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DIFFERENTIAL MODELS OF HEXAGONAL-TYPE GRID PLATES

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1. Introduction

The subject of the present paper is an analysis of various differential models approximating deformations of dense, elastic, hexagonal-type (honeycomb) plates in plane-stress state.

The simplest mathematical model describing honeycomb plate response is, so called in engineering literature, technical isotropy, cf. [1, 2]. Elastic properties are determined by two effective moduli e.g. effective Young modulus and effective Poisson's ratio. These characteristics have been found by Horvay (see [1]) in 1952; some adjustments concerning the deformability of nodes have been proposed in [2].

More accurate approximation yields from Woźniak's models of grid surface structures based on the two-dimensional Cosserats' media theory, [3]. Among many papers pertaining to the response of lattice-type plates of simple and complex layout (the list of them has been published in [3]) the only one [4] is devoted to hexagonal surface structures. Generalisation and extension of Klemm's and Woźniak's results are presented in [5]. However, in the latter work, some new questions occur concerning the existence of two different variants resulting from Woźniak's approach. One aim of the present work is to elucidate, why more than one version (in a frame of one Cosserats' model) can exist. In order to achieve the answer a new look at the problem is necessary.

„Phenomenological" approaches (resembling to that of Woźniak, for instance) will not be applied here. Differential approximations for difference equilibrium equations of the lattice will be found by means of Rogula and Kunin quasicontinuum method, [6, 10], analogy between the mentioned difference equations (yielded from the well known displacement method) and crystal lattice equations resulting from harmonic approximation [6, 7] being utilised. Such a method makes it feasible to carry out a consequent accuracy analysis of the proposed models and in particular allows a new look at Woźniak's theory; a separate paper will be devoted to the latter problem. Derivations performed via the Rogula-Kunin approach result from physically clear approximations. Nevertheless the obtained differential models of higher order than zero do not satisfy stability conditions (in the spirit of Kunin [6], for example). Thus the derived models cannot be used for analysis of boundary value problems. A simple method of formulating a stable, well established Cosserats' type model derived from Rogula-Kunin's differential approximations will be presented in a separate paper. In the prepared work a comparison of Woźniak's and modified Rogula-Kunin's Cosserat models will be carried out.

It is worth emphasising that more complicated (of higher order than one) continuum descriptions of hexagonal-type grid plates can be formulated as stable models via appropriate generalisation of Kunin's methods [6]; but the mentioned topics exceed the scope of the present paper.

2. Preleminaries. Basic assumptions

Consider elastic grid plate (in plane-stress state), cf. Fig. 2.1 in [5], axes of the rods constitute a honeycomb layout. A thickness of the plate is assumed to be of unit size. Rods'axes form hexagons the length of sides being equal to l. The rods are assumed to have two axes of symmetry, cross section areas and moments of inertia can vary. Lattice rods are made of elastic homogeneous and isotropic material whose elastic properties are determined by Young modulus E and Poisson's ratio ν. Considerations are confined to the grids composed of sufficiently slender bars so as to their deflections could be decribed by means of the improved theory of rods, where transverse shear deformations are taken into account. External loads are assumed to be subjected in-plane and concentrated in nodes only.

Notations, sign conventions of the external loads (forces and moments), of displacements and of internal forces as well as slope deflection equations are assumed as in the previous paper [5].

Proceeding analogously as in [4,5] two families of nodes: main and intermediate are distinguished, Fig. 1. To each main node a pair of integer numbers $\mathbf{m} = (m_1, m_2)$ is assigned. Cartesian coordinates $\mathbf{x}^{\mathbf{m}}$ of a node $\mathbf{m}$ and a vector $\mathbf{m}$ are interrelated by means of the formula

$$\mathbf{x}^{\mathbf{m}} = \mathbf{\Omega}\cdot\mathbf{m}, \quad \mathbf{\Omega} = b\cdot\begin{bmatrix}1 & 0{,}5\\ 0 & \sqrt{3}/2\end{bmatrix}, \quad b = l\sqrt{3}. \tag{2.1}$$

Main node displacements are denoted as follows

$$w^1_{\mathbf{m}} = u_{\mathbf{m}} = u^1(\mathbf{x}^{\mathbf{m}}), \quad w^2_{\mathbf{m}} = v_{\mathbf{m}} = u^2(\mathbf{x}^{\mathbf{m}}), \quad w^3_{\mathbf{m}} = \varphi_{\mathbf{m}} = \varphi(\mathbf{x}^{\mathbf{m}}). \tag{2.2}$$

Forces and moments subjected to main $\mathbf{m}$ and intermediate $\mathbf{m}'$ nodes are denoted by

$$F^{\mathbf{m}}_{\alpha} = F^{\alpha}(\mathbf{x}^{\mathbf{m}}), \quad F^{\mathbf{m}}_{3} = M(\mathbf{x}^{\mathbf{m}}), \quad \overset{*}{F}{}^{\mathbf{m}'}_{\alpha} = \overset{*}{F}{}^{\alpha}(\mathbf{x}^{\mathbf{m}'}), \quad \overset{*}{F}{}^{\mathbf{m}'}_{3} = \overset{*}{M}(\mathbf{x}^{\mathbf{m}'}), \quad \alpha = 1, 2. \tag{2.3}$$

Each main node $\mathbf{m}$ is surrouned by six main nodes $\mathbf{m}_J$, $J = \mathrm{I}, \ldots, \mathrm{VI}$

$$\mathbf{x}^{\mathbf{m}_J} = \mathbf{x}^{\mathbf{m}} - \mathbf{t}_J \tag{2.4}$$

which lie on the circumference of the circle $r = b = l\sqrt{3}$ ($\mathbf{t}_J$ vectors are shown in Fig. 1) and by intermediate nodes $\mathbf{m}'_J$, $J = \mathbf{a}, \mathbf{b}, \mathbf{c}$

$$\mathbf{x}^{\mathbf{m}'_J} = \mathbf{x}^{\mathbf{m}} - \mathbf{z}_J, \quad J = \mathbf{a}, \mathbf{b}, \mathbf{c}.$$

