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THERMAL STRESSES IN A TRANSVERSELY ISOTROPIC LAYER CONTAINING AN ANNULAR CRACK. TENSILE- AND SHEAR-TYPE CRACK

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The thermal stress problem for an annular crack contained in a transversely isotropic layer is investigated using the technique of Hankels transforms, triple integral equations and series solutions. For symmetrical constant temperature over the crack surfaces and for two cases of antisymmetrical thermal loadings namely a prescribed temperature applied to the surfaces of the layer and uniform heat flow disturbed by an insulated crack, expressions for the crack shapes, the normal and shearing thermal stresses in the crack plane and the mode I and II stress intensity factors are obtained. Results are presented illustrating the effect of the physical properties of the material on the stress intensity factors.

1. Introduction

The penny-shaped crack in a temperature field was treated by Olesiak and Sneddon, who showed that the crack opens; the problem was symmetrical with respect to the crack plane [1]. The features of antisymmetry were presented by Florence and Goodier in the linear thermoelastic problem of uniform heat flow disturbed by a penny-shaped insulated crack [2]. The thermal stress problem for an annular crack contained in a transversely isotropic medium [3 - 5] is investigated in the present work. The thermal conditions considered are symmetrical [1] and antisymmetrical [2] (two cases of thermal loadings) with respect to the crack plane. Assuming the heat flux on the crack surface as a Fourier cosine series (with singularities at the tips of the crack) in symmetrical problem or the temperature on the crack surface in the form of Fourier sine series in antisymmetrical one, and axial or radial displacement, respectively, as a Fourier sine series, we have reduced the triple integral equations of the heat conduction and thermoelastic problems to the solution of infinite sets of linear simultaneous equations for the coefficients introduced in the series representations. Thus, the mode I and II thermal stress intensity factors are easily evaluated in terms of the coefficients of series expansions of the physical quantities.

2. The potentials of thermoelastic displacements

The thermoelastic displacement potentials $\varphi_i(r, z)(i = 0, 1, 2)$ satisfying the foregoing equilibrium equations are defined [6] as:

$$u = \partial_r(k\varphi_1+\varphi_2+\varphi_0), \quad \vartheta = 0, \quad w = \partial_z(\varphi_1+k\varphi_2+k_1\varphi_0), \tag{2.1}$$

$$\begin{aligned}
\sigma_r &= -G_1(k+1)\partial_z^2(\varphi_1+\varphi_2)-2Gr^{-1}u-G_1 s_0^2\varkappa(1+k_1)T,\\
\sigma_\theta &= -G_1(k+1)\partial_z^2(\varphi_1+\varphi_2)-2G\partial_r u-G_1 s_0^2\varkappa(1+k_1)T,\\
\sigma_z &= G_1(k+1)\partial_z^2(s_1^{-2}\varphi_1+s_2^{-2}\varphi_2)+G_1\varkappa(1+k_1)T,\\
\sigma_{rz} &= G_1(k+1)\partial_r\partial_z(\varphi_1+\varphi_2)+G_1(1+k_1)\partial_r\partial_z\varphi_0,\\
\sigma_{r\theta} &= \sigma_{z\theta} = 0,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&(\partial_r^2+r^{-1}\partial_r+s_i^{-2}\partial_z^2)\varphi_i(r, z) = 0, \quad (i = 0, 1, 2),\\
&s_0^{-2}\partial_z^2\varphi_0(r, z) = \varkappa T(r, z),
\end{aligned} \tag{2.3}$$

where the symbols ∂_r, ∂_z etc. denote partial differentiation and

$$\begin{aligned}
k_1 &= \frac{\beta_r(c_{13}+c_{44})-\beta_z(c_{11}-s_0^2c_{44})}{\beta_r(c_{33}s_0^2-c_{44})-\beta_z s_0^2(c_{13}+c_{44})},\\
\varkappa &= \frac{\beta_r(c_{33}s_0^2-c_{44})-\beta_z s_0^2(c_{13}+c_{44})}{s_0^2(c_{13}+c_{44})^2-(c_{11}-s_0^2c_{44})(c_{33}s_0^2-c_{44})},
\end{aligned} \tag{2.4}$$

$$\beta_r = (c_{11}+c_{12})\alpha_r+c_{13}\alpha_z, \quad \beta_z = 2c_{13}\alpha_r+c_{33}\alpha_z, \quad s_0^{-2} = \lambda_z/\lambda_r.$$

The temperature field in the medium in a steady state is governed by the following equation:

$$(\partial_r^2+r^{-1}\partial_r+s_0^{-2}\partial_z^2)T(r, z) = 0. \tag{2.5}$$

In above equations, the z-axis is along the axis of symmetry of the material that has five elastic constants c_{ij} and u, v, w are the radial, circumferential and axial components of the displacement, σ_r, σ_θ etc. are the components of the tensor stress, T is the deviation of the absolute temperature from the temperature of the medium in a state of zero stress and strain, G and G_1 are the shear modulii in the planes of isotropy and along the z-axis respectively, λ_z, λ_r and α_z, α_r are the coefficients of thermal conductivity and the linear thermal expansion along the z-axis and in the planes of isotropy, respectively, and s_1, s_2, k are the material parameters [6].

3. Statement of the problem and temperature fields

The geometry is illustrated in Fig. 1. The middle plane of the layer is taken as the plane $z = 0$ ($\zeta = 0$) in cylindrical polar coordinates, the annular crack extending over the region $a \leqslant r \leqslant b$ ($\lambda \leqslant \varrho \leqslant 1$). The crack is opened by the application of constant temperature $-T_0$ to its flat surfaces and the heat flux is zero outside the crack in the middle plane (Fig. 1a — symmetrical problem). Two cases of antisymmetrical thermal loadings also are to be considered (Fig. 1b — antisymmetrical problem).

First case concerns the temperature, odd in z, applied on the layer surfaces. The pro-

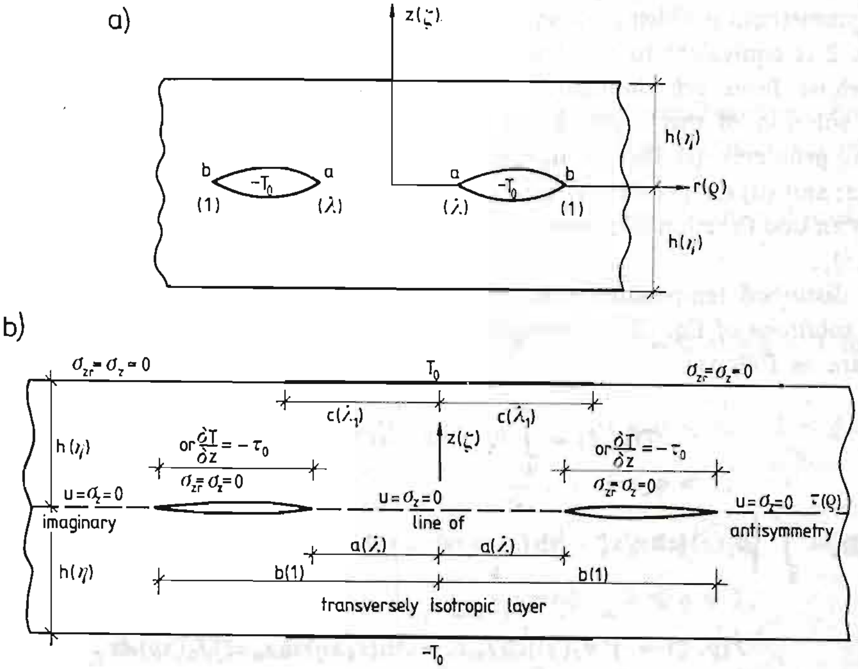

Fig. 1a, Geometry and coordinate system;
Fig. 1b, Geometry, coordinate system and boundary conditions

blem of uniform heat flow disturbed by a crack whose faces are thermally insulated in second case is considered. Because of symmetry, attention may be restricted to the region $0 \leqslant \zeta \leqslant \eta$.

