lambda_\alpha w_\lambda, \quad \gamma_{33} = \beta_3,\\
&\varkappa_{\alpha\beta} = \beta_{\alpha|\beta} - b_{\alpha\beta}\beta_3, \quad \varkappa_{\alpha 3} = \beta_{3,\alpha},
\end{aligned}
\tag{2.7}
$$

the equations of equilibrum

$$
\begin{aligned}
&N^{\beta\alpha}{}_{|\beta} - b^\alpha_\lambda N^{\lambda 3} + q^\alpha = 0, \quad N^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta} N^{\beta\alpha} + q^3 = 0,\\
&M^{\beta\alpha}{}_{|\beta} - N^{\alpha 3} + m^\alpha = 0, \quad M^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta} M^{\beta\alpha} - N_{33} + m^3 = 0,
\end{aligned}
\tag{2.8}
$$

the constitutive eqs.

$$
\begin{aligned}
N^{\beta\alpha} &= {}_0B^{\alpha\beta\lambda\eta}\gamma_{\lambda\eta} + {}_1B^{\alpha\beta\lambda\eta}\varkappa_{\lambda\eta} + {}_0B^{\alpha\beta 33}\gamma_{33},\\
M^{\beta\alpha} &= {}_1B^{\alpha\beta\lambda\eta}\gamma_{\lambda\eta} + {}_2B^{\alpha\beta\lambda\eta}\varkappa_{\lambda\eta} + {}_1B^{\alpha\beta 33}\gamma_{33},\\
N^{\alpha 3} &= {}_0B^{\alpha 3\lambda 3}\gamma_{\lambda 3} + {}_1B^{\alpha 3\lambda 3}\varkappa_{\lambda 3}, \quad M^{\alpha 3} = {}_1B^{\alpha 3\lambda 3}\gamma_\lambda + {}_2B^{\alpha 3\lambda 3}\varkappa_{\lambda 3},\\
N^{33} &= {}_0B^{3333}\gamma_{33} + {}_0B^{\alpha\beta 33}\gamma_{\alpha\beta} + {}_1B^{\alpha\beta 33}\varkappa_{\alpha\beta}
\end{aligned}
\tag{2.9}
$$

and the natural boundary conditions

$$
N^{\alpha i}\nu_\alpha = \overset{*}{N}{}^{\alpha i}\nu_\alpha, \quad M^{\alpha i}\nu_\alpha = \overset{*}{M}{}^{\alpha i}\nu_\alpha, \quad w_i = \overset{*}{w}_i, \quad \beta_i = \overset{*}{\beta}_i.
\tag{2.10}
$$

Six equations of equilibrum (2.8) can be readily expressed in terms of six generalized displacements w_i and β_i by subsequent usage of (2.9) and (2.7). The total order of the resulting differential equations amounts twelve in accordance with the number of boundary conditions (2.10).

Having solved the two-dimensional equations (2.7) - (2.10) one may seek an approximation to the exact distributions of the displacement and stress across the shell thickness. This problem cannot, of course, be answered uniquely. For example, displacements can be calculated from our original hypothesis $(2.2)_1$. This linear distribution is undoubtely the simplest possible but as shown in [8] not the most adequate. It is natural, that the stress distribution should from practical point of view be similar to that occuring in rods and plates. Furthermore, it ought to satisfy [6] the definition (2.4) of the stress resultants and couples, and the static boundary conditions at the shell faces $z = \pm h/2$. The following distributions

