

STRUCTURES ON THE PARTLY ELASTIC SUBSOIL

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Introduction

The aim of the present paper is the analysis of interactions between the structure footing and the subsoil, surface of which undergoes subsequent changes of configuration. The influence of the deformation of subsoil upon the structure behaviour has a deep significance in the problems concerning the foundation of buildings on the grounds suffering mining damage. Elastic plates and beams as well as arbitrary deformable structures of rigid footings were considered. The following two facts were taken into account: 1° — the possibility of losing the contact between the structure foundation and the subsoil, 2° — a different character of the reaction of subsoil in primary and secondary deformations of the subsoil, according to the concept of a partly elastic subsoil, given in [1]. The solutions were obtained for the special case of structures with a rigid footing.

Denotations. The symbol \tilde{a}_n stands for the sum $a_0 + a_1 + \dots + a_n$, at the same time $\tilde{a}_0 = a_0$ for $n = 0$. Thus, $a_n = \tilde{a}_n - \tilde{a}_{n-1}$ are increments of quantities \tilde{a}_{n-1} for $n = 1, 2, \dots, K-1$. The quantities a_0, a_1, \dots, a_n are referring to subsequent states of the structure foundation, the subsoil, as well as their mutual interactions. The subsequent states are distinguished by indices $0, 1, \dots, K$. The state distinguished by the index 0 is an initial state, the states denoted by the indices $1, 2, \dots, K-1$ are subsequent intermediate states, whereas the state denoted by the index K is a final state. It is assumed that with the passage from the n -th state to the state $n+1$, at least one of the quantities occurring in the problem under consideration is changed.

1. Deformations of the subsoil boundary surface

The subject of consideration will be now the deformations of the subsoil boundary surface, which will occur in geological processes or mining exploit i.e. without any outer loads acting at the subsoil from the structure. Let us consider Fig. 1, where Δ_0 is the boundary surface of the underformed subsoil in the vicinity of the building structure we are interested in. It is assumed that Ω is a regular region on the plane Ox_1x_3 . Each point $P(x_1, x_3)$ in the region Ω can be projected along x_2 — axis on the boundary surface Δ_0 of the undeformed subsoil. Hence Δ_0 will be given by the formula

$$x_2 = \tilde{u}_0(x_1, x_3); \quad (x_1, x_3) \in \Omega. \quad (1.1)$$

It is assumed that:

1° — Δ_0 is a smooth surface, i.e. there exist the derivatives

$$\tilde{u}_{0,1} \equiv \frac{\partial \tilde{u}_0}{\partial x_1}, \quad \tilde{u}_{0,3} \equiv \frac{\partial \tilde{u}_0}{\partial x_3}.$$

2° — Δ_0 is a shallow surface, i.e.

$$\frac{\partial \tilde{u}_0}{\partial x_\alpha} \ll 1, \quad \alpha = 1, 3.$$

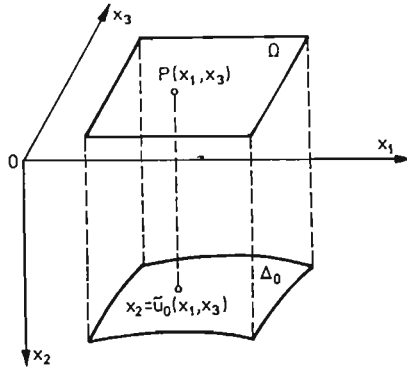


Fig. 1

The conditions given above concerned the boundary surface Δ_0 of the initially undeformed subsoil. Let us deal now with a description of the unloaded but deformed subsoil boundary surface, due to the effects of n subsequent changes of geological processes, where $n = 1, 2, \dots, K$. This boundary surface will be denoted by Δ_n , being the result of n subsequent subsoil deformations, cf. Fig. 2. The parametric equation of the surface Δ_n is:

$$x_2 = \tilde{u}_n(x_1, x_3), \quad (x_1, x_3) \in \Omega, \quad n = 1, 2, \dots, K. \quad (1.2)$$

Like in the case of the surface Δ_0 , it is demanded here, as well, that the surface Δ_0 be smooth, i.e.:

$$\tilde{u}_{n,1} \equiv \frac{\partial \tilde{u}_n}{\partial x_1}, \quad \tilde{u}_{n,3} \equiv \frac{\partial \tilde{u}_n}{\partial x_3},$$

and shallow

$$\frac{\partial u_n}{\partial x_\alpha} \ll 1, \quad \alpha = 1, 3.$$

Increments $\tilde{u}_n(x_1, x_3) - \tilde{u}_{n-1}(x_1, x_3)$ for $n = 1, 2, \dots, K$ will be designated by $u_n(x_1, x_3)$ i.e.

$$u_n(x_1, x_3) \equiv \tilde{u}_n(x_1, x_3) - \tilde{u}_{n-1}(x_1, x_3), \quad \text{for } n = 1, 2, \dots, K$$

assuming also that $u_0(x_1, x_3) = \tilde{u}_0(x_1, x_3)$ (compare the denotations in the beginning of the paper).

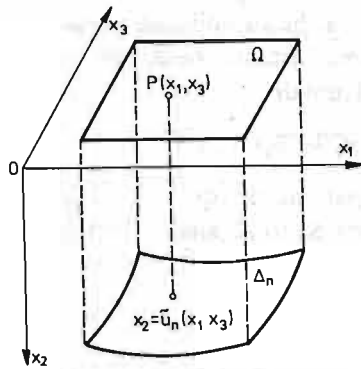


Fig. 2

2. Deformations of the structure foundation footing

Let us assume, on the basis of Fig. 3, that in the region Ω the building structure projection Π upon the plane $x_2 = 0$ was given. In the projection those structure elements are neglected which cannot have a contact with the subsoil. Moreover it is assumed that Π is a subregion of Ω , $\Pi \subset \Omega$. The structure footing after it is founded on the surface subsoil Δ_0 is designated by S_0 . The parametric equation of the footing S_0 will be

$$x_2 = \tilde{w}(x_1, x_3), \quad (x_1, x_3) \in \Pi. \quad (2.1)$$

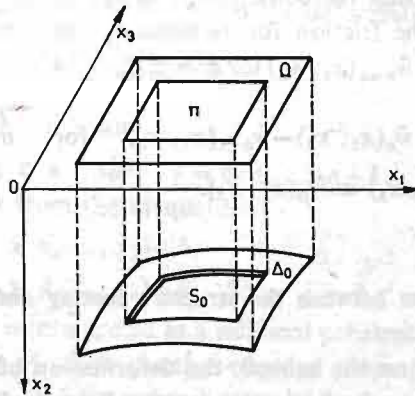


Fig. 3

Like in the case of deformations of the boundary subsoil surface, it will be assumed that:

1' — S_0 must constitute a smooth surface, i.e. there exist the derivatives

$$\tilde{w}_{0,1} \equiv \frac{\partial \tilde{w}_0}{\partial x_1}, \quad \tilde{w}_{0,3} \equiv \frac{\partial \tilde{w}_0}{\partial x_3},$$

2' — S_0 is a shallow surface, i.e.

$$\frac{\partial \tilde{w}_0}{\partial x_\alpha} \ll 1, \quad \alpha = 1, 3.$$

Founding a building structure on the subsoil and assuming that the possibility of its being loaded can occur at many time instants, deformations shown in Fig. 4 will take place and instead of (2.1) we shall obtain

$$x_2 = \tilde{\omega}_n(x_1, x_3), \quad (x_1, x_3) \in \Pi, \quad (2.2)$$

The above formula being the parametric equation of the surface S_n . It can be seen that the structure footing will pass from S_0 to S_n and the subsoil boundary surface will pass from Δ_0 to Δ_n . In both cases $n = 1, 2, \dots, K$. It is required here, as well, that each S_n should have the properties 1', 2'.

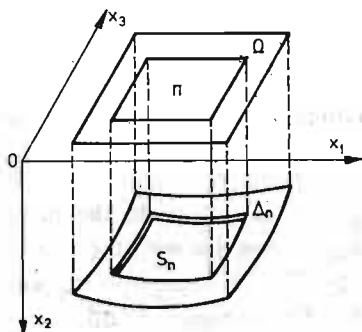


Fig. 4

It is to be pointed out that the consequence of the assumptions 2° and 2' will be the possibility of neglecting the friction forces between the subsoil and structure footing.

Increments $\tilde{w}_n(x_1, x_3) - \tilde{w}_{n-1}(x_1, x_3)$ for $n = 1, 2, \dots, K$ will be designated by $w_n(x_1, x_3)$ i.e.

$$w_n(x_1, x_3) = \tilde{w}_n(x_1, x_3) - \tilde{w}_{n-1}(x_1, x_3) \quad \text{for } n = 1, 2, \dots, K, \quad (2.3)$$

assuming also that $w_0(x_1, x_3) \equiv \tilde{w}_0(x_1, x_3)$.

3. Interaction between the structure footing and the subsoil

Founding the structure on the subsoil, the deformation of the subsoil is caused. The cause of the deformation can be both, the action of external forces transferred through the structure footing on the subsoil, as well as the deformations of the subsoil boundary surface resulting from the mining exploit. The first subsoil deformation due to the loadings is distinguished as a primary one and occurring after it, a secondary one. The notion of a subsoil primary deformation is usually treated as a certain stipulated notion, being not strictly defined. Namely, there appear some doubts whether the very small subsoil displacement, caused by the foundation transferring only small loads, can be recognized as primary. In such a case, the soil porosity remains practically the same after the deformation as it was before. Such cases occur seldom in practice. For practical purposes a minimal penetration h of the structure footing in a subsoil should be taken into account, which, in a sufficient way, would condition the possibility of treating further subsoil deformation

as secondary. Having taken into account the above remark, a definition of primary subsoil deformations will be formulated, cf. Fig. 5.

By the subsoil deformation at the point $P(x_1, x_3) \in \Pi$ after the n -th change of its configuration we shall mean the number $\tilde{\delta}_n(x_1, x_3)$ defined by

$$\tilde{\delta}_n(x_1, x_3) \equiv \tilde{w}_n(x_1, x_3) - \tilde{u}_n(x_1, x_3), \quad n = 0, 1, \dots, K, \quad (3.1)$$

assuming that it is not negative. When $\tilde{\delta}_n(x_1, x_3) < 0$ then the subsoil deformation is assumed to be equal to zero.

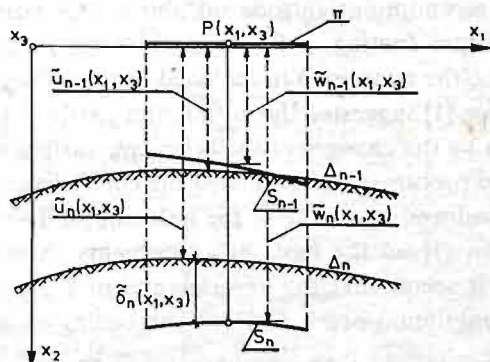


Fig. 5

By the primary deformation (or more precisely h -primary, where $h > 0$ is a given number) at the point $(x_1, x_3) \in \Pi$ we shall mean the deformation $\tilde{\delta}_n(x_1, x_3)$ fulfilling the conditions:

$$\text{for } n = 0 \quad \tilde{\delta}_0(x_1, x_3) > h$$

$$\text{for } n > 0 \quad \begin{cases} \tilde{\delta}_n(x_1, x_3) > h & \text{and} \\ \tilde{\delta}_l(x_1, x_3) \leq h & \text{for } l = 0, 1, \dots, n-1. \end{cases}$$

The above conditions result from the inequalities

$$\tilde{w}_n(x_1, x_3) > \tilde{u}_n(x_1, x_3) + h,$$

$$\tilde{w}_l(x_1, x_3) \leq \tilde{u}_l(x_1, x_3) + h \quad \text{for } l = 0, 1, \dots, n-1,$$

where the positive number h is interpreted as a minimal penetration of the structure footing in the subsoil, the exceeding of which changes physical properties of the foundation.

Let us designate by $p_n(x_1, x_3)$ the subsoil reaction to structure footing after the n -th change of its configuration, $\tilde{p}_n(x_1, x_3) \leq 0$. The reaction increments for $n = 1, 2, \dots, K$ will be denoted by $p_n(x_1, x_3)$ i.e. $p_n(x_1, x_3) \equiv \tilde{p}_n(x_1, x_3) - \tilde{p}_{n-1}(x_1, x_3)$ for $n = 1, 2, \dots, K$ assuming also that $p_0(x_1, x_3) \equiv \tilde{p}_0(x_1, x_3)$.