The thermal conditions inside and outside the annulus $\zeta = 0$, $\lambda \leqslant \varrho \leqslant 1$ and at $\zeta = \eta$ are as follows:

$$T(\varrho, 0) = -T_0, \quad \lambda \leqslant \varrho \leqslant 1, \tag{3.1}$$

$$\frac{\partial T}{\partial \zeta}(\varrho, 0) = 0, \quad 0 \leqslant \varrho < \lambda, \quad 1 < \varrho, \qquad T(\varrho, \eta) = 0, \quad \varrho \geqslant 0, \tag{3.2}$$

in symmetrical problem and

case 1:

$$\frac{\partial T}{\partial \zeta}(\varrho, 0) = 0, \quad \lambda < \varrho < 1, \qquad T(\varrho, 0) = 0, \quad 0 \leqslant \varrho < \lambda, \quad 1 < \varrho, \tag{3.3}$$

$$T(\varrho, \eta) = \begin{cases} T_0, & 0 \leqslant \varrho < \lambda_1, \\ 0, & \lambda_1 < \varrho, \end{cases} \tag{3.4}$$

case 2:

$$\frac{\partial T}{b\partial \zeta}(\varrho, 0) = -\tau_0, \quad \lambda < \varrho < 1, \qquad T(\varrho, 0) = 0, \quad 0 \leqslant \varrho < \lambda, \quad 1 < \varrho, \tag{3.5}$$

$$T(\varrho, \eta) = 0, \quad \varrho \geqslant 0, \tag{3.6}$$

in antisymmetrical problem, where T_0 and τ_0 are constants. The boundary condition for the case 2 is equivalent to the problem of uniform heat flow disturbed by the annular crack whose faces are thermally insulated.

The solution of this problem can be obtained by superposing the solutions of the following problems. (i) The problem of crack-free region having temperature distribution $T = \tau_0 z$; and (ii) the problem of the annular crack with a temperature distribution $T(r, z)$ which is an odd function of z, vanishes at the edges of the medium and satisfies the condition $(3.5)_1$.

The disturbed temperature field is $\tau_0 z + T(r, z)$.

The solutions of Eq. (2.5), appropriate to the conditions (3.2), (3.4) and (3.6), respectively, are as follows:

$$T(\varrho, \xi) = \int_0^\infty \theta_0(x) \operatorname{sh} s_0 x(\zeta - \eta) J_0(x\varrho)\,dx, \tag{3.7}$$

$$T(\varrho, \xi) = \int_0^\infty \left\{\theta_1(x)\left[\operatorname{ch} s_0 x\zeta - \operatorname{cth}(s_0 x\eta) \operatorname{sh} s_0 x\zeta\right] + \lambda_1 T_0 J_1(x\lambda_1) \frac{\operatorname{sh} s_0 x\zeta}{\operatorname{sh} s_0 x\eta}\right\} J_0(x\varrho)\,dx, \tag{3.8}$$

$$T(\varrho, \zeta) = \int_0^\infty \theta_2(x)\left[\operatorname{ch} s_0 x\zeta - \operatorname{cth}(s_0 x\eta) \operatorname{sh} s_0 x\zeta\right] J_0(x\varrho)\,dx, \tag{3.9}$$

where $J_n(x\varrho)$ is the Bessel function of the first kind in order n. The unknown functions $\theta_0(x), \theta_1(x), \theta_2(x)$, can be found from *Eqs.* (3.1), (3.3), (3.5), which lead to the following triple integral equations:

$$\begin{aligned} &\int_0^\infty \theta_0(x) \operatorname{sh}(s_0 x\eta) J_0(x\varrho)\,dx = T_0, \quad \lambda \leqslant \varrho \leqslant 1, \\ &\int_0^\infty x\theta_0(x) \operatorname{ch}(s_0 x\eta) J_0(x\varrho)\,dx = 0, \quad 0 \leqslant \varrho < \lambda, \quad 1 < \varrho, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\int_0^\infty x\theta_1(x)[1 + h_0(x\eta)] J_0(x\varrho)\,dx = \\ &\qquad = \lambda_1 T_0 \int_0^\infty \frac{x}{\operatorname{sh} s_0 x\eta} J_1(x\lambda_1) J_0(x\varrho)\,dx, \quad \lambda < \varrho < 1, \end{aligned} \tag{3.11}$$

$$\int_0^\infty \theta_1(x) J_0(x\varrho)\,dx = 0, \quad 0 \leqslant \varrho < \lambda, \quad 1 < \varrho,$$

$$\begin{aligned} &\int_0^\infty x\theta_2(x)[1 + h_0(x\eta)] J_0(x\varrho)\,dx = \tau_0 b s_0^{-1}, \quad \lambda < \varrho < 1, \\ &\int_0^\infty \theta_2(x) J_0(x\varrho)\,dx = 0, \quad 0 \leqslant \varrho < \lambda, \quad 1 < \varrho, \end{aligned} \tag{3.12}$$

with $h_0(x\eta)$ being defined as

$$h_0(x\eta) = 2[\exp(2s_0 x\eta)-1]^{-1}. \tag{3.13}$$

The above triple integral equations correspond to the symmetrical problem, Eqs. (3.10), and antisymmetrical one in the case 1, Eqs. (3.11), and in the case 2, Eqs. (2.12), of thermal conditions.

We use here the series expansion method to solve the above triple integral equations [7]. Interchanging the variable ϱ in $\lambda \leqslant \varrho \leqslant 1$ to ϕ in $0 \leqslant \phi \leqslant \pi$

$$2\varrho^2 = 1+\lambda^2-(1-\lambda^2)\cos\phi \tag{3.14}$$

the variables ϱ and ϕ correspond each other and $\varrho = \lambda$ is $\phi = 0$ and $\varrho = 1$ to $\phi = \pi$.

The following integral formulas are to be utilized [8]:

$$I_n = \int_0^\infty xZ_n(x)J_0(x\varrho)dx = \begin{cases} 0, & 0 \leqslant \varrho < \lambda, \quad 1 < \varrho, \\ \dfrac{4}{\pi(1-\lambda^2)}\cdot\dfrac{\cos n\phi}{\sin\phi}, & \lambda < \varrho < 1, \end{cases}$$
$$I_{n-1}-I_{n+1} = \int_0^\infty C_n(x)J_0(x\varrho)dx = \begin{cases} 0, & 0 \leqslant \varrho \leqslant \lambda, \quad 1 \leqslant \varrho, \\ \dfrac{8}{\pi(1-\lambda^2)}\sin n\phi & \lambda \leqslant \varrho \leqslant 1, \end{cases} \tag{3.15}$$

where

$$Z_n(x) = J_n\left(x\frac{1+\lambda}{2}\right)J_n\left(x\frac{1-\lambda}{2}\right), \quad C_n(x) = x[Z_{n-1}(x)-Z_{n+1}(x)]. \tag{3.16}$$

Then, the unknown functions $\theta_0(x)$, $\theta_1(x)$ and $\theta_2(x)$ can be expressed by series:

$$\theta_0(x)\mathrm{ch}(s_0 x\eta) = T_0 \sum_{n=0}^{\infty} a'_n Z_n(x), \tag{3.17}$$

$$\theta_1(x) = \lambda_1 T_0 \sum_{n=1}^{\infty} b'_n C_n(x), \tag{3.18}$$

$$\theta_2(x) = \tau_0 b s_0^{-1} \sum_{n=1}^{\infty} c'_n C_n(x), \tag{3.19}$$

where $a'_0, a'_1, \ldots, b'_1, b'_2, \ldots$ and $c'_1, c'_2, \ldots$ are unknown parameters.