$$
\begin{aligned}
\mu\delta^\beta_\varphi \mu^\lambda_\varphi \sigma^{\varphi\psi}(x^k) &= N^{\beta\lambda}/h + (12z/h^3)M^{\beta\lambda},\\
\mu\delta^\alpha_\varphi \sigma^{\varphi 3}(x^k) &= (N^{\alpha 3}3/2h + M^{\alpha 3}30z/h^3)[1-(2z/h)^2]\\
&\quad -(1/4)\{\overset{+}{\mu}\overset{+}{\sigma}{}^{\varphi 3}\delta^\alpha_\varphi[1+3(2z/h)-3(2z/h)^2-5(2z/h)^3]\\
&\quad +\overset{-}{\mu}\overset{-}{\sigma}{}^{\varphi 3}\delta^\alpha_\varphi[1-3(2z/h)-3(2z/h)^2+5(2z/h)^3]\},\\
\mu\sigma^{33}(x^k) &= (3/2h)[1-(2z/h)^2]N^{33} - (1/4)\{\overset{+}{\mu}\overset{+}{\sigma}{}^{33}[1-3(2z/h)-3(2z/h)^2+(2z/h)^3]\\
&\quad +\overset{-}{\mu}\overset{-}{\sigma}{}^{33}[1+3(2z/h)-3(2z/h)^2-(2z/h)^3]\},
\end{aligned}
\tag{2.11}
$$

possess the expected properties, where

$$
\begin{aligned}
&\overset{+}{\mu} = \mu(z = h/2), \quad \overset{+}{\mu} = \mu(z = -h/2), \quad \overset{+}{\sigma}{}^{a3} = \sigma^{a3}(z = h/2),\\
&\overset{-}{\sigma}{}^{a3} = \sigma^{a3}(z = -h/2).
\end{aligned}
\tag{2.12}
$$

Expressions similar to (2.11) were proposed in [6, 9, 10]. They, however, violate some of the requirements mentioned above.

For completness of our derivation the integrals $(2.5)_1$ should be calculated. We omit here this simple procedure (see e.g. [13]) assuming only that all the terms up to the order h/R are preserved in the resulting formulae, which is important in the case of not-so-thin shells.

3. Evaluation of the strain energy

In order to establish whether SP furnishes a consistent approximation to the three--dimensional elasticity we shall examine the strain energy density integrated with respect to the thickness coordinate z. Such a global evaluation of the strain energy has been proposed by KOITER [11] who proved the Kirchhoff-Love type theory to form the first approximation, and then used by PIETRASZKIEWICZ [4] for construction of an energy functional of the second approximation. For the purpose of the present analysis let us consider the following twodimensional strain energy expression

$$
\begin{aligned}
\Sigma = {} & (1/2)_0B^{\alpha\beta\lambda\eta}\gamma_{\alpha\beta}\gamma_{\lambda\eta}+{}_1B^{\alpha\beta\lambda\eta}\gamma_{\alpha\beta}\varkappa_{\lambda\eta}+(1/2)_2B^{\alpha\beta\lambda\eta}\varkappa_{\alpha\beta}\varkappa_{\lambda\eta}+(1/2)_0B^{\alpha3\eta3}\gamma_{\alpha3}\gamma_{\eta3} \\
& +{}_1B^{\alpha3\eta3}\gamma_{\alpha3}\varkappa_{\eta3}+(1/2)_2B^{\alpha3\eta3}\varkappa_{\alpha3}\varkappa_{\eta3}+(1/2)_0B^{3333}\gamma_{33}\gamma_{33} \\
& +{}_0B^{\alpha\beta33}\gamma_{\alpha\beta}\gamma_{33}+\underline{{}_2B^{\alpha\beta33}\varkappa_{\alpha\beta}\varkappa_{33}}+\underline{{}_2B^{\alpha\beta\lambda\eta}\gamma_{\alpha\beta}\mu_{\lambda\eta}}+\ \ldots,
\end{aligned}
\tag{3.1}
$$

where the underlined error terms should be neglected as for as SP is concerned.

The evaluation of (3.1) we start from observing, that for shells having symmetry of elastic properties with respect to the surfaces z = const. (which was assumed in deriving (2.3) and (3.1)) two groups of elastic moduli can be distinguished (e.g. $\{G, E, \nu\}$ and $\{G', E', \nu'\}$), where the non-primed and the primed quantities are accordingly related to the planes tangential and normal to the surfaces z = const, G denotes the shear modulus, E stands for the Young modulus and ν — the Poisson number. For a transversely isotropic material with its axis of isotropy coinciding with the z axis of the shell the components of the elasticity tensor C^{ijkl} (see e.g. [12]) have the following estimates

$$
C^{\alpha\beta\lambda\eta} \sim G \sim E, \quad C^{\alpha\beta33} \sim \nu'G, \quad C^{3333} \sim E', \quad C^{\alpha3\eta3} \sim G',
\tag{3.2}
$$

shoving that only four elastic moduli (e.g. G, G', E', ν') are of consequence in our approximate analysis (at this level of generality the estimates (3.2) remain valid for orthotropic shells).