Analogously, the deformation increments will be defined by $\delta_n(x_1, x_3) \equiv \tilde{\delta}_n(x_1, x_3) - \tilde{\delta}_{n-1}(x_1, x_3)$. Let us introduce the following basic assumptions about the interaction between the structure footing and the subsoil:

1° — if $\tilde{\delta}_n(x_1, x_3)$ is a primary deformation at the point $(x_1, x_3) \in \Pi$ then the reaction increment at this point is equal to:

$$p_n(x_1, x_3) = -k(x_1, x_3) \delta_n(x_1, x_3)$$

where k denotes the known Winkler coefficient.

2° — if $\tilde{\delta}_n(x_1, x_3)$ is not a primary deformation at the point but it is positive, $\tilde{\delta}_n(x_1, x_3) > 0$, then the reaction increment of this point is equal to:

$$p_n(x_1, x_3) = -\kappa k(x_1, x_3) \delta_n(x_1, x_3)$$

where κ , $\kappa > 1$ designates the subsoil inelasticity coefficient [1].

3° — if $\delta_n(x_1, x_3)$ is not positive at the point $(x_1, x_3) \in \Pi$, $\tilde{\delta}_n(x_1, x_3) \leq 0$, then the total reaction at this point is equal to:

$$\tilde{p}_n(x_1, x_3) = 0 \quad \text{i.e.} \quad p_n(x_1, x_3) = -\tilde{p}_{n-1}(x_1, x_3).$$

On the basis of the assumptions introduced above it is possible to call the subsoil interacted with the structure footing, a partly elastic subsoil (or more precisely the h -partly elastic subsoil since the quantity h is included in the definition of this subsoil).

J. Kwiatek in his paper [1] suggested the notion of a partly elastic subsoil affected both by outer loads as well as by the changes of the boundary surface subsoil curvatures under the structure footing, and proposed and evaluated the coefficient κ , which is to be applied both at the secondary loadings as well as at the unloadings. The numerical values of the coefficient κ was given in [1] on the basis of experiments. According to the theoretical model suggested above, it seems that the experimental model penetration should be defined to a conventional minimum depth h , $h > 0$, according to the interpretation of this quantity accepted above.

In many problems it can be assumed that the condition $\tilde{\delta}_0(x_1, x_3) > h$ occurs for each $(x_1, x_3) \in \Pi$ i.e. the primary deformation can only take place before the first change of the configuration i.e. for $n = 0$. It seems that such an assumption was silently accepted in the paper [1]. In this case the reaction $p(x_1, x_3)$ of the foundation after the n -th change of its configuration is determined by the conditions:

$$1' \quad \tilde{p}_0(x_1, x_3) = k(x_1, x_3) \tilde{\delta}_0(x_1, x_3) \quad \text{for each} \quad (x_1, x_3) \in \Pi,$$

$$2' \quad \text{if} \quad \tilde{\delta}_n(x_1, x_3) > 0 \quad \text{then} \quad p_n(x_1, x_3) = -\kappa k(x_1, x_3) \delta_n(x_1, x_3),$$

$$3' \quad \text{if} \quad \tilde{\delta}_n(x_1, x_3) \leq 0 \quad \text{then} \quad \tilde{p}_n(x_1, x_3) = 0 \quad \text{i.e.} \quad p_n(x_1, x_3) = -\tilde{p}_{n-1}(x_1, x_3).$$

Conditions 2', 3' hold only if $n = 1, 2, \dots, K$, where K is the number of all changes of the subsoil configurations and changes of the loadings.

4. Elastic plates on the partly elastic subsoil

We are to confine ourselves to the linear theory of thin elastic plates, cf. [2]. It is assumed here that the plate deflections are relatively small as compared with the plate thickness. We also assume that the plate edges can shift freely in the plate middle plane. We restrict ourselves to the equilibrium plate problems. The scheme of plate loadings and deformations is given on Fig. 6, i.e. after the n -th change of the subsoil configuration, q_n being the n -th change of the plate load (thus the possibility of subsequent K changes of the plate load is assumed).

The governing equation of the plate normal displacement, plate being made of homogeneous isotropic material, has a known form

$$\nabla^4 w = \frac{q}{D},$$

where: w is the plate deflection, q is the load, $D = Eg^3/12(1-\nu^2)$ is the rigidity of the plate in which g is the thickness of the plate, E is Young's modulus, and ν is the Poisson's ratio. In the Cartesian coordinates x_1, x_3 this equation is

$$w_{,3333} + 2w_{,3311} + w_{,1111} = \frac{q}{D},$$

where $w = w(x_1, x_3)$; $(x_1, x_3) \in \Pi$. The deflection w is to fulfill the above equation at each point of the region Π . This equation has to be complemented with the boundary conditions on the free edge $\partial\Pi$, of the region Π , given at the end of this section.

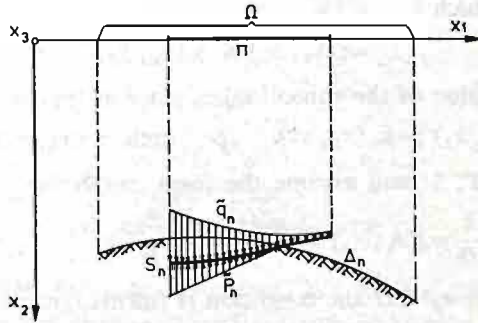


Fig. 6

In the case of taking into consideration the reaction of the subsoil, the plate equation will have the form:

$$\nabla^4 \tilde{w}_n(x_1, x_3) = \frac{\tilde{q}_n(x_1, x_3) + \tilde{p}_n(x_1, x_3)}{D} \quad \text{for } n = 1, 2, \dots, K, \quad (x_1, x_3) \in \Pi,$$

thus, it will also hold for the increment

$$\nabla^4 w_n(x_1, x_3) = \frac{q_n(x_1, x_3) + p_n(x_1, x_3)}{D} \quad \text{for } n = 0, 1, \dots, K, \quad (x_1, x_3) \in \Pi$$

together with the conditions on the free edge $\partial\Pi$.