Thus, the appropriate series representations (3.17) - (3.19) satisfied two of the three equations exactly. Substituting Eqs. (3.17), (3.18) and (3.19) into Eqs. $(3.10)_1$, $(3.11)_1$ and $(3.12)_1$, respectively, and using the Neumann's formula [9]

$$J_0(x\varrho) = Z_0(x)+2\sum_{m=1}^{\infty} Z_m(x)\cos m\phi, \quad \lambda \leqslant \varrho \leqslant 1, \tag{3.20}$$

we get the following infinite sets of simultaneous equations for the determination of the coefficients a'_n, b'_n, c'_n:

$$\sum_{n=0}^{\infty} a'_n \int_0^\infty [1-g_0(x\eta)]Z_n(x)Z_m(x)dx = \delta_{0m}, \quad (m = 0, 1, 2, \ldots), \tag{3.21}$$

$$\sum_{n=1}^{\infty} b'_n \int_0^{\infty} [1+h_0(x\eta)]\, C_n(x)\, C_m(x)\, dx =$$

$$= \int_0^{\infty} \frac{1}{\operatorname{sh} s_0 x\eta} J_1(x\lambda_1)\, C_m(x)\, dx, \qquad (m = 1, 2, \ldots), \tag{3.22}$$

$$\sum_{n=1}^{\infty} c'_n \int_0^{\infty} [1+h_0(x\eta)]\, C_n(x)\, C_m(x)\, dx = \delta_{1m}, \qquad (m = 1, 2, \ldots), \tag{3.23}$$

where δ_{0m} and δ_{1m} are Kronecker's deltas and

$$g_0(x\eta) = 2[1+\exp(2x\eta s_0)]^{-1}. \tag{3.24}$$

In terms of coefficients a'_n, b'_n, c'_n it is found that

$$T(\varrho, \zeta) = T_0 \sum_{n=0}^{\infty} a'_n \int_0^{\infty} \frac{\operatorname{sh} s_0 x(\zeta-\eta)}{\operatorname{ch} s_0 x\eta} Z_n(x) J_0(x\varrho)\, dx, \tag{3.25}$$

$$T(\varrho, \zeta) = \lambda_1 T_0 \int_0^{\infty} \Big\{ \sum_{n=1}^{\infty} b'_n C_n(x) [\operatorname{ch} s_0 x\zeta - \operatorname{cth}(s_0 x\eta) \operatorname{sh} s_0 x\zeta] +$$

$$+ J_1(x\lambda_1) \frac{\operatorname{sh} s_0 x\zeta}{\operatorname{sh} s_0 x\eta} \Big\} J_0(x\varrho)\, dx, \tag{3.26}$$

$$T(\varrho, \zeta) = \tau_0 b s_0^{-1} \sum_{n=1}^{\infty} c'_n \int_0^{\infty} C_n(x)\, [\operatorname{ch} s_0 x\zeta - \operatorname{cth}(s_0 x\eta) \operatorname{sh} s_0 x\zeta] J_0(x\varrho)\, dx, \tag{3.27}$$

in the symmetrical and antisymmetrical (case 1 and 2) problems, respectively.

At the plane $\zeta = 0$ we have:

$$\begin{aligned} T(\varrho, 0) &= -T_0 \sum_{n=0}^{\infty} a'_n [I_0^n - 2G_0^n], \qquad 0 \leqslant \varrho \leqslant \lambda, \quad 1 \leqslant \varrho, \\ \frac{\partial T}{\partial \zeta}(\varrho, 0) &= \frac{4T_0 s_0}{\pi(1-\lambda^2)} \cdot \frac{1}{\sin\phi} \sum_{n=0}^{\infty} a'_n \cos(n\phi), \qquad \lambda < \varrho < 1, \end{aligned} \tag{3.28}$$

for symmetrical thermal loading and

$$\begin{aligned} T(\varrho, 0) &= \frac{8T_0\lambda_1}{\pi(1-\lambda^2)} \sum_{n=1}^{\infty} b'_n \sin(n\phi), \qquad \lambda \leqslant \varrho \leqslant 1, \\ \frac{\partial T}{\partial \zeta}(\varrho, 0) &= \frac{8T_0\lambda_1 s_0}{\pi(1-\lambda^2)} \sum_{n=1}^{\infty} n b'_n \left[I_0^n + \varrho \frac{\partial}{\partial \varrho} I_0^n - 2H_0^n \right] + \\ &+ \lambda_1 T_0 s_0 \int_0^{\infty} \frac{x}{\operatorname{sh} s_0 x\eta} J_1(x\lambda_1) J_0(x\varrho)\, dx, \qquad 0 \leqslant \varrho < \lambda, \quad 1 < \varrho, \end{aligned} \tag{3.29}$$

$$T(\varrho, 0) = \frac{8\tau_0 b}{\pi(1-\lambda^2)s_0} \sum_{n=1}^{\infty} c_n' \sin(n\phi), \quad \lambda \leqslant \varrho \leqslant 1,$$

$$\frac{\partial T}{\partial \zeta}(\varrho, 0) = \frac{8\tau_0 b}{\pi(1-\lambda^2)} \sum_{n=1}^{\infty} nc_n' \left[I_0^n + \varrho \frac{\partial}{\partial \varrho} I_0^n - 2H_0^n\right], \quad 0 \leqslant \varrho < \lambda, \quad 1 < \varrho \tag{3.30}$$

in the case 1 and 2 of the antisymmetrical problem.
In above equations:

$$I_0^n = \int_0^{\infty} J_0(x\varrho) Z_n(x) dx,$$

$$G_0^n = \int_0^{\infty} [\exp(2x\eta s_0)+1]^{-1} J_0(x\varrho) Z_n(x) dx, \quad (n = 0, 1, 2, \ldots), \tag{3.31}$$

$$H_0^n = \int_0^{\infty} [\exp(2x\eta s_0)-1]^{-1} J_0(x\varrho) x \frac{\partial Z_n(x)}{\partial x} dx, \quad (n = 1, 2, \ldots).$$

The integrals I_0^n can be presented analytically by Gaussian hypergeometric series and a Gamma function [10]. The integrals G_0^n and H_0^n can be easily evaluated numerically, because those integrands decrease exponentially to zero. The heat-flux on the plane $\zeta = 0$ has singularities of the form $(\varrho-\lambda)^{-1/2}$ at $\varrho = \lambda$ and $(1-\varrho)^{-1/2}$ at $\varrho = 1$ in the symmetrical problem and $(\lambda-\varrho)^{-1/2}$ and $(\varrho-1)^{-1/2}$ in antisymmetrical one (see Appendix to [10]).

In the limiting case of a penny-shaped crack problem ($\lambda = 0$) in the infinite medium ($\eta \to \infty$, $g_0(x\eta) = h_0(x\eta) = 0$) we obtain from the above results the closed-form solutions:

$$a_n' = -\frac{4}{\pi(1+\delta_{n0})} \cdot \frac{1}{4n^2-1}, \quad b_n' = 0, \quad c_n' = \frac{2}{\pi} \cdot \frac{n}{4n^2-1} \tag{3.32}$$

and at $\zeta = 0$

$$T(\varrho, 0) = -\frac{2}{\pi} T_0 \arcsin\left(\frac{1}{\varrho}\right), \quad \varrho \geqslant 1,$$

$$\frac{\partial T}{\partial \zeta}(\varrho, 0) = \frac{2}{\pi} T_0 s_0 \frac{1}{\sqrt{1-\varrho^2}}, \quad 0 \leqslant \varrho < 1, \tag{3.33}$$

in the symmetrical problem and

$$T(\varrho, 0) = \frac{2}{\pi} \tau_0 b s_0^{-1} \sqrt{1-\varrho^2}, \quad 0 \leqslant \varrho \leqslant 1,$$

$$\frac{\partial T}{\partial \zeta}(\varrho, 0) = \frac{2}{\pi} \tau_0 b \left[\frac{1}{\sqrt{\varrho^2-1}} - \arcsin\left(\frac{1}{\varrho}\right)\right], \quad 1 < \varrho, \tag{3.34}$$

n the uniform heat-flux problem.

4. The thermoelastic problem

The solutions of Eqs. (2.3) appropriate to our problems are:

$$\varphi_i(\varrho,\zeta) = \frac{s_{i\pm1}b^2}{G_1(k+1)(s_1-s_2)}\int\limits_0^\infty x^{-2}[A_i(x)\operatorname{sh}s_ix\zeta+B_i(x)\operatorname{ch}s_ix\zeta]J_0(x\varrho)\,dx, \tag{4.1}$$

$$\varphi_0(\varrho,\zeta) = b^2\varkappa\int\limits_0^\infty x^{-2}\theta_0(x)\operatorname{sh}s_0x(\zeta-\eta)J_0(x\varrho)\,dx, \tag{4.2}$$

$$\varphi_0(\varrho,\zeta) = b^2\varkappa\int\limits_0^\infty x^{-2}\Big\{\theta_1(x)[\operatorname{ch}s_0x\zeta-\operatorname{cth}(s_0x\eta)\operatorname{sh}s_0x\zeta]+ \\ +\lambda_1T_0J_1(x\lambda_1)\frac{\operatorname{sh}s_0x\zeta}{\operatorname{sh}s_0x\eta}\Big\}J_0(x\varrho)\,dx, \tag{4.3}$$

$$\varphi_0(\varrho,\zeta) = b^2\varkappa\int\limits_0^\infty x^{-2}\theta_2(x)[\operatorname{ch}s_0x\zeta-\operatorname{cth}(s_0x\eta)\operatorname{sh}s_0x\zeta]J_0(x\varrho)\,dx. \tag{4.4}$$