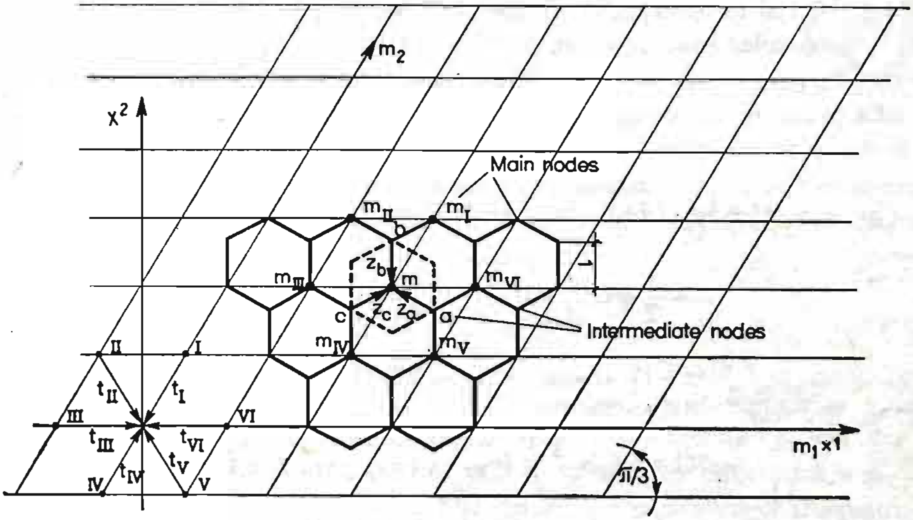

Fig. 1

Without afraid of misunderstandings one can write also

$$\mathbf{m}-\mathbf{m}_J = \mathbf{t}_J, \quad \mathbf{m}-\mathbf{m}'_J = \mathbf{z}_J,$$

where

$$\mathbf{t}_I = (0, -1), \quad \mathbf{t}_{II} = (+1, -1), \quad \mathbf{t}_{III} = (1, 0),$$
$$\mathbf{t}_{IV} = (0, 1), \quad \mathbf{t}_V = (-1, 1), \quad \mathbf{t}_{VI} = (-1, 0),$$

and

$$\mathbf{z}_a = (-2/3, 1/3), \quad \mathbf{z}_b = (0, -2/3), \quad \mathbf{z}_c = (2/3, 1/3).$$

In the course of the procedure a discrete Fourier transform (cf. [9], [10]) will be applied. Discrete Fourier transform of a discrete argument function $f^{\mathbf{m}}$ is defined with the aid of the formula*[1]

$$\hat{f}(\mathbf{k}) = P \cdot \sum_{\mathbf{m}} e^{-ikx^{\mathbf{m}}} \cdot f^{\mathbf{m}}, \quad \mathbf{k} = (k_1, k_2) \tag{2.5}$$

where $P = 1{,}5\sqrt{3}l^2$ denotes a hexagon's area indicated by a dot line in Fig. 1.

3. Difference equilibrium equations referred to mains nodes

Slope deflection equations (which express internal forces in terms of displacements, see (2.6), [5]) make it possible to find equilibrium equations of each node of the grid. However, these difference formulae vary depending on intermediate nodes. By utilising equilibrium conditions of the latters it is feasible to eliminate displacements and rotations of the intermediate nodes and then to arrive at main nodes'equilibrium equations involving displacements of main nodes only. These formulae will be called difference

[1] Notations used in Rogula's paper included in [10].

equations referred to main nodes. A brief derivation of these equations is presented beneeth; more detailed procedure can be found in [12].

A starting point of the derivation is a set of equilibrium conditions of the intermediate **a**, **b** and **c** nodes which surround the main node **m**. Equations of equilibrium of the node **a** have the form (the proof is omitted here)

$$
\begin{gathered}
-\frac{1}{2}(1+3\bar{\eta})\cdot(\tilde{u}+\tilde{u}_{VI})-2\tilde{u}_V+3\cdot(1+\bar{\eta})\tilde{u}_{\mathbf{a}}+\frac{\sqrt{3}}{2}(\bar{\eta}-1)(\tilde{v}-\tilde{v}_{VI})+\\
-\frac{1}{2}\varphi-\frac{1}{2}\varphi_{VI}+\varphi_V-\frac{\bar{\eta}}{6\eta}\cdot\frac{l^2}{EJ}\cdot\overset{*}{F}{}^1_{\mathbf{a}}=0,\\
\frac{\sqrt{3}}{2}(\bar{\eta}-1)\cdot(\tilde{u}-\tilde{u}_{VI})-\frac{1}{2}(\bar{\eta}+3)\cdot(\tilde{v}+\tilde{v}_{VI})-2\bar{\eta}\cdot\tilde{v}_V+\\
+3(1+\bar{\eta})\tilde{v}_{\mathbf{a}}+\frac{\sqrt{3}}{2}(\varphi_{VI}-\varphi)-\frac{\bar{\eta}}{6\eta}\cdot\frac{l^2}{EJ}\cdot\overset{*}{F}{}^2_{\mathbf{a}}=0,\\
\frac{1}{2}(\tilde{u}+\tilde{u}_{VI}-2\tilde{u}_V)+\frac{\sqrt{3}}{2}(\tilde{v}-\tilde{v}_{VI})+\frac{3\eta+\bar{\eta}}{2\eta}\cdot\varphi_{\mathbf{a}}+\\
+\frac{3\eta-\bar{\eta}}{6\eta}\cdot(\varphi+\varphi_{VI}+\varphi_V)-\frac{\bar{\eta}}{6\eta}\cdot\frac{l}{EJ}\cdot\overset{*}{M}_{\mathbf{a}}=0,
\end{gathered}
\tag{3.1}
$$

where

$$
\begin{gathered}
(\tilde{u},\tilde{u}_i,\tilde{u}_j;\tilde{v},\tilde{v}_i,\tilde{v}_j)=(u_{\mathbf{m}},u_{\mathbf{m}_i},u_j;v_{\mathbf{m}},v_{\mathbf{m}_i},v_j)/l,\\
\varphi=\varphi_{\mathbf{m}},\qquad \varphi_i=\varphi_{\mathbf{m}_i},\qquad i=I,\dots,VI,\qquad j=\mathbf{a},\mathbf{b},\mathbf{c}.
\end{gathered}
\tag{3.2}
$$

Quantities η and $\bar{\eta}$ stand for slenderness ratios of grid bars, EJ denotes an effective flexural stiffness (cf. [5], Sec. 2.1).

Note that the set of equations (3.1) is decoupled with respect to $u_{\mathbf{a}}$, $v_{\mathbf{a}}$ and $\varphi_{\mathbf{a}}$ unknowns. Thus it is easy to express these quantities in terms of displacements of main nodes **m**, $\mathbf{m}_V$ and $\mathbf{m}_{VI}$ and in terms of the loads subjected to **a**.