On the basis of the formulae cited above, we shall determine the equation for the elastic plate on the partly elastic subsoil. And so we shall get:

1° — if the deformation $\tilde{\delta}_n(x_1, x_3)$ is primary at the point $(x_1, x_3) \in \Pi$, then

$$\nabla^4 w_n(x_1, x_3) + \frac{k}{D} w_n(x_1, x_3) = \frac{q_n(x_1, x_3) + k u_n(x_1, x_3)}{D},$$

for such $(x_1, x_3) \in \Pi$, for which

$$\begin{aligned} \tilde{w}_n(x_1, x_3) &> \tilde{u}_n(x_1, x_3) + h \quad \text{and} \\ \tilde{w}_l(x_1, x_3) &\leq \tilde{u}_l(x_1, x_3) + h \quad \text{for } l = 0, 1, \dots, n-1. \end{aligned}$$

2° — if the deformation $\tilde{\delta}_n(x_1, x_3)$ is not primary at the point $(x_1, x_3) \in \Pi$ and it is positive, then

$$\nabla^4 \omega_n(x_1, x_3) + \frac{\kappa k}{D} w_n(x_1, x_3) = \frac{q_n(x_1, x_3) + \kappa k u_n(x_1, x_3)}{D},$$

for such $(x_1, x_3) \in \Pi$, for which

$$\tilde{\omega}_n(x_1, x_3) > \tilde{u}_n(x_1, x_3) + h$$

and for which there is such l , $0 \leq l < n$, that

$$\tilde{\omega}_l(x_1, x_3) > \tilde{u}_l(x_1, x_3) + h.$$

3° — if the deformation at the point $(x_1, x_3) \in \Pi$ does not occur i.e. $\tilde{\delta}_n(x_1, x_3) \leq 0$, then

$$\nabla^4 w(x_1, x_3) = \frac{q_n(x_1, x_3) - p_{n-1}(x_1, x_3)}{D}$$

for such (x_1, x_3) for which

$$\tilde{\omega}_n(x_1, x_3) \leq \tilde{u}_n(x_1, x_3).$$

If a primary deformation of the subsoil takes place at the first loading, i.e. if

$$\tilde{\omega}_0(x_1, x_3) > \tilde{u}_0(x_1, x_3) \quad \text{for each } (x_1, x_3) \in \Pi,$$

then the conditions 1°, 2°, 3° will assume the form, respectively:

$$1^\circ - \nabla^4 w_0(x_1, x_3) + \frac{k}{D} w_0(x_1, x_3) = \frac{q_0(x_1, x_3) + k u_0(x_1, x_3)}{D}$$

assuming that for all $(x_1, x_3) \in \Pi$ the condition is fulfilled

$$\omega_0(x_1, x_3) \geq u_0(x_1, x_3) + h$$

which should be checked after the solution of the problem (after finding $w_0(x_1, x_3)$, $(x_1, x_3) \in \Pi$).

$$2^\circ - \nabla^4 \omega_n(x_1, x_3) + \frac{\kappa k}{D} \omega_n(x_1, x_3) = \frac{q_n(x_1, x_3) + \kappa k u_n(x_1, x_3)}{D}$$

for such $(x_1, x_3) \in \Pi$ and $n = 1, 2, \dots, K$ for which

$$\tilde{w}_n(x_1, x_3) > \tilde{u}_n(x_1, x_3).$$

$$3^\circ - \nabla^4 w_n(x_1, x_3) = \frac{q_n(x_1, x_3) - \tilde{p}_{n-1}(x_1, x_3)}{D}$$

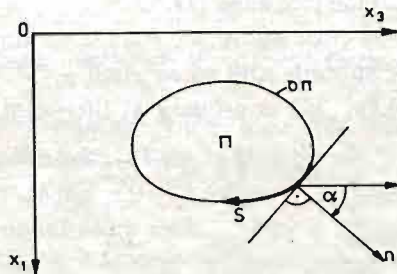


Fig. 7

for such $(x_1, x_3) \in \Pi$ and $n = 1, 2, \dots, K$ for which

$$\tilde{w}_n(x_1, x_3) \leq \tilde{u}_n(x_1, x_3).$$

Each of the cases considered is to be complemented, according to the theory of plates, with the following two boundary conditions (viz. Fig. 7):

$$M_3^n \cos^2 \alpha + M_1^n \sin^2 \alpha - 2M_{31}^n \sin \alpha \cos \alpha = 0$$

$$(Q_3^n \cos \alpha + Q_1^n \sin \alpha) - \frac{\partial}{\partial S} [M_{31}^n (\cos^2 \alpha - \sin^2 \alpha) + (M_3^n - M_1^n) \sin \alpha \cos \alpha] = 0 \quad (4.1)$$

in which the following were designated

$$M_3^n \equiv -D \left(\frac{\partial^2 w_n}{\partial x_3^2} + \nu \frac{\partial w_n}{\partial x_1^2} \right),$$

$$M_1^n \equiv -D \left(\frac{\partial^2 w_n}{\partial x_1^2} + \nu \frac{\partial w_n}{\partial x_3^2} \right),$$

$$M_{31}^n \equiv -M_{13}^n = D(1-\nu) \frac{\partial^2 w_n}{\partial x_1 \partial x_3}, \quad (4.2)$$

$$Q_3^n \equiv -D \frac{\partial}{\partial x_3} \left(\frac{\partial^2 w_n}{\partial x_3^2} + \frac{\partial^2 w_n}{\partial x_1^2} \right),$$

$$Q_1^n \equiv -D \frac{\partial}{\partial x_1} \left(\frac{\partial^2 w_n}{\partial x_3^2} + \frac{\partial^2 w_n}{\partial x_1^2} \right),$$

S being on are co-ordinate along the boundary. We are still maintaining the convention that the quantity after the n -th change of configuration or loading is overlined by a „tilde” over it, while the quantity increments do not have a „tilde” over them.

The solution of the plate problem lies in determining the function $w_n = w_n(x_1, x_3)$, $n = 0, 1, \dots, K$, fulfilling for each $(x_1, x_3) \in \Pi$ one of the conditions: 1°, 2° or 3°, and such that the boundary conditions (4.1) are fulfilled for all those $(x_1, x_3) \in \Pi$ for which the unit normal n exist. After determining $w_n(x_1, x_3)$ from the formulae (4.2), bending couples \tilde{M}_1^n and \tilde{M}_3^n and torques \tilde{M}_{13}^n can be calculated from Eqs. (4.2). If, having determined $w_0(x_1, x_3)$ it will turn out that for each $(x_1, x_3) \in \Pi$ exists $w_0(x_1, x_3) > u_0(x_1, x_3) + h$ then we can only apply the conditions 2' and 3' further on, instead of 1°, 2°, 3°.