The potentials (4.1) and (4.2) are used to solve the boundary conditions (Fig. 1*a*):

$$\begin{aligned} \sigma_z(\varrho,0) &= 0, \quad \lambda<\varrho<1,\\ w(\varrho,0) &= 0, \quad 0\leqslant\varrho\leqslant\lambda, \quad 1\leqslant\varrho, \end{aligned} \tag{4.5}$$

$$\sigma_{zr}(\varrho,0) = \sigma_z(\varrho,\eta) = \sigma_{zr}(\varrho,\eta) = 0, \quad \varrho\geqslant0, \tag{4.6}$$

which correspond to tensile-type crack problem in the layer with traction-free surfaces and the potentials (4.1) and (4.3) or (4.4) to solve the boundary conditions (Fig. 1*b*):

$$\begin{aligned} \sigma_{zr}(\varrho,0) &= 0, \quad \lambda<\varrho<1,\\ u(\varrho,0) &= 0, \quad 0\leqslant\varrho\leqslant\lambda, \quad 1\leqslant\varrho, \end{aligned} \tag{4.7}$$

$$\sigma_z(\varrho,0) = \sigma_z(\varrho,\eta) = \sigma_{zr}(\varrho,\eta) = 0, \quad \varrho\geqslant0, \tag{4.8}$$

which correspond to shear-type crack problem.

5. The triple integral equations

At first we consider the mode I tensile-type crack problem. Substituting Eqs. (4.1) and (4.2) into Eqs. (2.2) and (2.1) the stress and displacement components can be obtained in terms of the four unknown functions $A_1(x)$, $A_2(x)$ etc. The conditions (4.6) yield equations:

$$\begin{aligned} A_2(x) &= -A_1(x)-\beta s_0(s_1s_2)^{-1}G_1\varkappa(1+k_1)\theta_0(x)\operatorname{ch}s_0x\eta,\\ \Delta_0(x\eta)B_1(x) &= -A_1(x)(\operatorname{ch}\alpha x\eta+\alpha\beta^{-1}\operatorname{ch}\beta x\eta-2s_1\beta^{-1}]-\\ &\quad -2\varkappa s_0s_2^{-1}G_1(1+k_1)\theta_0(x)(\operatorname{ch}s_2x\eta-\operatorname{ch}s_0x\eta),\\ \Delta_0(x\eta)B_2(x) &= A_1(x)(\operatorname{ch}\alpha x\eta-\alpha\beta^{-1}\operatorname{ch}\beta x\eta+2s_2\beta^{-1})+s_0(s_1s_2)^{-1}\varkappa G_1\times\\ &\quad\times(1+k_1)\theta_0(x)[\operatorname{ch}(s_0x\eta)(\beta\operatorname{ch}\alpha x\eta-\alpha\operatorname{ch}\beta x\eta)+2s_2\operatorname{ch}s_1x\eta], \end{aligned} \tag{5.1}$$

where

$$\Delta_0(x) = \operatorname{sh}\alpha x + \alpha\beta^{-1}\operatorname{sh}\beta x, \quad \{\alpha, \beta\} = s_1 \pm s_2. \tag{5.2}$$

The functions $A_2(x)$, $B_1(x)$ and $B_2(x)$ are expressed in terms of $A_1(x)$, and hence they can be eliminated from the rest of analysis. On the crack plane $\xi = 0$ Eqs. (5.1), (4.1), (4.2), (2.1) and (2.2) give:

$$w(\varrho, 0) = -(G_1 C)^{-1} b \int_0^\infty \{A_1(x) + \varkappa s_0 C G_1 (k-k_1)(k+1)^{-1}\theta_0(x)\operatorname{ch}s_0 x\eta\} x^{-1} J_0(x\varrho)\,dx,$$

$$\sigma_z(\varrho, 0) = \int_0^\infty \{[1-g_1(x\eta)]A_1(x) + s_0 s_2^{-1}\varkappa G_1(1+k_1)\theta_0(x)\operatorname{ch}s_0 x\eta \times \tag{5.3}$$

$$\times [1-g_2(x\eta)-g_3(x\eta)] - G_1\varkappa(1+k_1)\theta_0(x)\operatorname{sh}s_0 x\eta\} J_0(x\varrho)\,dx,$$

where

$$g_i(x) = \frac{1}{\Delta_0(x)}\begin{cases} 1+\alpha^2\beta^{-2}(\operatorname{ch}\beta x - 1) + \alpha\beta^{-1}\operatorname{sh}\beta x - e^{-\alpha x}, & i = 1,\\ \alpha\beta^{-1}(\operatorname{sh}\beta x + \operatorname{ch}\beta x) - 2s_2\beta^{-1} - e^{-\alpha x}, & i = 2,\\ 2s_2\beta^{-1}(\operatorname{ch}s_1 x - \operatorname{ch}s_2 x)/\operatorname{ch}s_0 x, & i = 3,\end{cases} \tag{5.4}$$

$$C = (k+1)(k-1)^{-1}(s_2^{-1} - s_1^{-1}). \tag{5.5}$$

Substituting Eqs. (5.3) into Eqs. (4.5) we obtain

$$\int_0^\infty [A_1(x) + \varkappa s_0 G_1 C(k-k_1)(k+1)^{-1}\theta_0(x)\operatorname{ch}(s_0 x\eta)] x^{-1} J_0(x\varrho)\,dx = 0,$$
$$0 \leqslant \varrho \leqslant \lambda, \quad 1 \leqslant \varrho, \tag{5.6}$$

$$\int_0^\infty \{[1-g_1(x\eta)]A_1(x) + s_0 s_2^{-1}\varkappa G_1(1+k_1)\theta_0(x)\operatorname{ch}(s_0 x\eta)[1-g_2(x\eta)-$$
$$-g_3(x\eta)]\} J_0(x\varrho)\,dx = G_1\varkappa(1+k_1)T_0, \quad \lambda < \varrho < 1, \tag{5.7}$$

where in the second equation the relation $(3.10)_1$ is used. Next, for the case 1 of the thermal conditions in antisymmetrical problem we have:

$$\begin{aligned} s_1 B_2(x) &= -s_2 B_1(x) - G_1\varkappa(1+k_1)\beta\theta_1(x),\\ \Delta_1(x\eta)A_1(x) &= -B_1(x)(\operatorname{ch}\alpha x\eta - \alpha\beta^{-1}\operatorname{ch}\beta x\eta + 2s_2\beta^{-1}) +\\ &\quad + G_1\varkappa(1+k_1)(s_2\operatorname{sh}s_0 x\eta)^{-1}[2(s_0\operatorname{sh}s_2 x\eta - s_2\operatorname{sh}s_0 x\eta)\theta_1(x) +\\ &\quad + T_0\lambda_1 J_1(x\lambda_1)(\gamma_1\operatorname{sh}\gamma_2 x\eta - \gamma_2\operatorname{sh}\gamma_1 x\eta)],\\ \Delta_1(x\eta)A_2(x) &= B_1(x)s_2 s_1^{-1}(\operatorname{ch}\alpha x\eta + \alpha\beta^{-1}\operatorname{ch}\beta x\eta - 2s_1\beta^{-1}) -\\ &\quad - G_1\varkappa(1+k_1)(s_1\operatorname{sh}s_0 x\eta)^{-1}\{[2s_0\operatorname{sh}s_1 x\eta - \operatorname{sh}s_0 x\eta(\alpha\operatorname{ch}\beta x\eta +\\ &\quad + \beta\operatorname{ch}\alpha x\eta)]\theta_1(x) + T_0\lambda_1 J_1(x\lambda_1)(\delta_1\operatorname{sh}\delta_2 x\eta - \delta_2\operatorname{sh}\delta_1 x\eta)\}, \end{aligned} \tag{5.8}$$

where

$$\Delta_1(x) = \operatorname{sh}\alpha x - \alpha\beta^{-1}\operatorname{sh}\beta x, \quad \gamma_{1,2} = s_2 \mp s_0, \quad \delta_{1,2} = s_1 \mp s_0. \tag{5.9}$$