The equations of equilibrium of **b** and **c** nodes assume an analogous form (which will not be reported here). Thus the intermediate nodes'displacements w_j, $j=\mathbf{a},\mathbf{b},\mathbf{c}$, can be expressed by means of main nodes'displacements $w_{\mathbf{m}}$ and $w_{\mathbf{m}_i}$, $i=I,\dots,VI$ and with the aid of the loads subjected to intermediate joints, i.e. the functions h^α_j

$$
w^\alpha_j=h^\alpha_j(w^\sigma_{\mathbf{m}};w^\sigma_{\mathbf{m}_i};\overset{*}{F}{}^\sigma_j,\overset{*}{M}_j),\quad \alpha=1,2,3,\quad \sigma=1,2,\quad j=\mathbf{a},\mathbf{b},\mathbf{c},\quad i=I,\dots,VI
\tag{3.2}
$$

are known, where, according to (2.2), $w^1=u$, $w^2=v$, $w^3=\varphi$.

Let us write equilibrium equations of the **m**'s node

$$
\begin{gathered}
-\frac{1}{2}(1+3\bar{\eta})(\tilde{u}_{\mathbf{a}}+\tilde{u}_{\mathbf{c}})-2\tilde{u}_{\mathbf{b}}+3(1+\bar{\eta})\tilde{u}+\frac{\sqrt{3}}{2}(\bar{\eta}-1)(\tilde{v}_{\mathbf{a}}-\tilde{v}_{\mathbf{c}})+\\
+\frac{1}{2}(\varphi_{\mathbf{a}}+\varphi_{\mathbf{c}}-2\varphi_{\mathbf{b}})-\frac{\bar{\eta}}{6\eta}\cdot\frac{l^2}{EJ}\cdot F^1=0,
\end{gathered}
\tag{3.3}
$$

$$
\frac{\sqrt{3}}{2}(\tilde{u}_{\mathbf{a}}-\tilde{u}_{\mathbf{c}})-\frac{1}{2}(\bar{\eta}+3)\cdot(\tilde{v}_{\mathbf{a}}+\tilde{v}_{\mathbf{c}})+3(1+\bar{\eta})\tilde{v}-2\bar{\eta}\cdot\tilde{v}_{\mathbf{b}}+\frac{\sqrt{3}}{2}(\varphi_{\mathbf{a}}-\varphi_{\mathbf{c}})-\frac{\bar{\eta}}{6\eta}\cdot\frac{l^2}{EJ}\cdot F^2=0,
$$

$$-\frac{1}{2}(\tilde{u}_a-2\tilde{u}_b+\tilde{u}_c)-\frac{\sqrt{3}}{2}\cdot(\tilde{v}_a-\tilde{v}_c)+\frac{3\eta+\bar{\eta}}{2\eta}\cdot\varphi+ \tag{3.3 [cont.]}$$
$$+\frac{3\eta-\bar{\eta}}{6\eta}(\varphi_a+\varphi_b+\varphi_c)-\frac{\bar{\eta}}{6\eta}\cdot\frac{l}{EJ}\cdot M=0.$$

By inserting (3.2) into (3.3) the sought equilibrium equations of the **m**'s node referred to main nodes only (i.e. to **m** and $\mathbf{m}_i$, $i = I, \ldots, VI$ nodes) are arrived at. These equations can be displayed in the following discrete-convolution form

$$-P\sum_{\beta=1}^{3}\sum_{\mathbf{n}}\Phi_{\alpha\beta}^{(\mathbf{m}-\mathbf{n})}w_{\mathbf{n}}^{\beta}+P\sum_{\beta=1}^{3}\sum_{\mathbf{m}'}S_{\alpha\beta}^{(\mathbf{m}-\mathbf{m}')}\cdot\overset{*}{F}{}_{\beta}^{\mathbf{m}'}+F_{\alpha}^{\mathbf{m}}=0, \tag{3.4}$$

where $\alpha, \beta = 1, 2, 3$; **n**, **m** denote main nodes. Summation with respect to **n** extends (for the **m** fixed) on seven vectors: $\mathbf{n} = \mathbf{m}$ and six vectors such that $|\mathbf{m}-\mathbf{n}| = b$. For other pairs $(\mathbf{m}, \mathbf{n})$ $\Phi_{\alpha\beta}^{(\mathbf{m}-\mathbf{n})} = 0$. Summation with respect to **m**'concerns three vectors $\mathbf{m}' = \mathbf{m}-z_J$, J = **a**, **b**, **c**. For others $S_{\alpha\beta}^{(\mathbf{m}-\mathbf{m}')} = 0$. Nonvanishing components of the matrix $\Phi_{\alpha\beta}^{(\mathbf{m})}$ read

$$\Phi_{11}^{(0)}=\frac{4}{\sqrt{3}}\cdot\frac{\eta}{\bar{\eta}}\left(\frac{\bar{\eta}^2+6\bar{\eta}+1}{\bar{\eta}+1}-\frac{3\eta}{\bar{\eta}+3\eta}\right)\cdot\frac{EJ}{l^5},$$

$$\Phi_{11}^{(t_I)}=\Phi_{11}^{(t_{IV})}=\Phi_{11}^{(t_{II})}=\Phi_{11}^{(t_V)}=\frac{4\eta}{\sqrt{3}\cdot\bar{\eta}}\left(\frac{\eta}{\bar{\eta}+3\eta}-\frac{1+3\bar{\eta}}{3(1+\bar{\eta})}\right)\cdot\frac{EJ}{l^5},$$

$$\Phi_{11}^{(t_{III})}=\Phi_{11}^{(t_{VI})}=\left[-\frac{2}{\sqrt{3}}\frac{\eta^2}{\bar{\eta}(\bar{\eta}+3\eta)}+\frac{4\eta}{6\sqrt{3}\,\bar{\eta}}\cdot\frac{(-3\bar{\eta}^2-6\bar{\eta}+1)}{1+\bar{\eta}}\right]\frac{EJ}{l^5},$$

$$\Phi_{12}^{(t_{III})}=-\Phi_{12}^{(t_{VI})}=\frac{-4\eta}{\bar{\eta}}\cdot\left[\frac{\eta}{2(\bar{\eta}+3\eta)}-\frac{(\bar{\eta}-1)^2}{6(\bar{\eta}+1)}\right]\cdot\frac{EJ}{l^5},$$