Before estimating the strains occuring in (3.1) let us define a dimensionless coefficient δ

$$
\gamma_{\alpha\beta} \sim \delta h\varkappa_{\alpha\beta}, \quad \gamma_{33} \sim \delta h\varkappa_{33}, \quad \delta\gamma_{\alpha3} \sim h\varkappa_{\alpha3},
\tag{3.3}
$$

allowing for the specification of the bending theory $\delta \sim 1$, the membrane theory $\delta \gg 1$ and the inextensional bending theory $\delta \ll 1$. Defining by $\gamma \sim \gamma_{\alpha\beta}$ a typical value of the shell deformation the strain components can be estimated as below

$$
\begin{aligned}
& \gamma_{\alpha\beta} \sim \gamma, \quad \gamma_{33} \sim (\nu'+\vartheta^2)(G/E')\gamma, \quad \gamma_{\alpha3} \sim (G/G')(h/L)\gamma, \\
& h^2\mu_{\alpha\beta} \sim [h/R+(G/G')(h/L)^2+(\nu'+\vartheta^2)(G/E')\vartheta^2]\gamma,
\end{aligned}
\tag{3.4}
$$

where R is the typical radius of curvature of the middle surface, L — the characteristic wavelength of the deformation pattern of that surface and ϑ — the small parameter, given as follows

$$b^{\beta}_{\alpha} \sim 1/R, \quad (\ )|_{\alpha} \sim (\ )/L, \quad \vartheta \sim (\sqrt{h/R}+h/L) \tag{3.5}$$

The estimation $(3.4)_3$ results from $(2.8)_3$ and $(2.9)_{2,3}$ with the help of (2.5), (3.2), $(3.4)_1$ and (3.5). The relation $(3.4)_2$ follows from $(2.9)_5$ with $N^{33} \sim Gh\gamma\vartheta^2$ — implied by $(2.8)_4$. Having $(3.4)_{1-3}$ the estimation (3.4) can be deduced using the three-dimensional compatibility equations as done in [11] for isotropic shells.

Introduction of (2.5), (3.2), (3.4) and (3.3) into (3.1) yields

$$\begin{gathered}\Sigma/(Gh\gamma^2) \sim 1+(h/R)\delta^{-1}+\delta^{-2}+(G/G')(h/L)^2\delta^{-2}+\\ +(G/G')(h/L)^2(h/R)\delta^{-1}+(G/G')(h/L)^2+(\nu'+\vartheta^2)(G/E')+\nu'(\nu'+\vartheta^2)(G/E')+\\ +\underline{\nu'(\nu'+\vartheta^2)(G/E')\delta^{-2}}+\underline{[h/R+(G/G')(h/L)^2+(\nu'+\vartheta^2)(G/E')\vartheta^2]}+\ldots,\end{gathered} \tag{3.6}$$