The boundary conditions (4.7) lead to the triple integral equations:

$$u(\varrho, 0) = -b(G_1 C s_1)^{-1}\int_0^\infty [B_1(x) - s_1 G_1 C\varkappa(k_1-k)\times$$
$$\times (k+1)^{-1}\theta_1(x)] x^{-1} J_1(x\varrho)\,dx = 0, \quad 0 \leqslant \varrho \leqslant \lambda, \quad 1 \leqslant \varrho, \tag{5.10}$$

$$\sigma_{zr}(\varrho, 0) = s_2 \int_0^\infty B_1(x)[1-h_1(x\eta)]J_1(x\varrho)dx -$$

$$-G_1 \varkappa (1+k_1) \int_0^\infty \{\theta_1(x)[s_2(1+h_3(x\eta)) - s_0(1+h_0(x\eta))] +$$

$$+\lambda_1 T_0 J_1(x\lambda_1) h_2(x\eta)\} J_1(x\varrho)dx = 0, \quad \lambda < \varrho < 1, \tag{5.11}$$

where

$$h_i(x) = \frac{1}{\Delta_1(x)} \begin{cases} 1-\alpha^2\beta^{-2}(1-\mathrm{ch}\beta x)-\alpha\beta^{-1}\mathrm{sh}\beta x - e^{-\alpha x}, & i=1, \\ (\mathrm{sh} s_0 x)^{-1}[s_1\beta^{-1}(\gamma_1 \mathrm{sh}\gamma_2 x - \gamma_2 \mathrm{sh}\gamma_1 x) - \\ \qquad -s_2\beta^{-1}(\delta_1 \mathrm{sh}\delta_2 x - \delta_2 \mathrm{sh}\delta_1 x) + s_0], & i=2, \\ e^{-\alpha x} + \alpha\beta^{-1}(\mathrm{ch}\beta x + \mathrm{sh}\beta x - 2s_1\alpha^{-1}) + \\ \qquad +2s_0(s_2\beta \mathrm{sh} s_0 x)^{-1}(s_1 \mathrm{sh} s_2 x - s_2 \mathrm{sh} s_1 x), & i=3. \end{cases} \tag{5.12}$$

The unknown functions $B_2(x)$, $A_1(x)$ and $A_2(x)$ and the triple integral equations of the case 2 of thermal conditions can be obtained by replacing $\theta_1(x)$ by $\theta_2(x)$ in Eqs. (5.8), (5.10) and (5.11) and setting $J_1(x\lambda_1) = 0$.

6. Infinite sets of linear simultaneous equations

Using the formula $(3.15)_2$, the unknown function $A_1(x)$ in Eq. (5.6) can be expressed by series

$$A_1(x) = G_1 M\varkappa T_0 x \sum_{n=1}^{\infty}{}' a_n C_n(x) - \varkappa s_0 G_1 C(k-k_1)(k+1)^{-1}\theta_0(x)\mathrm{ch}(s_0 x\eta), \tag{6.1}$$

where a_n are the arbitrary coefficients and

$$M = (1+k_1)(1-s_0 s_2^{-1}) + C(k-k_1)(k+1)^{-1}s_0 \tag{6.2}$$

is the material parameter.

Thus, the equations (5.6) are satisfied exactly, while the equation (5.7) leads to

$$\sum_{n=1}^{\infty} a_n \int_0^\infty [1-g_1(x\eta)]xC_n(x)J_0(x\varrho)dx = 1 + \sum_{n=1}^{\infty} a_n' \int_0^\infty \{(1-m_0)g_0(x\eta) -$$

$$-m_1 g_1(x\eta) + m_2[g_2(x\eta)+g_3(x\eta)]\} Z_n(x)J_0(x\varrho)dx, \quad \lambda < \varrho < 1 \tag{6.3}$$

where the result (3.17) is used and

$$m_0 = (1+k_1)M^{-1}, \quad m_1 = Cs_0(k-k_1)[(1+k)M]^{-1}, \quad m_2 = s_0(1+k_1)(s_2 M)^{-1} \tag{6.4}$$

are the material parameters.

Substituting the relation (3.20) into Eq. (6.3) and equating the coefficients of $\cos(m\phi)$ in both sides we obtain infinite system of algebraic equations with respect to the unknown coefficients a_n. Substracting the $(m+2)$-th of these from the m-th, we obtain

$$\sum_{n=1}^{\infty} a_n \int_0^\infty [1-g_1(x\eta)]C_n(x)C_m(x)dx = \delta_{1m} + \tag{6.5}$$

$$+\sum_{n=0}^{\infty} a_n' \int_0^{\infty} \{(1-m_0)g_0(x\eta)-m_1 g_1(x\eta)+m_2[g_2(x\eta)+ \qquad (6.5) [cont.]$$

$$+g_3(x\eta)]\}Z_n(x)[Z_{m-1}(x)-Z_{m+1}(x)]dx, \quad (m = 1, 2, \ldots).$$

The matrix of the system (6.5) in l.h.s. is symmetrical with respect to m and n and in r.h.s. is not. The functions $g_0(x\eta)$, $g_1(x\eta)$ etc. are continuous for all of x and tend exponentially to zero as $x\eta$ tends to infinity. For the case of infinite medium these functions are indentically zero and the system (6.5) reduces to

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} C_n(x)C_m(x)dx = \delta_{1m}, \quad (m = 1, 2, 3, \ldots). \tag{6.6}$$

Next, in antisymmetrical problem, using [8]

$$\int_0^{\infty} J_1(x\varrho)Z_n(x)dx = \begin{cases} 0, & 0 \leqslant \varrho \leqslant \lambda, \quad 1 \leqslant \varrho, \\ \dfrac{\sin n\phi}{\pi n\varrho}, & \lambda \leqslant \varrho \leqslant 1, \quad (n = 1, 2, \ldots), \end{cases} \tag{6.7}$$

from Eq. (5.10) we obtain

$$B_1(x) = s_1 CG_1 \varkappa(k_1-k)(k+1)^{-1}\left[\theta_1(x)+\lambda_1 T_0 x \sum_{n=1}^{\infty} b_n Z_n(x)\right], \tag{6.8}$$

where b_n are the arbitrary coefficients.

Substituting Eq. (6.8) into Eq. (5.11), using the solution for $\theta_1(x)$ and the formula corresponding to Neumann's addition theorem [9]

$$xJ_1(x\varrho) = \frac{8\varrho}{(1-\lambda^2)\sin\phi} \sum_{m=1}^{\infty} mZ_m(x)\sin m\phi, \quad \lambda < \varrho < 1, \tag{6.9}$$

it is found that the set of linear simultaneous equations of the thermoelastic problem of the shear-type crack in the case 1 of thermal conditions is

$$\sum_{n=1}^{\infty} b_n \int_0^{\infty} [1-h_1(x\eta)]Z_n(x)Z_m(x)dx = \sum_{n=1}^{\infty} b_n' \int_0^{\infty} [Z_{n-1}(x)-$$

$$-Z_{n+1}(x)]Z_m(x)[-1+m_3\gamma_1+h_1(x\eta)+s_2 m_3 h_3(x\eta)-$$

$$-s_0 m_3 h_0(x\eta)]dx+m_3\int_0^{\infty} x^{-1}J_1(x\lambda_1)h_2(x\eta)Z_m(x)dx, \quad (m = 1, 2, 3, \ldots), \tag{6.10}$$

where

$$m_3 = (1+k_1)(k-1)\beta^{-1}(k_1-k)^{-1}. \tag{6.11}$$

For the case 2 of thermal conditions the set of simultaneous equation can be obtained by replacing b_n and b_n' by c_n and c_n', respectively, and setting $J_1(x\lambda_1) = 0$ in Eqs. (6.10). The unknown function $B_1(x)$ in this case can be obtained by replacing $\theta_1(x)$, $\lambda_1 T_0$ and b_n by $\theta_2(x)$, $\tau_0 bs_0^{-1}$ and c_n, respectively, in Eq. (6.8).

7. Thermal crack shape

The Eqs. $(5.3)_1$, (6.1) and the formula $(3.15)_2$ give the following crack shape in the case when the applied temperature is symmetrical with respect to the crack plane

$$w(\varrho, 0) = -\frac{8T_0 b\varkappa M}{\pi(1-\lambda^2)C}\sum_{n=1}^{\infty} a_n \sin(n\phi), \qquad \lambda \leqslant \varrho \leqslant 1. \tag{7.1}$$

For the cases of antisymmetrical problem the following thermal crack shapes we obtain:

$$u(\varrho, 0) = -\frac{1}{\pi} T_0 b\lambda_1 \varkappa(k_1 - k)(k+1)^{-1}\varrho^{-1}\sum_{n=1}^{\infty}\frac{b_n}{n}\sin(n\phi), \qquad \lambda \leqslant \varrho \leqslant 1, \tag{7.2}$$

$$u(\varrho, 0) = -\frac{1}{\pi}\tau_0 b^2 s_0^{-1}\varkappa(k_1 - k)(k+1)^{-1}\varrho^{-1}\sum_{n=1}^{\infty}\frac{c_n}{n}\sin(n\phi), \qquad \lambda \leqslant \varrho \leqslant 1, \tag{7.3}$$

in the case 1 and 2 of the thermal conditions, respectively.