$$\Phi_{12}^{(t_I)}=-\Phi_{12}^{(t_{II})}=-\frac{4\eta}{\bar{\eta}}\left[\frac{\eta}{3\eta+\bar{\eta}}+\frac{\bar{\eta}-1}{3(\bar{\eta}+1)}\right]\cdot\frac{EJ}{l^5},$$

$$\Phi_{12}^{(t_{IV})}=-\Phi_{12}^{(t_V)}=\frac{-4\eta(\bar{\eta}-1)}{3(\bar{\eta}+1)}\cdot\frac{EJ}{l^5}, \tag{3.5}$$

$$\Phi_{13}^{(t_{III})}=\Phi_{13}^{(t_{VI})}=\frac{2\eta}{3\sqrt{3}\,\bar{\eta}}\left[\frac{\bar{\eta}-3\eta}{\bar{\eta}+3\eta}+\frac{1-3\bar{\eta}}{1+\bar{\eta}}\right]\cdot\frac{EJ}{l^4},$$

$$\Phi_{13}^{(t_{IV})}=\Phi_{13}^{(t_V)}=\frac{2\eta}{3\sqrt{3}\,\bar{\eta}}\left[\frac{\bar{\eta}-3\eta}{\bar{\eta}+3\eta}+\frac{1+3\bar{\eta}}{1+\bar{\eta}}\right]\cdot\frac{EJ}{l^4},$$

$$\Phi_{13}^{(t_I)}=\Phi_{13}^{(t_{II})}=\frac{4\eta}{3\sqrt{3}\cdot\bar{\eta}}\left[\frac{3\eta-\bar{\eta}}{3\eta+\bar{\eta}}-\frac{1}{1+\bar{\eta}}\right]\cdot\frac{EJ}{l^4},$$

$$\Phi_{21}^{(t_J)}=\begin{cases}\Phi_{12}^{(t_{J+III})} & \text{for} \quad J=I,II,III\\ \Phi_{12}^{(t_{J-III})} & \text{for} \quad J=IV,V,VI,\end{cases}$$

$$\Phi_{22}^{(0)}=\frac{4\eta}{\sqrt{3}\,\bar{\eta}}\left[\frac{\bar{\eta}^2+6\bar{\eta}+1}{\bar{\eta}+1}-\frac{3\eta}{3\eta+\bar{\eta}}\right]\cdot\frac{EJ}{l^5},$$

$$\Phi_{22}^{(t_I)} = \Phi_{22}^{(t_{II})} = \Phi_{22}^{(t_{IV})} = \Phi_{22}^{(t_V)} = \frac{-4\eta(\bar{\eta}+3)}{3\sqrt{3}\cdot(\bar{\eta}+1)}\cdot\frac{EJ}{l^5},$$

$$\Phi_{22}^{(t_{III})} = \Phi_{22}^{(t_{VI})} = \frac{2\eta}{\sqrt{3}\cdot\bar{\eta}}\cdot\left[\frac{3\eta}{3\eta+\bar{\eta}} + \frac{\bar{\eta}^2-6\bar{\eta}-3}{3\cdot(1+\bar{\eta})}\right]\cdot\frac{EJ}{l^5},$$

$$\Phi_{23}^{(t_{VI})} = -\Phi_{23}^{(t_{III})} = \frac{+4\eta}{3\cdot(\bar{\eta}+3\eta)}\cdot\frac{EJ}{l^4}, \tag{3.5 [cont.]}$$

$$\Phi_{23}^{(t_{IV})} = -\Phi_{23}^{(t_V)} = \frac{2\eta}{3\bar{\eta}}\cdot\left[\frac{\bar{\eta}-1}{\bar{\eta}+1} - \frac{\bar{\eta}-3\eta}{\bar{\eta}+3\eta}\right]\cdot\frac{EJ}{l^4},$$

$$\Phi_{23}^{(t_I)} = -\Phi_{23}^{(t_{II})} = \frac{4\eta}{3(\bar{\eta}+1)}\cdot\frac{EJ}{l^4},$$

$$\Phi_{3\beta}^{(t_J)} = \begin{cases}\Phi_{\beta 3}^{(t_{J+III})} & \text{for} \quad J = I, II, III\\ \Phi_{\beta 3}^{(t_{J-III})} & \text{for} \quad J = IV, V, VI\end{cases}$$

$$\beta = 1, 2,$$

$$\Phi_{33}^{(0)} = \left[\frac{-4\eta}{\sqrt{3}\cdot\bar{\eta}\cdot(1+\bar{\eta})} + \frac{2(\bar{\eta}+3\eta)}{\sqrt{3}\cdot\bar{\eta}} - \frac{2(\bar{\eta}-3\eta)^2}{3\sqrt{3}\,\bar{\eta}(\bar{\eta}+3\eta)}\right]\cdot\frac{EJ}{l^3},$$

$$\Phi_{33}^{(t_J)} = \left[\frac{2\eta}{3\sqrt{3}\cdot\bar{\eta}\cdot(1+\bar{\eta})} - \frac{2(\bar{\eta}-3\eta)^2}{9\sqrt{3}\cdot\bar{\eta}\cdot(\bar{\eta}+3\eta)}\right]\cdot\frac{EJ}{l^3}, \qquad J = I, \ldots, VI$$

Nonvanishing components of $S_{\alpha\beta}^{(n')}$ have the form

$$S_{11}^{(Z_a)} = S_{11}^{(Z_c)} = \frac{(1+3\bar{\eta})}{9\sqrt{3}\cdot(1+\bar{\eta})}\cdot l^{-2}, \qquad S_{11}^{(Z_b)} = \frac{4}{9\sqrt{3}\cdot(1+\bar{\eta})}\cdot l^{-2},$$

$$S_{12}^{(Z_a)} = -S_{12}^{(Z_c)} = \frac{(1-\bar{\eta})}{9(1+\bar{\eta})}\cdot l^{-2}, \qquad S_{21}^{(Z_a)} = -S_{12}^{(Z_c)} = S_{12}^{(Z_a)},$$

$$S_{13}^{(Z_a)} = S_{13}^{(Z_c)} = -2S_{13}^{(Z_b)} = \frac{2\eta}{3\sqrt{3}\cdot(\bar{\eta}+3\eta)}\cdot l^{-3},$$