which with the sequence of terms correspoding to that of (3.1) expresses an approximation to the strain energy in terms of the nondimensional parameters: geometric h/R, h/L, ϑ, δ and elastic G/G', G/E' and ν'. Inspecting in (3.6) possible rates of the above listed parameters one can establish global energetical cosistency of shell theories. It turns out that the KL theory forms (as well known [11]) within the relative error ϑ^2 the first approximation in the case of bending of isotropic thin shells subjected to uniform loads; accordingly the first and third term in (3.6) are of primary importance. The Reissner-Naghdi (RN) theory [9] and the Timoshenko-type (T) theory [3] (each including the transverse shear strain $\gamma_{\alpha 3}$) prove energetically consistent with regard to the inextensional bending of thin anisotropic shells, with a large ($G/G' \gg 1$) transverse shearing deformability; here only the third and fourth term in (3.6) should be retained. The SP theory owing to the absence in (3.6) of the two underlined terms cannot be consistent in general, i.e. when the analysis is solely based on the rather rough parameters involved in (3.6). Yet in some specific problems SP may, perhaps, yield a cosistent approximation to the strain energy which conjecture, however, we are not able to prove rigorously. Instead of that observe that an inconsistent theory can still be expected to furnish with a desired accuracy selected components of the stress and displacement. We shall elaborate on that point of view and show by a physical argument and ensuing numerical example that SP compared with more elementary theories (e.g. RN, T, KL) offers a distinctly improved approximation of the transverse shear $\sigma^{\rho 3}$ and normal σ^{33} stress in certain shell regions such as the vicinity of the load discontinuity.

To this end let us focus attention on the equation of equilibrium $(2.8)_4$, the transverse shear couple $M^{\alpha 3}$ (also called [1] the splitting force) and the transverse normal stress resultant N^{33}. Since in planes normal to the middle surface N^{33} does not occur and $M^{\alpha 3}$ is self-equilibrated (to be exact, $M^{\alpha 3}$ tends to be self-equilibrated as h/R approaches zero) thus neither N^{33} nor $M^{\alpha 3}$ can affect significantly the global equilibrum of a shell element cut out across the thickness. Therefore in passing from SP to the more elementary theories which may be reached by the assumption $M^{\alpha 3} = 0$, the ensuing simplification of $(2.8)_4$ to the form

$$b_{\alpha\beta}M^{\beta\alpha}-N^{33}+m^3 = 0 \tag{3.7}$$

and direct determination of N^{33} from (3.7), one cannot expect a noticeable disturbation of the global shell behaviour (characterized for example by the normal deflection of the middle surface). Yet, as regards N^{33} and $M^{\alpha 3}$ (and consequently the stresses σ^{33} and $\sigma^{\varphi 3}$) the foregoing simplifications may obviously lead to a considerable change. In fact, it is readily verified that all the elements in $(2.8)_4$ have in general the same order of magnitude (see [4]), with the $M^{\alpha 3}{}_{|\alpha}$ contribution increasing in proportion with $(1/\delta)$ and $(1/L)$. Thus in regions characterized by a large index $(1/L)$ of variation of the deformation (such as the vicinity of the load discontinuity) we cannot expect the theories (e.g. RN) utilising (3.7) to approximate $\sigma^{\varphi 3}$ and σ^{33} with sufficient accuracy, whereas SP using the exact equation $(2.8)_4$ seems here far more promising.

Return to the first underlined term in (3.6). In most elementary theories (e.g. KL, RN, T) this term is implicitely taken into account by a simple algebraic elimination (see [12, 13]) of the transverse normal strain e_{33}. Such a procedure undoubtely improving the strain energy approximation in those theories makes, however, the variational derivation of SP extremely awkward and precludes the possibility of improved approximation of $\sigma^{\varphi 3}$ and σ^{33} because of destruction of the crucial equation $(2.8)_4$. Thus we omit the relevant underlined term in (3.6) assuming that $(\nu')^2 \ll 1$, which holds for numerous elastic media.

It should be stressed that the foregoing equations of SP and their analysis refer to the interior shell problem, i.e. they lose their meaning in the boundary layer zone.

4. Numerical example

Let us consider (Fig. 1) an infinitely long circular cylindrical shell of constant thickness h, the outer surface radius r, made from an isotropic material characterized by the Poisson number ν and the Young's modulus E.