5. Elastic beams on a partly elastic subsoil

The equilibrium of elastic beams on a partly elastic subsoil is treated like the equilibrium of elastic plates, the basic equation has the form (Fig. 8 for $x = x_1$)

$$\frac{d^4 w(x)}{dx^4} = \frac{q(x)}{EJ} \quad \text{for } l_1 < x < l_2$$

in which the plate rigidity D was replaced by the beam rigidity EJ , and

$$w = w(x) = w(x_1, 0).$$

The boundary conditions have now the form

$$\frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^3 w}{\partial x^3} = 0,$$

for $x_1 = l_1$ and $x_2 = l_2$ (Fig. 8). The interaction conditions 1°–3° and 1'–3' for the plate

also hold for the beam if $x_1 \equiv x$ and if the dependence of the function $w_n, q_n, u_n, n = 0, 1, \dots, K$ on the variable x_3 is neglected.

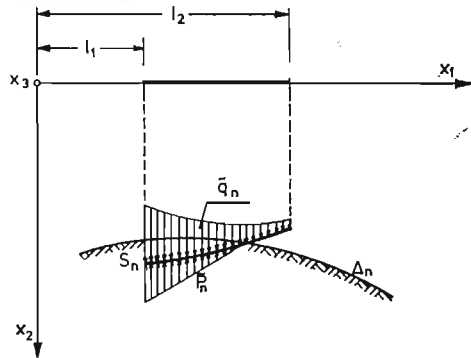


Fig. 8

6. Structures with rigid footing on a partly elastic subsoil

Let us assume according to Fig. 9 that any point $P(x_1, x_3) \in II$, of the structure on a rigid footing, is displaced under the influence of structure load and subsoil configuration changes. Since the structure undergoes only rigid deformations, the position of the point $P(x_1, x_3)$ is determined by

$$\tilde{w}_n(x_1, x_3) = \tilde{\vartheta}_n + x_3 \tilde{\varphi}_n - x_1 \tilde{\psi}_n. \tag{6.1}$$

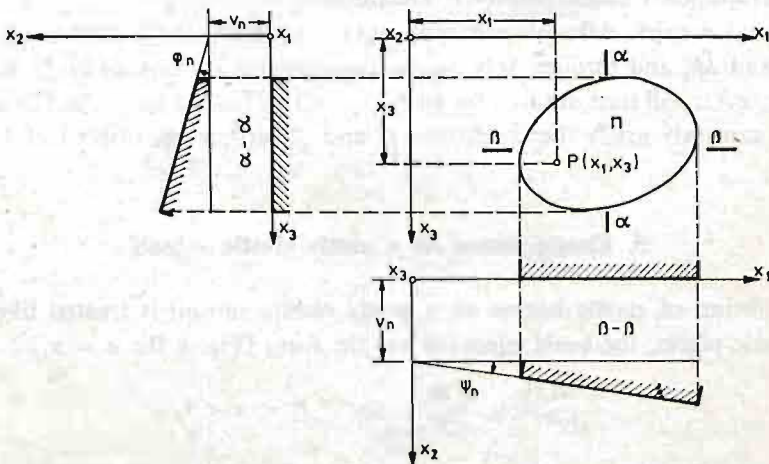


Fig. 9

The equilibrium condition of the forces acting upon the structure footing has the form

$$\tilde{R}_n + \int_{II} \tilde{p}_n(x_1, x_3) dx_1 dx_3 = 0, \tag{6.2}$$

where \tilde{R}_n is a resultant (in the direction of the axis x_2) of the loads acting from the structure on its footing.

The remaining equilibrium conditions are

$$\begin{aligned}\tilde{M}_n^{(3)} + \int_{\Pi} \tilde{p}_n(x_1, x_3) x_1 dx_1 dx_3 &= 0, \\ \tilde{M}_n^{(1)} + \int_{\Pi} \tilde{p}_n(x_1, x_3) x_3 dx_1 dx_3 &= 0,\end{aligned}\quad (6.3)$$

where $\tilde{M}_n^{(3)}$, $\tilde{M}_n^{(1)}$ are the couples (with respect to the axis x_3 and x_1 respectively) due to the loadings acting from the structure upon its footing. It is assumed that that equations of equilibrium not mentioned here are fulfilled identically, i.e. all the forces act parallel to the axis x_2 . Equilibrium conditions analogical to (6.1) and (6.2) as well as (6.3) also hold for the increments:

$$w_n(x_1, x_3) = \vartheta_n + x_3 \varphi_n - x_1 \psi_n, \quad (6.4)$$

and

$$\begin{aligned}R_n + \int_{\Pi} p_n(x_1, x_3) dx_1 dx_3 &= 0, \\ M_n^{(3)} + \int_{\Pi} p_n(x_1, x_3) x_1 dx_1 dx_3 &= 0, \\ M_n^{(1)} + \int_{\Pi} p_n(x_1, x_3) x_3 dx_1 dx_3 &= 0.\end{aligned}\quad (6.5)$$

The basic unknowns are now the numbers ϑ_n , φ_n , ψ_n for each $n = 0, 1, \dots, K$. Each of the triples of these number will be determined from the three equations (6.5) assuming that R_n (the increment of the resultant of normal loads acting upon the structure footing) and $M_n^{(3)}$ and $M_n^{(1)}$ (the couples increments acting upon the structure footing after the n -th change of its configuration) are known. The increment of the subsoil reaction $p_n(x_1, x_3)$ is to be distinguished here, by the conditions 1°, 2°, 3° included in section 3. It leads to the equations:

$$\begin{aligned}R_n + \int_{\Pi_n^1} -k[\vartheta + x_3 \varphi_n - x_1 \psi_n - u_n(x_1, x_3)] dx_1 dx_3 + \\ + \int_{\Pi_n^2} -\kappa k[\vartheta + x_3 \varphi_n - x_1 \psi_n - u_n(x_1, x_3)] dx_1 dx_3 - \int_{\Pi_n^3} \tilde{p}_{n-1}(x_1, x_3) dx_1 dx_3 &= 0, \\ M_n^{(3)} + \int_{\Pi_n^1} -kx_1[\vartheta + x_3 \varphi_n - x_1 \psi_n - u_n(x_1, x_3)] dx_1 dx_3 + \\ + \int_{\Pi_n^2} -\kappa kx_1[\vartheta + x_3 \varphi_n - x_1 \psi_n - u_n(x_1, x_3)] dx_1 dx_3 - \int_{\Pi_n^3} \tilde{p}_{n-1}(x_1, x_3) x_1 dx_1 dx_3 &= 0, \\ M_n^{(1)} + \int_{\Pi_n^1} -kx_3[\vartheta + x_3 \varphi_n - x_1 \psi_n - u_n(x_1, x_3)] dx_1 dx_3 + \\ + \int_{\Pi_n^2} -\kappa kx_3[\vartheta + x_3 \varphi_n - x_1 \psi_n - u_n(x_1, x_3)] dx_1 dx_3 - \int_{\Pi_n^3} \tilde{p}_{n-1}(x_1, x_3) x_3 dx_1 dx_3 &= 0,\end{aligned}$$

where Π_n^1 is a set of all those $(x_1, x_3) \in \Pi$ for which $\tilde{\delta}_n(x_1, x_3)$ is primary and M_n^2 is a set of all those $(x_1, x_3) \in \Pi$ for which $\tilde{\delta}_n(x_1, x_3)$ is not primary but is positive, and M_n^3

is a set of all those $(x_1, x_3) \in \Pi$ for which $\tilde{\delta}_n(x_1, x_3) \leq 0$. It is to be pointed out that in the general case the regions $\Pi_n^1, \Pi_n^2, \Pi_n^3$ are disjoint and not known a priori. For all $(x_1, x_3) \in \Pi_n^2$ there is no contact of the structure footing with the subsoil, i.e. $\tilde{p}(x_1, x_3) = 0$ if $n = 0$ then $R_0 = \tilde{R}_0, M_0^{(3)} = \tilde{M}_0^{(3)}, M_0^{(1)} = \tilde{M}_0^{(1)}$ and for $n = 0$ it is to be assumed $\tilde{p}_{n-1} \equiv 0$, both in the above and further equations. The above conditions lead to the set of equations:

$$\begin{aligned}
 & \vartheta_n \left(\int_{\Pi_n^1} k(x_1, x_3) dx_1 dx_3 + \kappa \int_{\Pi_n^2} k(x_1, x_3) dx_1 dx_3 \right) + \varphi_n \left(\int_{\Pi_n^1} k(x_1, x_3) x_3 dx_1 dx_3 + \right. \\
 & \quad + \kappa \int_{\Pi_n^2} k(x_1, x_3) x_3 dx_1 dx_3 \left. \right) + \psi_n \left(- \int_{\Pi_n^1} k(x_1, x_3) x_1 dx_1 dx_3 - \right. \\
 & \quad - \kappa \int_{\Pi_n^2} k(x_1, x_3) x_1 dx_1 dx_3 \left. \right) = R_n + \int_{\Pi_n^1} k(x_1, x_3) u_n(x_1, x_3) dx_1 dx_3 + \\
 & \quad + \kappa \int_{\Pi_n^2} k(x_1, x_3) u_n(x_1, x_3) dx_1 dx_3 - \int_{\Pi_n^3} \tilde{p}_{n-1}(x_1, x_3) dx_1 dx_3. \\
 & \vartheta_n \left(\int_{\Pi_n^1} k(x_1, x_3) x_1 dx_1 dx_3 + \kappa \int_{\Pi_n^2} k(x_1, x_3) x_1 dx_1 dx_3 \right) + \\
 & \quad + \varphi_n \left(\int_{\Pi_n^3} k(x_1, x_3) x_1 x_3 dx_1 dx_3 + \kappa \int_{\Pi_n^2} k(x_1, x_3) x_1 x_3 dx_1 dx_3 \right) + \\
 & \quad + \varphi_n \left(- \int_{\Pi_n^1} k(x_1, x_3) (x_1)^2 dx_1 dx_3 - \kappa \int_{\Pi_n^2} k(x_1, x_3) (x_1)^2 dx_1 dx_3 \right) = M_n^{(3)} + \\
 & \quad + \int_{\Pi_n^1} k(x_1, x_3) x_1 u_n(x_1, x_3) dx_1 dx_3 + \kappa \int_{\Pi_n^2} k(x_1, x_3) x_1 u_n(x_1, x_3) dx_1 dx_3 - \\
 & \quad - \int_{\Pi_n^3} \tilde{p}_{n-1}(x_1, x_3) x_1 dx_1 dx_3. \\
 & \vartheta_n \left(\int_{\Pi_n^1} k(x_1, x_3) x_3 dx_1 dx_3 + \kappa \int_{\Pi_n^2} k(x_1, x_3) x_3 dx_1 dx_3 \right) + \\
 & \quad + \varphi_n \left(\int_{\Pi_n^1} k(x_1, x_3) (x_3)^2 dx_1 dx_3 + \kappa \int_{\Pi_n^2} k(x_1, x_3) (x_3)^2 dx_1 dx_3 \right) + \\
 & \quad + \psi_n \left(- \int_{\Pi_n^1} k(x_1, x_3) x_1 x_3 dx_1 dx_3 - \kappa \int_{\Pi_n^2} k(x_1, x_3) x_1 x_3 dx_1 dx_3 \right) = M_n^{(1)} + \\
 & \quad + \int_{\Pi_n^1} k(x_1, x_3) x_3 u_n(x_1, x_3) dx_1 dx_3 + \kappa \int_{\Pi_n^2} k(x_1, x_3) x_3 u_n(x_1, x_3) dx_1 dx_3 - \\
 & \quad - \int_{\Pi_n^3} \tilde{p}_{n-1}(x_1, x_3) x_3 dx_1 dx_3
 \end{aligned} \tag{6.6}$$

for $\vartheta_n, \varphi_n, \psi_n$. The total values of the displacements and rotations are equal to $\tilde{\vartheta}_n = \vartheta_0 + \vartheta_1 + \dots + \vartheta_n$; $\tilde{\varphi}_n = \varphi_0 + \varphi_1 + \dots + \varphi_n$; $\tilde{\psi}_n = \psi_0 + \psi_1 + \dots + \psi_n$, respectively. The final position of the structure footing will be determined by Eq. (6.1).

Essential difficulty in solving the above three equations lies in the fact that regions

$\Pi_n^1, \Pi_n^2, \Pi_n^3$ are not known in advance but are expressed by the conditions given in Sec. 3.