$$S_{22}^{(Z_a)} = S_{22}^{(Z_c)} = \frac{(\bar{\eta}+3)}{9\sqrt{3}\cdot(\bar{\eta}+1)}\cdot l^{-2}, \qquad S_{22}^{(Z_b)} = \frac{4\bar{\eta}}{9\sqrt{3}\cdot(\bar{\eta}+1)}\cdot l^{-2}, \tag{3.6}$$

$$S_{23}^{(Z_a)} = -S_{23}^{(Z_c)} = \frac{2\eta}{3(\bar{\eta}+3\eta)}\cdot l^{-3},$$

$$S_{31}^{(Z_a)} = S_{31}^{(Z_c)} = -2S_{31}^{(Z_b)} = \frac{1}{9\sqrt{3}\cdot(1+\bar{\eta})}\cdot l^{-1},$$

$$S_{32}^{(Z_a)} = -S_{32}^{(Z_c)} = 1/9l,$$

$$S_{33}^{(Z_a)} = S_{33}^{(Z_b)} = S_{33}^{(Z_c)} = \frac{2}{9\sqrt{3}}\cdot\frac{3\eta-\bar{\eta}}{3\eta+\bar{\eta}}\cdot l^{-2}.$$

4. Main node equilibrium equations in k-representation

A formal derivation of equilibrium equations in k-representation, similar to that of Rogula and Kunin, see [6, 10], concerning crystal lattices, will be presented herein. On performing the discrete Fourier transform (cf. (2.5)) of Eqs. (3.4), algebraic equations

$$-\sum_{\beta=1}^{3}\hat{\Phi}_{\alpha\gamma\beta}(\mathbf{k})\hat{w}^{\beta}(\mathbf{k})+\sum_{\beta=1}^{3}\overline{S}_{\alpha\beta}(\mathbf{k})\tilde{F}_{\beta}(\mathbf{k})+\hat{F}_{\alpha}(\mathbf{k})=0, \tag{4.1}$$

where

$$\check{\Phi}_{\alpha\beta}(\mathbf{k}) = P\sum_{\mathbf{m}} e^{-i\mathbf{k}\cdot\mathbf{x}^{\mathbf{m}}}\Phi_{\alpha\beta}^{(\mathbf{m})}, \qquad \hat{w}^{\beta}(\mathbf{k}) = P\sum_{\mathbf{m}} e^{-i\mathbf{k}\cdot\mathbf{x}^{\mathbf{m}}}\cdot w_{\mathbf{m}}^{\beta},$$

$$\overline{S}_{\alpha\beta}(\mathbf{k}) = P\sum_{\mathbf{s}} e^{-i\mathbf{k}\cdot\mathbf{x}^{\mathbf{s}}}S_{\alpha\beta}^{(\mathbf{s})}, \qquad F_{\beta}(\mathbf{k}) = P\cdot\sum_{\mathbf{m}'} e^{-i\mathbf{k}\cdot\mathbf{x}^{\mathbf{m}'}}\cdot \overset{*}{F}{}_{\beta}^{(\mathbf{m}')}, \tag{4.2}$$

$$\hat{F}_{\alpha}(\mathbf{k}) = P\sum_{\mathbf{m}} e^{-i\mathbf{k}\cdot\mathbf{x}^{\mathbf{m}}}\cdot F_{\alpha}^{(\mathbf{m})},$$

are obtained. Vectors $\mathbf{s}$ assume all the values $\mathbf{m}-\mathbf{m}'$. The Eqs. (4.1) have been found with the aid of the theorem on the transform of convolution equations, cf. [10]. The summations in definitions of $\hat{\Phi}_{\alpha\beta}$ and $\overline{S}_{\alpha\beta}$ are finite. By virtue of (2.4) we arrive at

$$P^{-1}\cdot\hat{\Phi}_{\alpha\beta}(\mathbf{k}) = \sum_{J=I}^{VI} e^{i\mathbf{k}\cdot\mathbf{t}_J}\cdot\Phi_{\alpha\beta}^{(-\mathbf{t}_J)}+\Phi_{\alpha\beta}^{(0)}. \tag{4.3}$$

Similarly

$$P^{-1}\cdot\overline{S}_{\alpha\beta}(\mathbf{k}) = \sum_{J=\mathbf{a},\mathbf{b},\mathbf{c}} e^{i\mathbf{k}\cdot\mathbf{z}_J}\cdot S_{\alpha\beta}^{(-\mathbf{z}_J)}. \tag{4.4}$$

5. Formulation of differential approximate models

A set of k-representation Eqs. (4.1) is a starting point (cf. [6]) to obtain differential equations approximating discrete argument functions being solutions of (3.4). The known functions $\hat{\Phi}_{\alpha\beta}$ and $\overline{S}_{\alpha\beta}$ can be expanded in power series with respect to the variables ik_1, ik_2

$$\begin{aligned} -P^{-1}\cdot\hat{\Phi}_{\alpha\beta}(k_\sigma) &= C_{\alpha\beta}^{\psi\mu}w_{\psi^\mu}, \qquad S_{\alpha\beta}(k_\sigma) = s_{\alpha\beta}^{\psi\mu}w_{\psi^\mu},\\ w_{\psi^\mu} &= i^\mu k_{\psi^\mu}, \qquad i=\sqrt{-1}, \qquad \text{(not summed)}\end{aligned} \tag{5.1}$$

where μ denotes a multiindex, cf. [11] p. 77. Substituting Eqs. (5.1) into (4.1) and then carrying out an inverse integral Fourier transformation, differential equations of equilibrium in x-representation

$$C_{\alpha\beta}^{\psi\mu}\partial_{\psi^\mu}w^{\beta}(x^\sigma)+P\cdot s_{\alpha\beta}^{\psi\mu}\partial_{\psi^\mu}\overset{*}{p}{}^{\beta}(x^\sigma)+p^{\beta}(x^\sigma) = 0, \qquad \alpha,\beta,\sigma = 1,2,3 \tag{5.2}$$

where

$$\overset{*}{p}{}^{\beta} = P^{-1}\cdot\overset{*}{F}_{\beta}, \qquad p^{\beta} = P^{-1}\cdot F_{\beta} \tag{5.3}$$

are arrived at. In order to avoid misunderstandings let us display first few terms of the expansion

$$C^{\varphi\mu}_{\alpha\beta}\partial_{\varphi\mu} w^{\beta}(x^{\sigma}) = \sum_{\beta=1}^{3} C_{\alpha\beta} w^{\beta}(x^{\sigma}) + \sum_{\beta=1}^{3} (C^{1}_{\alpha\beta}\partial_1 w^{\beta} + C^{2}_{\alpha\beta}\partial_2 w^{\beta}) +$$
$$+ \sum_{\beta=1}^{3} (C^{11}_{\alpha\beta}\partial_1^2 w^{\beta}_{\kappa} + (C^{12}_{\alpha\beta} + C^{21}_{\alpha\beta})\partial_1\partial_2 w^{\beta} + C^{22}_{\alpha\beta}\partial_2^2 w^{\beta}) +$$
$$+ \sum_{\beta=1}^{3}\sum_{k,l,m} C^{klm}_{\alpha\beta}\partial_k\partial_l\partial_m w^{\beta} + \sum_{\beta=1}^{3}\sum_{k,l,m,n} C^{klmn}_{\alpha\beta}\partial_k\partial_l\partial_m\partial_n w^{\beta} + \ldots .$$