8. Thermal stress distribution

The normal stress in Eq. (5.3), with the aid of Eq. (6.1) and the solution (3.17) may be rewritten to the form

$$\sigma_z(\varrho, 0) = G_1 M\varkappa \Big\{T(\varrho, 0) + T_0 \sum_{n=1}^{\infty} a_n [1 - g_1(x\eta)] x C_n(x) J_0(x\varrho)\, dx -$$
$$- T_0 \sum_{=0}^{\infty} a_n' \int_0^{\infty} [(1-m_0) g_0(x\eta) - m_1 g_1(x\eta) +$$
$$+ m_2 (g_2(x\eta) + g_3(x\eta))]\Big\}. Z_n(x) J_0(x\varrho)\, dx\Big\}. \tag{8.1}$$

Using the formulas

$$C_n(x) = \frac{8n}{1-\lambda^2}\cdot\frac{\partial Z_n(x)}{\partial x},$$
$$\int_0^{\infty} x C_n(x) J_0(x\varrho)\, dx = -\frac{8n}{1-\lambda^2}\left(I_0^n + \varrho\frac{\partial}{\partial\varrho} I_0^n\right) \tag{8.2}$$

and the result $(3.28)_1$, we obtain

$$\sigma_z(\varrho, 0) = -G_1 M\varkappa T_0 \left\{\frac{8}{1-\lambda^2}\sum_{n=1}^{\infty} n a_n \left(I_0^n + \varrho\frac{\partial}{\partial\varrho} I_0^n + \bar{G}_1^n\right) + \right.$$
$$\left. + \sum_{n=0}^{\infty} a_n' [I_0^n - 2m_0 G_0^n - m_1 G_1^n + m_2 (G_2^n + G_3^n)]\right\}, \tag{8.3}$$

where

$$\begin{aligned} G_j^n &= \int_0^\infty g_j(x\eta)Z_n(x)J_0(x\varrho)dx, \quad j = 1,2,3, \quad (n = 0,1,2,\ldots), \\ \bar{G}_1^n &= \int_0^\infty g_1(x\eta)xJ_0(x\varrho)\frac{\partial Z_n(x)}{\partial x}dx, \quad (n = 1,2,\ldots), \end{aligned} \tag{8.4}$$

are the convergent integrals.

The normal stress $\sigma_z(\varrho, 0)$ involves singularities of the form $(\lambda-\varrho)^{-1/2}$ and $(\varrho-1)^{-1/2}$ ($\partial I_0^n/\partial\varrho$, see Appendix to [10]).

Employing Eqs. (6.8) and (3.18) in Eq. (5.11), using the formula $(8.2)_1$ and integrating by parts, we obtain the shearing stress for the case 1 of antisymmetrical of thermal loading

$$\begin{aligned} \sigma_{zr}(\varrho,0) = & -T_0\lambda_1 s_1 s_2 G_1 C\varkappa(k_1-k)(k+1)^{-1}\left\{\sum_{n=1}^{\infty} b_n\left(\frac{\partial I_0^n}{\partial\varrho}+H_1^n\right)+\right. \\ & +\frac{4(1-m_3\gamma_1)}{1-\lambda^2}\varrho\sum_{n=1}^{\infty} nb_n'(I_0^n-I_2^n)+\frac{8m_3}{1-\lambda^2}\sum_{n=1}^{\infty} nb_n'h_1^n+ \\ & \left.+m_3\int_0^\infty J_1(x\lambda_1)h_2(x\eta)J_1(x\varrho)dx\right\}, \end{aligned} \tag{8.5}$$

where

$$\begin{aligned} I_l^n &= \int_0^\infty J_l(x\varrho)Z_n(x)dx, \quad l = 0,2, \\ H_1^n &= \int_0^\infty h_1(x\eta)xZ_n(x)J_1(x\varrho)dx, \\ h_1^n &= \int_0^\infty [s_2h_3(x\eta)+m_3^{-1}h_1(x\eta)-s_0h_0(x\eta)]J_1(x\varrho)\frac{\partial Z_n(x)}{\partial x}dx, \end{aligned} \tag{8.6}$$

are convergent integrals.

The shearing stress involves a square-root singularity at the crack tips. The singularity of $\sigma_{zr}(\varrho, 0)$ is included in term $\partial I_0^n/\partial\varrho$, while the remaining terms are nonsingular.

Replacying, in Eq. (8.5), $\lambda_1 T_0$, b_n, b_n' by $\tau_0 bs_0^{-1}$, c_n, c_n', respectively and setting, in Eq. (8.5), $J_1(x\lambda_1) = 0$ we obtain the shearing stress for the case 2 of antisymmetrical problem.

9. The stress intensity factors (SIFs)

To determine the singularities of the normal stress, the transform in Eq. (8.1) is calculated for C_n given in Eq. $(8.2)_1$ and the asymptotic formula [9]

$$x\frac{\partial Z_n(x)}{\partial x} \doteq -\frac{2}{\pi\sqrt{1-\lambda^2}}[\lambda\sin x\lambda-(-1)^n\cos x]+0(x^{-1}), \tag{9.1}$$

as follows:

$$\sigma_z(\varrho, 0) \sim -\frac{16G_1 M\varkappa T_0}{(1-\lambda^2)^{3/2}} \sum_{n=1}^{\infty} na_n \left[\frac{\lambda H(\lambda-\varrho)}{\sqrt{\lambda^2-\varrho^2}} + (-1)^{n+1}\frac{H(\varrho-1)}{\sqrt{\varrho^2-1}}\right], \tag{9.2}$$

where the nonsingular terms have been neglected and $H(\cdot)$ denotes Heaviside unit functional.

Defining the mode I „thermal" stress intensity factors as:

$$\begin{aligned} K^T_{\mathrm{I\,in.}} &= \sqrt{2b} \lim_{\varrho\to\lambda^-} \sqrt{\lambda-\varrho}\, \{\sigma_z(\varrho, 0)\}_{\varrho<\lambda}, \\ K^T_{\mathrm{I\,out.}} &= \sqrt{2b} \lim_{\varrho\to 1^+} \sqrt{\varrho-1}\, \{\sigma_z(\varrho, 0)\}_{\varrho>1}, \end{aligned} \tag{9.3}$$

we obtain

$$\begin{aligned} K^T_{\mathrm{I\,in.}} &= -\frac{16G_1 M\varkappa T_0 \sqrt{a}}{\pi(1-\lambda^2)^{3/2}} \sum_{n=1}^{\infty} na_n, \\ K^T_{\mathrm{I\,out.}} &= -\frac{16G_1 M\varkappa T_0 \sqrt{b}}{\pi(1-\lambda^2)^{3/2}} \sum_{n=1}^{\infty} (-1)^{n+1} na_n. \end{aligned} \tag{9.4}$$

The SIFs in shear at $\varrho = \lambda^-$ and $\varrho = 1^+$ can be obtained from equation (8.5) Using

$$\begin{aligned} \int_0^{\infty} xJ_1(x\varrho) Z_n(x)dx &= -\frac{\partial}{\partial\varrho} I_0^n, \\ xZ_n(x) &\doteq \frac{2}{\pi\sqrt{1-\lambda^2}} [\cos\lambda x + (-1)^n \sin x] + 0(x^{-1}), \end{aligned} \tag{9.5}$$

we have as $\varrho \to 1^+$ and $\varrho \to \lambda^-$

$$\begin{aligned} \sigma_{zr}(\varrho, 0) \sim &-\frac{2}{\pi\sqrt{1-\lambda^2}} s_1 s_2 G_1 C\varkappa(k_1-k)(k+1)^{-1}\lambda_1 T_0 \varrho^{-1} \times \\ &\times \sum_{n=1}^{\infty} b_n \left\{\frac{\lambda H(\lambda-\varrho)}{\sqrt{\lambda^2-\varrho^2}} + (-1)^{n+1}\frac{H(\varrho-1)}{\sqrt{\varrho^2-1}}\right\}. \end{aligned} \tag{9.6}$$