This means that:

- 1 — $(x_1, x_3) \in \Pi_n^1$
for $n = 0$ $\tilde{w}_0(x_1, x_3) > \tilde{u}_0(x_1, x_3) + h$
for $n \geq 0$ $\tilde{w}_n(x_1, x_3) > \tilde{u}_n(x_1, x_3)$ and $\tilde{w}_l(x_1, x_3) \leq \tilde{w}_l(x_1, x_3) + h$.
- 2 — $(x_1, x_3) \in \Pi_n^2$
if $\tilde{w}_n(x_1, x_3) > \tilde{u}_n(x_1, x_3)$ and (x_1, x_3) does not belong to Π_n^1 .
- 3 — $(x_1, x_3) \in \Pi_n^3$

if $\tilde{w}_n(x_1, x_3) < \tilde{u}_n(x_1, x_3)$ i.e. the contact between the footing and subsoil does not take place.

The above problem can be solved in an elementary way, if the deformations are primary for each $(x_1, x_3) \in \Pi$ before the first change of subsoil configuration. Thus

$$\vartheta_0 + x_3 \varphi_0 - x_1 \psi_0 \geq u_0(x_1, x_3) + h \text{ for each } (x_1, x_3) \in \Pi. \quad (6.7)$$

The second condition is the lack of contact loss of the structure footing with the subsoil i.e.

$$\tilde{\vartheta}_n + x_3 \tilde{\varphi}_n - x_1 \tilde{\psi}_n \geq \tilde{u}_n(x_1, x_3) \text{ for each } (x_1, x_3) \in \Pi, \quad n = 1, 2, \dots, K. \quad (6.8)$$

These conditions can be checked, however, only after the problem has been solved, since $\vartheta_0, \varphi_0, \psi_0$ and $\tilde{\vartheta}_n, \tilde{\varphi}_n, \tilde{\psi}_n, n = 1, 2, \dots, K$ occurring only in the conditions (6.7), (6.8) are unknown. When the conditions (6.7), (6.8) are fulfilled, then in Eqs. (6.6) for $n = 0$ integrals over Π_n^2 do not occur (because for $n = 0$ only primary deformations take place) and $\Pi_n^1 = \Pi$ as there is no loss of contact of the structure footing with the subsoil. By analogy, for $n = 1, 2, \dots, K$ integrals over Π_n^1 do not occur and $\Pi_n^2 = \Pi$.

Let's assume that the assumptions (6.7) and (6.8) are fulfilled and let's take the coordinate system $0x_1x_3$ on the plane in such a way that:

$$\begin{aligned} \int_{\Pi} k(x_1, x_3) x_3 dx_1 dx_3 &= 0, & \int_{\Pi} k(x_1, x_3) x_1 dx_1 dx_3 &= 0, \\ \int_{\Pi} k(x_1, x_3) x_1 x_3 dx_1 dx_3 &= 0. \end{aligned} \quad (6.9)$$

Moreover, let us denote

$$\begin{aligned} A &\equiv \int_{\Pi} k(x_1, x_3) dx_1 dx_3, & I_1 &\equiv \int_{\Pi} k(x_1, x_3) (x_3)^2 dx_1 dx_3, \\ I_3 &\equiv \int_{\Pi} k(x_1, x_3) (x_1)^2 dx_1 dx_3. \end{aligned} \quad (6.10)$$

If $k(x_1, x_3) = k = \text{const.}$ for each $(x_1, x_3) \in \Pi$ i.e. if Winkler's coefficient is constant, then the coordinate axes on the plane $x_2 = 0$ are obviously the main central axes of the region Π and $A/k, I_1/k, I_3/k$ are a surface of the region Π , and the main central inertia moments of this region, respectively. The co-ordinate system x_1x_3 fulfilling the above conditions always exists because of $k(x_1, x_3) > 0$ the numbers A, I_1, I_3 are always po-

sitive. The set of equilibrium equations (6.6) for $n = 0$ will have the following form now

$$\begin{aligned}\vartheta_0 A &= \tilde{R}_0 + \int_{\Pi} k(x_1, x_3) u_0(x_1, x_3) dx_1 dx_3, \\ \varphi_0 I_1 &= \tilde{M}_1^{(0)} + \int_{\Pi} k(x_1, x_3) x_3 u_0(x_1, x_3) dx_1 dx_3, \\ -\psi_0 I_3 &= \tilde{M}_3^{(0)} + \int_{\Pi} k(x_1, x_3) x_1 u_0(x_1, x_3) dx_1 dx_3\end{aligned}$$

and for the increments $n = 1, 2, \dots, K$

$$\begin{aligned}\vartheta_n \kappa A &= R_n + \kappa \int_{\Pi} k(x_1, x_3) u_n(x_1, x_3) dx_1 dx_3, \\ \varphi_n \kappa I_1 &= M_n^{(3)} + \kappa \int_{\Pi} k(x_1, x_3) x_3 u_n(x_1, x_3) dx_1 dx_3, \\ -\psi_n \kappa I_3 &= M_n^{(1)} + \kappa \int_{\Pi} k(x_1, x_3) x_1 u_n(x_1, x_3) dx_1 dx_3.\end{aligned}$$

From the above formulae we shall obtain

$$\begin{aligned}\vartheta_0 &= \tilde{\vartheta}_0 = \frac{1}{A} \left[\tilde{R}_0 + \int_{\Pi} k(x_1, x_3) u_0(x_1, x_3) dx_1 dx_3 \right], \\ \varphi_0 &= \tilde{\varphi}_0 = \frac{1}{I_1} \left[\tilde{M}_0^{(3)} + \int_{\Pi} k(x_1, x_3) x_3 u_0(x_1, x_3) dx_1 dx_3 \right], \\ \psi_0 &= \tilde{\psi}_0 = -\frac{1}{I_3} \left[\tilde{M}_0^{(1)} + \int_{\Pi} k(x_1, x_3) x_1 u_0(x_1, x_3) dx_1 dx_3 \right].\end{aligned}\tag{6.11}$$