Coefficients $C^{\varphi\mu}_{\alpha\beta}$ are proportional to consecutive powers of the quantity b which express a spacing of main nodes of the grid. The Eqs. (5.2) will be assumed to be of p-order provided the coefficients $C^{\varphi\mu}_{\alpha\beta}$ proportional to b^s, $s \leqslant p$, are retained. It will be said that Eqs. (5.2) are of p-order with respect to the displacement $u(v$ or $\varphi)$ provided all the terms involving u (v or φ) proportional to b^s, $s \leqslant p$, are taken into account and the other terms are assumed to be negligible.

Substitution of infinite series of Eq. (5.2) by polynomials of p-order with respect to differential operators ∂_α amounts to assuming that deformation patterns of wave lengths being shorter than some value L_p have a negligible effect on resulting lattice plate response. It is always required here that $L_p \geqslant 2b$, hence $|k_\alpha b| \leqslant \pi$. Thus physical facet of the problem restricts a domain of variation of the wave vector $\mathbf{k}$ to a certain circular neighbourhood of point $\mathbf{k} = \mathbf{0}$.

The smaller the parameter p is, the longer the deformation waves can be admitted. In the limiting case of $p = 0$ a zero-order approximation, so-called long-wave approximation, is obtained the solutions of which are quantitatively different from those yielding from the more complex models. In particular, the simplest model does not describe dispersion of waves, cf. [6]. It will be shown below that in this model the hexagonal lattice is considered as a point-wise centrosymmetrical structure so that an interchange of main and intermediate nodes do not change the governing equations of the theory. Nevertheless, the formulation of this model is not a main goal of the paper. This work ought to be treated rather as an introduction to further considerations (see [13]) pertaining to Cosserat-type models of hexagonal grids, i.e. to the models of the same mathematical structure as those of Woźniak's-type outlined in [5].

6. Second order approximation equations

By neglecting in (5.2) the terms dependent on the powers b^s, $s \geqslant 3$, second-order equations (with respect to all displacements) are found. Appropriate rearrangements give

$$[(\mu+\alpha)\nabla^2 u + (\lambda+\mu-\alpha)\partial_1^2 u] + l^2 \underline{\left[\frac{3}{16}(\mu+\alpha)\nabla^4 u + \frac{1}{4}(\lambda+\mu-\alpha)\partial_1^4 u\right]} +$$

$$+(\lambda+\mu-\alpha)\partial_1\partial_2 v+l[\delta\cdot\partial_1(\partial_1^2-3\partial_2^2)v]+l^2\left[\frac{1}{8}(\lambda+\mu-\alpha)\partial_1\partial_2(\partial_1^2+3\partial_2^2)v\right]+$$

$$+2\alpha\partial_2\varphi+l\beta(\partial_1^2-\partial_2^2)\varphi+l^2\left(\frac{3}{4}\alpha\partial_2\nabla^2\varphi\right)+{}'p^1=0,$$

$$(\lambda+\mu-\alpha)\partial_1\partial_2 u+l[-\delta\partial_1(\partial_1^2-3\partial_2^2)]u+l^2\left[\frac{1}{8}(\lambda+\mu-\alpha)\partial_1\partial_2(\partial_1^2+3\partial_2^2)u\right]+ \tag{6.1}$$

$$+(\mu+\alpha)\nabla^2 v+(\lambda+\mu-\alpha)\partial_2^2 v+\frac{3}{16}l^2\left[(\mu+\alpha)\nabla^4 v+(\lambda+\mu-\alpha)\left(-\frac{1}{3}\partial_1^4+\partial_2^4+2\partial_1^2\partial_2^2\right)v\right]-$$

$$-2\alpha\partial_1\varphi+l(-2\beta\partial_1\partial_2\varphi)+l^2\left(-\frac{3}{4}\alpha\partial_1\nabla^2\varphi\right)+{}'p^2=0,$$

$$-2\alpha\partial_2 u+l\beta(\partial_1^2-\partial_2^2)u-l^2\left(\frac{3}{4}\alpha\partial_2\nabla^2 u\right)+2\alpha\partial_1 v+l[-2\beta\partial_1\partial_2 v]+$$

$$+l^2\left(\frac{3}{4}\alpha\partial_1\nabla^2 v\right)-4\alpha\cdot\varphi+l^2(\gamma\nabla^2\varphi)+{}'Y^3=0,$$

where functions $u(x^\sigma)$, $v(x^\sigma)$ and $\varphi(x^\sigma)$ are equal to $w^1(x^\sigma)$, $w^2(x^\sigma)$ and $w^3(x^\sigma)$, respectively. The following definitions of effective elastic moduli, depending on slenderness ratios η and $\bar{\eta}$ only,

$$\lambda=\frac{2\sqrt{3}\,\eta(\bar{\eta}-1)}{(\bar{\eta}+1)}\cdot\frac{EJ}{l^3},\quad \mu=\frac{4\sqrt{3}}{(1+\bar{\eta})}\cdot\frac{EJ}{l^3},\quad \alpha=\frac{2\sqrt{3}\cdot\eta}{\bar{\eta}+3\eta}\cdot\frac{EJ}{l^3},$$

$$\beta=\frac{\sqrt{3}}{2}\cdot\frac{\eta}{\bar{\eta}}\left[\frac{3\eta-\bar{\eta}}{3\eta+\bar{\eta}}+\frac{3\bar{\eta}-1}{\bar{\eta}+1}\right]\cdot\frac{EJ}{l^3},$$

$$\delta=\frac{3\sqrt{3}}{2}\cdot\frac{\eta}{\bar{\eta}}\cdot\left[\frac{(\bar{\eta}-1)^2}{3(\bar{\eta}+1)}-\frac{\eta}{\bar{\eta}+3\eta}\right]\cdot\frac{EJ}{l^3}, \tag{6.2}$$

$$\gamma=\frac{\sqrt{3}}{\eta}\left[\frac{(3\eta-\bar{\eta})^2}{3(\bar{\eta}+3\eta)}-\frac{\eta}{\bar{\eta}+1}\right]\cdot\frac{EJ}{l^3},$$

are introduced, where, in the case of prismatic rods, see [5], Eq. $(2.9)_4$

$$\frac{EJ}{l^3}=\frac{E}{12\eta\sqrt{\eta}}.$$

Two first definitions expressing effective Lamé moduli λ and μ are exactly consistent with Horvay's results [1]. Moreover, the same expressions for λ and μ have been obtained in [5] by means of two different approaches resulting from the general concept of Woźniak.