Both case 1 and case 2 thermal loadings give the following SIFs:

$$\begin{aligned} K^T_{\mathrm{II\,in.}} &= -\frac{2\sqrt{a}}{\pi\lambda\sqrt{1-\lambda^2}} G_1 C s_1 s_2 \varkappa(k_1-k)(k+1)^{-1}\lambda_1 T_0 \sum_{n=1}^{\infty} b_n, \\ K^T_{\mathrm{II\,out.}} &= -\frac{2\sqrt{b}}{\pi\sqrt{1-\lambda^2}} G_1 C s_1 s_2 \varkappa(k_1-k)(k+1)^{-1}\lambda_1 T_0 \sum_{n=1}^{\infty} (-1)^{n+1} b_n, \end{aligned} \tag{9.7}$$

for the temperature T_0 applied at the surfaces of the layer and

$$K^T_{\text{II in.}} = -\frac{2\sqrt{a}}{\pi\lambda\sqrt{1-\lambda^2}} G_1 C s_1 s_2 \varkappa(k_1-k)(k+1)^{-1}\tau_0 b s_0^{-1} \sum_{n=1}^{\infty} c_n,$$

$$K^T_{\text{II out.}} = -\frac{2\sqrt{b}}{\pi\sqrt{1-\lambda^2}} G_1 C s_1 s_2 \varkappa(k_1-k)(k+1)^{-1}\tau_0 b s_0^{-1} \sum_{n=1}^{\infty} (-1)^{n+1} c_n, \tag{9.8}$$

for uniform heat flow τ_0 disturbed by an annular crack.

10. The special cases

(a) Cracks in an infinite medium

Particulary, taking $\eta \to \infty$, we have $g_i(x\eta) = G_i^n = 0$ $(i = 0, 1, 2, 3)$ and $\operatorname{sh} s_0 x(\zeta-\eta)/$ $/\operatorname{ch} s_0 x\eta = -\exp(-s_0 x\xi)$.

Using preceding results in Eqs. (3.21), (3.25) and (3.28) we obtain easily the temperature field in infinite medium under symmetrical loading. From Eqs. (6.6), (7.1), (8.3) and (9.4) we obtain the solution of thermoelastic problem of the mode I tensile-typ crack.

Next, taking $\eta \to \infty$ in antisymmetrical problem, we have $h_i(x\eta) = h_i^n = H_i^n = 0$ $(i = 0, 1, 2, 3)$. In this case the r.h.s. in Eq. (3.22) degenerates to zero and b'_n are identically zero. Consequently, the solutions b_n of Eqs. (6.10) are equal zero and both the SIFs approache zero for infinite medium in the case 1 of thermal loading. For uniform heat flow disturbed by an annular crack the infinite sets of linear algebraic equations are:

$$\sum_{n=1}^{\infty} c'_n \int_0^{\infty} C_n(x) C_m(x)\,dx = \delta_{1m}, \qquad (m = 1, 2, \ldots),$$

$$\sum_{n=1}^{\infty} c_n \int_0^{\infty} Z_n(x) Z_m(x)\,dx = (-1+m_1\gamma_1) \sum_{n=1}^{\infty} c'_n [Z_{n-1}(x) - Z_{n+1}(x)] Z_m(x)\,dx. \tag{10.1}$$

In the classical case of a penny-shaped crack problem ($\lambda = 0$) in the infinite medium the thermal fields and the coefficients a'_n, b'_n, c'_n are given in closed-form by formulas (3.32)-(3.34). For the mode I tensile-type penny-shaped crack problem the equations (6.6) have a closed-form solution: $a_n = 2n/\pi(4n^2-1)$ and the normal components $w(\varrho, 0)$, $\sigma_z(\varrho, 0)$ and the SIF are:

$$\begin{aligned} w(\varrho, 0) &= -2\pi^{-1} T_0 b \varkappa M C^{-1}(1-\varrho^2)^{1/2}, \quad && 0 \leqslant \varrho \leqslant 1,\\ \sigma_z(\varrho, 0) &= -2\pi^{-1} T_0 \varkappa G_1 M(\varrho^2-1)^{-1/2}, && \varrho > 1,\\ K_I^T &= -2\pi^{-1} T_0 b^{1/2} \varkappa G_1 M. \end{aligned} \tag{10.2}$$

On the other hand, if $\lambda = a/b \to 0$ but b is bounded, the SIF $K^T_{\text{I out.}}$ tends to the result of a penny-shaped crack and $K^T_{\text{I in.}}$ tends to infinity, while if $\lambda \to 1-2\varepsilon$($\varepsilon - a$ small value) the „thermal" SIFs become equal: $K^T_{\text{I out}} \to K^T_{\text{I in}} \approx -G_1 \varkappa M T_0 \sqrt{b\varepsilon}$ and tend to zero as ε is zero.

For the uniform heat flow disturbed by a penny-shaped insulated crack in an infinite transversely isotropic medium the temperature field is given by Eqs. (3.34) and the SIF assume the value

$$K_{II}^{T} = \frac{2}{3\pi}\tau_0 b^{3/2} G_1 C s_1 s_2 \varkappa (k_1 - k)(k+1)^{-1} s_0^{-1}(1 - m_3\gamma_1). \tag{10.3}$$

(b) Isotropic medium

A limiting case of isotropic state gives:

$s_1 \to 1, \quad s_2 \to 1, \quad s_0 = 1, \quad k \to 1, \quad \alpha = \gamma_2 = \delta_2 = 2, \quad \beta \to 0, \quad \gamma_1 \to 0,$
$\delta_1 \to 0, \quad \beta_r = \beta_z, \quad \alpha_r = \alpha_z, \quad G_1 = G, \quad k_1 \to 3 - 2\nu, \quad \varkappa \to (1+\nu)\alpha_r/(1-\nu),$
$m_0 = m_2 = -m_3 = -2(2-\nu), \quad m_1 = 1, \quad C = (1-\nu)^{-1}, \quad M = -1, \quad \varkappa(k_1 - k) \times (k+1)^{-1} = (1+\nu)\alpha_r, \quad G_1 C s_1 s_2 \varkappa (k_1 - k)(k+1)^{-1} = G\alpha_r(1+\nu)/(1-\nu)$ and the boundary function are:

$$\begin{aligned} g_0(x) &= 2[\exp(2x)+1]^{-1}, \\ g_i(x) &= \frac{1}{\operatorname{sh}2x + 2x} \begin{cases} 1 + 2x(1+x) - \exp(-2x), & i = 1 \\ 1 + 2x - \exp(-2x), & i = 2 \\ 2x + \operatorname{th}x, & i = 3 \end{cases} \\ h_0(x) &= 2[\exp(2x) - 1]^{-1} \\ h_2(x) &= 0 \\ h_i(x) &= \frac{1}{\operatorname{sh}2x - 2x} \begin{cases} 1 + 2x(x-1) - \exp(-2x), & i = 1 \\ 1 + 2x(1 - \operatorname{cth}x) + \exp(-2x), & i = 3. \end{cases} \end{aligned} \tag{10.4}$$

From preceding equations, we note that the boundary functions do not depend on the physical properties of the material in the case of isotropic medium, while in the transversely isotropic case depend, as shown Eqs. (5.4) and (5.12).

The results (10.2) and (10.3) agree with those in the case of the penny-shaped crack in isotropic medium, where are given by Olesiak and Sneddon [1], and Florence and Goodier [2].

11. Numerical results and discussion

Solving the infinite systems of linear algebraic equations (3.21)–(3.23) and (6.5), (6.10) in both cases numerically and by truncation, we have examined the effects of geometrically parameters and physical properties of the medium on the stress intensity factors. The integrals involving the product of four Bessel function can be evaluated by the similar method as in the previous papers [10, 11]. The first ten roots of the sets of algebraic equations give the desired accuracy.

The value of normalized stress intensity factors $-K_I^T/G_1 M \varkappa T_0 \sqrt{b}$ versus $\lambda = a/b$ are shown in Fig. 2, for infinite medium. Both SIFs increases monotonically with the decrease a/b and become infinity or $2/\pi$, respectively, as $a/b \to 0$. Since $K_{I\,in.}^T > K_{I\,out.}^T$, growth of the crack under mode I thermal loading would tend to occur at the inner edge suggesting that an annular crack will develop into a penny-shaped crack. In contrast to the results of the crack in infinite medium under mode I [11] and III [10] mechanical loading, the intensity of the local thermal stresses depends on the physical properties of the material. The effect of the material dissimilarity is used by the parameters $\varkappa G_1 M$. The crack opens if $T_0 \varkappa M < 0$.