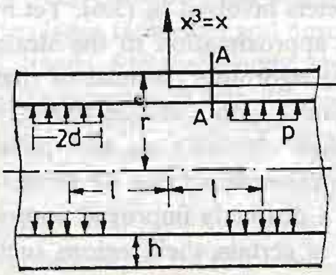

Fig. 1

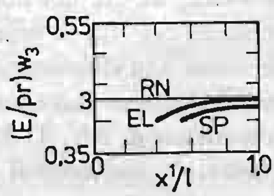

Fig. 2

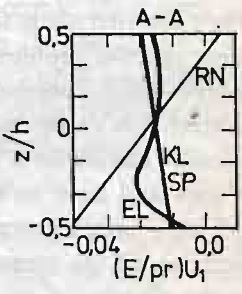

Fig. 3

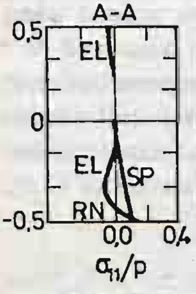

Fig. 4

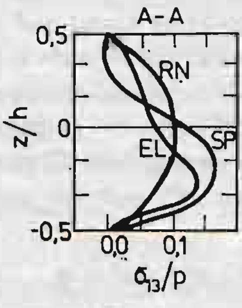

Fig. 5

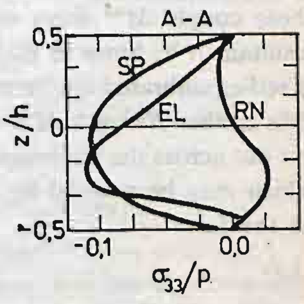

Fig. 6

The internal surface of the cylinder is subjected to a rotationally-symmetric band pressure p spaced in the longitudinal direction with the period $2l$ and having the band width $2d$. The normal force N_{11} directed along the generator is assumed to vanish throughout the shell and the longitudinal displacement U_1 to be zero in the plane $x^1 = 0$. The solution of the relevant equations of SP can be readily found in terms of the Fourier series expansions but we omit it here for the sake of brevity (the details are analogous to that given in [14, 15]). Computations have been carried out with the following data: $h/r = 0.3$, $d/l = 0.2$, $l/r = 0.2$ and $\nu = 0.3$ which describe a nonthin shell under a local load ($d/h = 0.13$). The results depicted in Fig. 2 - 6 (with distributions EL, RN and KL taken from [14], where EL denotes the three-dimensional elasticity solution) evidently confirm our expectations (sec. 3). To wit, in a vicinity (the cross-section A-A in Fig. 1, having the coordinate $x^1/l = 0.6$) of the load discontinuity (having the coordinate $x^1/l = 0.8$) SP approximates the transverse shear (Fig. 5) and normal (Fig. 6) stress distinctly more accurately than RN. At the same time, SP is only slightly more adequate than RN (or KL) in the case of the displacements (Fig. 2 and 3) and the normal stress along the generator (Fig. 4); the surprisingly poor approximation of the cross section rotation by RN (Fig. 3), disclosed in [14], does not occur virtually [15], i.e. is caused by some errors in [14].

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Резюме

О ЛИНЕЙНОЙ ТЕОРИИ АНИЗОТРОПНЫХ ОБОЛОЧЕК СРЕДНЕЙ ТОЛЩИНЫ

Рассматривается в линейной постановке статика относительно толстых, упругих, анизотропных оболочек под действием быстроизменяющихся нагрузок. Вариационные и дифференциальные уравнения двумерной теории оболочек выведены из трехмерных уравнений упругости на основе предположения о линейном распределении вектора перемещений по толщине оболочки. Исследована точность аппроксимации упругой энергии для принятой модели оболочки и определена область применения уравнений этой модели. Дан пример расчета.

Streszczenie

O LINIOWEJ TEORII ANIZOTROPOWYCH POWŁOK O ŚREDNIEJ GRUBOŚCI

W pracy rozważono statyczne zagadnienie wewnętrzne liniowej teorii niezbyt cienkich, sprężystych powłok anizotropowych, poddanych szybkozmiennym obciążeniom. Równania wariacyjne i różniczkowe teorii dwuwymiarowej wyprowadzono z równań teorii sprężystości na podstawie założenia liniowego rozkładu wektora przemieszczenia na grubości powłoki. Zbadano dokładność aproksymacji energii sprężystej w przyjętym modelu powłoki i określono zakres stosowalności równań tego modelu. Podano przykład liczbowy.

Praca została złożona w Redakcji dnia 11 stycznia 1983 roku