Functions ${}'p^\alpha$ and ${}'Y^3$ depend on the loads subjected to both intermediate and main nodes. Their form is complex (see [12]) and will not be given here.

Note that displacements u, v and rotations φ are involved in different ways in the second order equilibrium equations (6.1). Two first equations involve the fourth order derivatives of functions u and v at coefficients proportional to l^2, whereas the fourth order derivatives of φ do not occur in (6.1). Thus the considered set of equations is not consequent with respect to orders of powers of the parameter 1. In order to make the system of Eqs. (6.1) consistent in the mentioned meaning the last Eq. $(6.1)_3$ should be substituted by the relation of order three with respect to u, v and of fourth order with respect to φ:

$$-2\alpha\partial_2 u+l\cdot\beta(\partial_1^2-\partial_2^2)u-l^2\left(\frac{3}{4}\alpha\partial_2\nabla^2 u\right)+l^3\left[\frac{\beta}{16}(5\partial_1^4-3\partial_2^4-6\partial_1^2\partial_2^2)u\right]+$$
$$+2\alpha\partial_1 v+l[-2\beta\partial_1\partial_2 v]+l^2\left[\frac{3}{4}\alpha\partial_1\nabla^2 v\right]+l^3\left[-\frac{1}{4}\beta\partial_1\partial_2(\partial_1^2+3\partial_2^2)v\right]+ \tag{6.3}$$
$$-4\alpha\cdot\varphi+l^2(\gamma\nabla^2\varphi)+l^4\left(\frac{3}{16}\gamma\nabla^4\varphi\right)+'Y^3 = 0.$$

Stability

It will be shown that both systems of Eqs. (6.1) and $(6.1)_{1,2}$, (6.3) do not allow us to formulate boundary value problems, e.g. these sets are not well-established since they do not satisfy stability conditions. The stability Kunin's criterion [6], means positive determination of the matrix $\Phi^{(2)}_{\alpha\beta}(\mathbf{k})$ (for the arbitrary wave vector $\mathbf{k}$), associated with the second order approximation. One of the necessary conditions reads

$$P^{-1}\cdot\Phi^{(2)}_{11} = (\mu+\alpha)\cdot(k_1^2+k_2^2)+(\lambda+\mu-\alpha)\cdot k_1^2-l^2\cdot\left[\frac{3}{16}\cdot(\mu+\alpha)\cdot(k_1^2+k_2^2)^2+\right.$$
$$\left.+\frac{1}{4}(\lambda+\mu-\alpha)\cdot k_1^4\right]>0 \quad \forall k_1, k_2\in R. \tag{6.4}$$

Let $k_1 = |\mathbf{k}|\cos\theta$, $k_2 = |\mathbf{k}|\sin\theta$, $\tilde{\varrho} = |\mathbf{k}|l$. The condition (6.4) takes the form

$$\tilde{\varrho}^2\left\{(\mu+\alpha)+(\lambda+\mu-\alpha)\cos^2\theta-\frac{\tilde{\varrho}^2}{4}\left[\frac{3}{4}(\mu+\alpha)+(\lambda+\mu-\alpha)\cos^4\theta\right]\right\}>0$$

for arbitrary $\theta\in(0, 2\pi)$ and $\tilde{\varrho}>0$. Inserting $\theta=\pi/2$, we have $\tilde{\varrho}<4\sqrt{3}/3$, $|\mathbf{k}|\cdot l<2{,}31$. Thus, the analysed inequality is satisfied in some vicinity of $\mathbf{k}=\mathbf{0}$ vector: $|\mathbf{k}|<k_{\mathrm{crit}}$. Moreover it can be proved that such $\mathbf{k}_{\mathrm{crit}}$ exists that in the region $|\mathbf{k}|<k_{\mathrm{crit}}$ the stability condition of second order equations is satisfied.

In the case of sufficiently long wave deformation patterns (sufficiently small $|\mathbf{k}|$), an application of the second order equations is justified. However, the mentioned equations are not correct in general so that they lose their sense in the case of particularly short wave lengths.

Elimination of rotation unknowns

Proceeding similarly to the Kunin's method (cf. [10], Sec. III, p. 134), function (which stands for rotations of nodes) will be eliminated from Eqs. (6.1). To this end the last of the latter equations is expressed in k — representation

$$\hat{\varphi} = \frac{1}{(4\alpha+l^2|k|^2\cdot\gamma)}\left\{\left[-2\alpha ik_2+l\beta(k_2^2-k_1^2)+\frac{3}{4}l^2\alpha k_2 i(k_1^2+k_2^2)\right]\hat{u}+\right.$$
$$\left.+\left[2\alpha\cdot k_1 i+2\beta l\cdot k_1\cdot k_2-\frac{3}{4}l^2\cdot\alpha\cdot k_1\cdot i\cdot(k_1^2+k_2^2)\right]\hat{v}+{}'Y^3\right\}.$$

Provided $|\mathbf{l}\cdot\mathbf{k}| < 2\sqrt{\alpha/\gamma}$ the RHS of the above equation can be expanded in convergent power series with respect to k_α. Retaining terms of lower order than second and transforming the obtained formula into x-representation, we arrive at

$$\varphi(x^\sigma) \approx \left[\frac{1}{2}(\partial_1 v-\partial_2 u)+\frac{1}{4\alpha}\cdot{}'Y^3\right]+l\cdot\frac{\beta}{4\alpha}[(\partial_1^2-\partial_2^2)u-2\partial_1\partial_2 v]+$$
$$+l^2\left\{\left(\frac{3}{16}+\frac{\gamma}{8\alpha}\right)\nabla^2(\partial_1 v-\partial_2 u)+\frac{\gamma}{16\alpha^2}\cdot\nabla^2{}'Y^3\right\}.$$