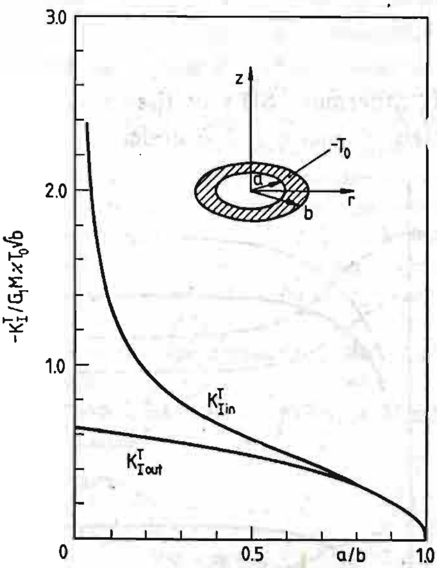

Fig. 2 Stress intensity factors for annular crack under uniform temperature

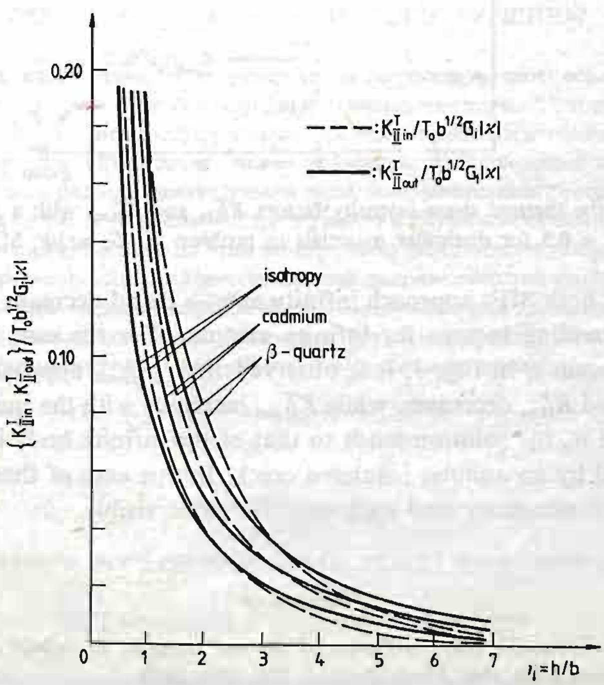

Fig. 3 Variation of the thermal stress intensity factors $K_{II\,in}^T$ and $K_{II\,out}^T$ with a layer thickness under $\lambda = 0{,}5$ and $\lambda_1 = 2{,}0$ for dissimilar materials in problem I; $G_i = 10^4$MPa

For the mode II shear-type crack problem numerical computations were done for cadmium and β-quartz crystals and comparative isotropic material ($G_t = 10^4$MPa, $\nu = 0{,}30$). The values of elastic constants c_{ij}, coefficients of linear expansion α_r, α_z and the ratio of the thermal conductivity coefficients s_0^{-2} are taken from references [12]. The anisotropic thermal and elastic properties of materials were presented in [13] for metallic substances and in [14] for composite materials. The results for the antisymmetrical thermal conditions of the case 1 are shown in Fig. 3 and those for the case 2 shown in Fig. 4. Fig. 3 shows the mode II „thermal" SIFs at the inner and outer tips of the annular crack, in presented materials, versus $\eta = h/b$ under $\lambda = a/b = 0{,}5$ and $\lambda_1 = c/b = 2{,}0$.

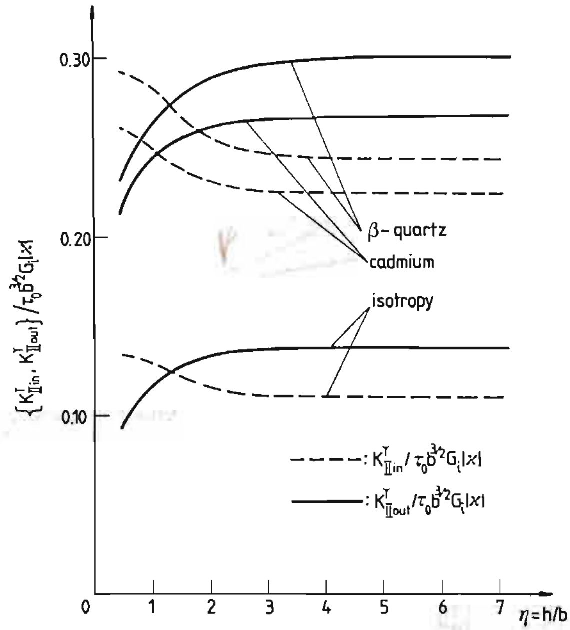

Fig. 4 Variation of the thermal stress intensity factors $K^T_{II\,in}$ and $K^T_{II\,out}$ with a layer thickness under $\lambda = 0{,}5$ for dissimilar materials in problem II; $G_t = 10^4$ MPa

It is observed, that both SIFs approach infinity as $\eta \to 0$ and decrease monotonically with the increase of η tending to zero for infinite medium. For the case 2 and $\lambda = 0{,}5$, the SIFs are plotted versus η in Fig. 4. It is observed that, $K^T_{II\,out}$ approaches zero as $\eta \to 0$ but $K^T_{II\,in}$ doesn't and $K^T_{II\,in}$ decreases, while $K^T_{II\,out}$ increases with the increase of η. When η is larger then some η_0 the solution tends to that of the infinite body in which is uniform heat flow disturbed by an annular insulated crack. In this case of thermal conditions the effect of material dissimilarity and anisotropy is more visible.

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Резюме

ТЕПЛОВЫЕ НАПРЯЖЕНИЯ В ТРАНСВЕРСАЛЬНО-ИЗОТРОПНОМ СЛОЕ С КОЛЬЦЕВОЙ ТРЕЩИНОЙ. РАСКРЫТИЕ И СДВИГ ТРЕЩИНЫ

Задачу тепловых напряжений в трансверсально-изотропном слое ослабленном кольцевой трещиной и подверженном действию симметричной и антисимметричной тепловой нагрузки сформулировано как решение тройных интегральных уравнений соответсвующих задачам теплопроводности и термоупругости. До решения выше указанных предположено такие представления искомых функций в виде рядов с неизвестными пока коэффициентами, которые удовлетворяют два из трех уравнений точно, в то время как третие водут к двум бесконечным системам линейных алгебраических уравнений относительно этих коэффициентов. Соответсвующие суммы этих коэффициентов определяют коэффициенты интенсивности напряжений при раскрыти и сдвиге трещины, а образобанные при их помощи ряды Фурье определяют вид трещины в задачах. Результаты бычислений, представлены графически, иллюстрируют коэффийиенты интенсивности напряжений и эффект физических свойств материалов в процесе разрушения.

Streszczenie

NAPRĘŻENIA CIEPLNE W POPRZECZNIE IZOTROPOWEJ WARSTWIE Z PIERŚCIENIOWĄ SZCZELINĄ. ROZWIERANIE I ŚCINANIE SZCZELINY

Zagadnienie naprężeń cieplnych w warstwie poprzecznie izotropowej osłabionej pierścieniową szczeliną i poddanej działaniu symetrycznego i antysymetrycznych obciążeń termicznych sformułowano jako rozwiązanie dwóch układów potrójnych równań całkowych, odpowiadających zagadnieniom przepływu ciepła i termosprężystemu. W celu rozwiązania tych ostatnich zaproponowano takie przedstawienia poszukiwanych funkcji w postaci szeregów z nieznanymi współczynnikami, które spełniają dwa z trzech równań ściśle, podczas gdy trzecie prowadzą do dwóch nieskończonych układów równań algebraicznych liniowych względem tych współczynników. Odpowiednie sumy tych współczynników wyznaczają współczynniki intensywności naprężenia przy rozwieraniu i ścinaniu szczeliny, a utworzone z ich pomocą szeregi Fouriera określają kształt szczeliny w obu zagadnieniach. Przedstawiono graficznie wyniki ilustrujące współczynniki intensywności naprężenia i efekt fizycznych właściwości materiałów w procesie pękania.

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