For $n = 1, 2, \dots, K$, on the other hand, we shall get

$$\begin{aligned}\tilde{\vartheta}_n &= \vartheta_0 \left(1 - \frac{1}{\kappa} \right) + \frac{1}{\kappa A} \left[\tilde{R}_n + \kappa \int_{\Pi} k(x_1, x_3) \tilde{u}_n(x_1, x_3) dx_1 dx_3 \right], \\ \tilde{\varphi}_n &= \varphi_0 \left(1 - \frac{1}{\kappa} \right) + \frac{1}{\kappa I_1} \left[\tilde{M}_n^{(3)} + \kappa \int_{\Pi} k(x_1, x_3) x_1 \tilde{u}_n(x_1, x_3) dx_1, dx_3 \right], \\ \tilde{\psi}_n &= \psi_0 \left(1 - \frac{1}{\kappa} \right) + \frac{1}{\kappa I_3} \left[\tilde{M}_n^{(1)} + \kappa \int_{\Pi} k(x_1, x_3) x_3 \tilde{u}_n(x_1, x_3) dx_1, dx_3 \right].\end{aligned}\tag{6.12}$$

It is to be emphasized that the formulae (6.11) and (6.12) are to hold only when the conditions (6.7) and (6.8) are fulfilled. If the conditions (6.7) and (6.8) are not satisfied, then the formulae (6.11) and (6.12) do not represent the solution of the problem and cannot be applied.

In order to draw attention to inelastic features of the subsoil let us consider a case in which $\tilde{u}_K = 0$ and let us assume that $\tilde{R}_K = 0$, $\tilde{M}_K^{(1)} = 0$ and $M_K^{(3)} = 0$.

This means that we have taken the whole load off the structure under consideration.

Then, according to the formulae (6.12), for $n = K$ we shall get

$$\begin{aligned}\tilde{\vartheta}_K &= \vartheta_0 \left(1 - \frac{1}{\varkappa}\right), \\ \tilde{\varphi}_K &= \varphi_0 \left(1 - \frac{1}{\varkappa}\right), \\ \tilde{\psi}_K &= \psi_0 \left(1 - \frac{1}{\varkappa}\right).\end{aligned}\tag{6.13}$$

In the above formulae $\vartheta_0, \varphi_0, \psi_0$ characterize primary deformations of the subsoil caused by the action of the rigid construction loaded with the resultant force R_0 and the resultant couples $M_0^{(1)}, M_0^{(3)}$ (according to the formulae (6.11) while $\tilde{\vartheta}_K, \tilde{\varphi}_K, \tilde{\psi}_K$ characterize analogous deformation of the not loaded structure (as we have assumed $\tilde{R}_K = 0, \tilde{M}_K^{(1)} = 0, \tilde{M}_K^{(3)} = 0$). Thus, it can be seen that $\tilde{\vartheta}_K, \tilde{\varphi}_K, \tilde{\psi}_K$ characterize inelastic deformations, i.e. deformations that are not accompanied by any forces. Thus, the formulae (6.13) characterize a certain effect of inelasticity of the subsoil which will always occur when $\varkappa \neq 1$.

If x_1 is the symmetry axis of the region II and when the loads are symmetric in relation to that axis, and the subsoil surfaces after deformations are cylindrical surfaces of constant curvatures and independent of x_3 , then $\tilde{M}_n^{(1)} = 0, \tilde{\psi}_n = 0, n = 1, 2, \dots, K$ and we shall get the special case of the considered problem, which was discussed in [1], where four states of the subsoil were considered (i.e. $K = 3$), assuming $u_0 = 0, u_1 = u_1(x_1), u_2 = u_2(x_1), u_3 = 0$, where $u_1(x_1)$ and $u_2(x_1)$ being assumed as the cylindrical surfaces of constant curvature of a different sign.

7. Finals remarks

In sections 4,5 and 6 general formulae were introduced describing the footing structure behaviour on a partly elastic subsoil which undergoes subsequent deformations. The solution of the problem was obtained only for a special case of the rigid footing, the interaction of which with the subsoil is described by the conditions (6.7) and (6.8). These conditions can be checked, however only obtaining after the solution. If the conditions (6.7), (6.8) are not fulfilled, then it is generally not possible to get a solution with elementary methods of the considered problem. Similarly, with elementary methods one is not able to obtain solutions for elastic plates and beams on a partly elastic subsoil; the formulae given in Sections 4 and 5 are limited only to formulate the problem, leaving open the methods of its solution. The analytic difficulties in obtaining solutions result from the fact that the problems discussed are characterized not only by equations, but also by inequalities. The analysis of the problems described with equations and inequalities simultaneously can be found e.g. in the monograph [3]. It is to be pointed out, however, that methods of solving problems of that kind numerically are known, and can be successfully applied for the equations and inequalities given in Secs. 4, 5, 6. Furthermore the case solved at the end of Sec. 6, in spite of a very special character, has a practical meaning in many engineering problems. In other cases an approximate approaches

can be applied, for example those discussed in the monograph [4] and making use of the notion of discretization of problems.

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Резюме

ВЗАИМОДЕЙСТВИЕ КОНСТРУКЦИЙ С ЧАСТИЧНО УПРУГИМ ОСНОВАНИЕМ

Целью работы является описание поведения конструкций посаженных на основании, которое изменяет конфигурацию. Учет влияния деформаций основания на поведение конструкций имеет практическое значение в задачах посадки объектов в районах убытков угольной промышленности. Рассмотрено упругие плиты и балки а также деформируемые конструкции, имеющие жесткие основания. Учтено: 1 — возможность потери контакта основы конструкций с основанием, 2 — разное качество отпора основания описанного в [1]. Получено решения в частном случае конструкций с жестким основанием имеющим практическое значение.

Streszczenie

WSPÓLPRACA KONSTRUKCJI Z CZĘŚCIOWO SPRĘŻYSTYM PODŁOŻEM

Celem pracy jest opis zachowania się konstrukcji spoczywających na podłożu doznającym zmian konfiguracji. Uwzględnienie wpływu deformacji podłoża na zachowanie się konstrukcji ma bowiem znaczenie praktyczne w problemach posadowienia obiektów na terenach objętych uszkodzeniami górniczymi. Rozpatrywano sprężyste płyty i belki oraz konstrukcje o sztywnych podstawach. Uwzględniono możliwość utraty podstawy konstrukcji z podłożem oraz różny charakter odporu podłoża przy odkształceniach pierwotnych i wtórnych, zgodnie z koncepcją podaną w [1]. Otrzymano rozwiązania dla przypadku szczególnego konstrukcji o sztywnej podstawie, mającego znaczenie praktyczne.

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