Substituting the RHS of the above equation into two first of Eqs. (6.1) and neglecting the terms involving the powers l^s, $s \geqslant 3$, we finally find

$$[(2\mu+\lambda)\partial_1^2+\mu\partial_2^2]u+\frac{l^2}{16}[(7\mu+4\lambda-\alpha+4\beta^2\alpha^{-1})\partial_1^4+(3\mu-8\alpha-4\gamma+4\beta^2\alpha^{-1})\partial_2^4+$$
$$+(6\mu-6\alpha-4\gamma-8\beta^2\alpha^{-1})\partial_1^2\partial_2^2]u+[(\lambda+\mu)\partial_1\partial_2]v+l(\delta+\beta/2)\partial_1(\partial_1^2-3\partial_2^2)v+$$
$$+\frac{l^2}{8}\cdot\partial_1\partial_2[(5\alpha+\lambda+\mu+2\gamma-4\beta^2\alpha^{-1})\partial_1^2+(3\lambda+3\mu+3\alpha+2\gamma+4\beta^2\alpha^{-1})\partial_2^2]v+\check{p}^1 = 0_0$$

$$(\lambda+\mu)\partial_1\partial_2 u-l(\delta+\beta/2)\partial_1(\partial_1^2-3\partial_2^2)u+\frac{l^2}{8}\partial_1\partial_2[(5\alpha+\lambda+\mu+2\gamma-4\beta^2\alpha^{-1})\cdot\partial_1^2+$$
$$+(3\lambda+3\mu+3\alpha+2\gamma+4\beta^2\alpha^{-1})\partial_2^2]u+[(2\mu+\lambda)\partial_2^2+\mu\partial_1^2]v+\frac{l^2}{16}[(2\mu-\lambda-8\alpha-4\gamma)\partial_1^4+$$
$$+(6\mu+3\lambda)\partial_2^4+(12\mu+6\lambda-12\alpha-4\gamma+16\beta^2\alpha^{-1})\partial_1^2\partial_2^2]v+\check{p}^2 = 0, \tag{6.5}$$

where

$$\check{p}^\alpha = {}'p^\alpha+\frac{1}{2}\epsilon^{\alpha\beta}\partial'_\beta Y^3+l\cdot\frac{\beta}{4\alpha}\cdot G^\alpha_{\sigma\mu}\partial_\sigma\partial'_\mu Y^3-l^2\left(\frac{\gamma}{8\alpha}+\frac{3}{16}\right)\epsilon^{\alpha\beta}\partial_\alpha\nabla^{2\prime}Y^3 \tag{6.6}$$

and $G^1_{11} = -G^1_{22} = G^2_{12} = G^2_{21} = 1$, the other $G^\alpha_{\sigma\mu} = 0$, $\epsilon_{\alpha\beta}$ denotes a permutation symbol.

It can be shown that the obtained system of Eqs. (6.5) is not stable.

7. First order approximation

By neglecting the underlined terms in Eqs. (6.1) we arrive at the first order approximation equations. The functions $'p^\alpha$, $'Y^3$ take the form

$$'p^1 = (p^1+\overset{*}{p}{}^1)+\frac{3\eta}{3\eta+\overline{\eta}}\partial_2\overset{*}{Y}{}^3+l\left[-\frac{(\overline{\eta}-1)}{2(\overline{\eta}+1)}\partial_2\overset{*}{p}{}^1-\right.$$
$$\left.-\frac{(\overline{\eta}-1)}{2(\overline{\eta}+1)}\partial_1\overset{*}{p}{}^2-\frac{3}{4}\frac{\eta}{\overline{\eta}+3\eta}(\partial_1^2-\partial_2^2)\overset{*}{Y}{}^3\right], \tag{7.1}$$

$$'p^2 = (p^2+\overset{*}{p}^2)-\frac{3\eta}{3\eta+\bar{\eta}}\partial_1\overset{*}{Y}^3+l\left[-\frac{\bar{\eta}-1}{2(\bar{\eta}+1)}\partial_1\overset{*}{p}^1+\frac{(\bar{\eta}-1)}{2(\bar{\eta}+1)}\partial_2\overset{*}{p}^2+\frac{3}{2}\frac{\eta}{\bar{\eta}+3\eta}\partial_1\partial_2\overset{*}{Y}_3\right],$$

$$'Y^3 = -\frac{3\eta-\bar{\eta}}{3\eta+\bar{\eta}}\overset{*}{Y}^3+Y^3, \qquad (7.1)\ [\text{cont.}]$$

where $\overset{*}{Y}^3 = \overset{*}{p}^3$, $Y^3 = p^3$, see Eqs. (5.3).

The last equation allows us to express the function φ in terms of functions u, v, their derivatives and — function$'Y^3$ depending on moment loads. The elimination of rotations does not require here any additional assumptions and leads to equations involving two functions u and v only

$$[(2\mu+\lambda)\partial_1^2+\mu\partial_2^2]u+(\lambda+\mu)\partial_1\partial_2 v+l(\delta+\beta/2)\partial_1(\partial_1^2-3\partial_2^2)v+''p^1 = 0,$$
$$[(\lambda+\mu)\partial_1\partial_2]u-l(\delta+\beta/2)\partial_1(\partial_1^2-3\partial_2^2)u+[(2\mu+\lambda)\partial_2^2+\mu\partial_1^2]v+''p^2 = 0, \qquad (7.2)$$
$$''p^\alpha = 'p^\alpha+\frac{1}{2}\in^{\alpha\beta}\partial_\beta' Y^3+l\frac{\beta}{4\alpha}G^\alpha_{\sigma\mu}\partial_\sigma\partial_\mu{}'Y^3.$$

However, it can be proved that Eqs. (7.2) are not stable.

The derived model (and the obtained before too) takes into account the lack of centro-symmetry of the neighbourhoods of nodes. This is revealed in Eqs. (7.2) by terms involving the third derivatives of the displacement functions. These terms include constants δ and β, the signs of which depend on the choice of main nodes. Thus the first order equations are sensitive to the division of the nodes on two families of intermediate and main nodes.

8. Zero-order equations (Horvay's model)

Zero-oder equations are obtained by neglecting of all the terms of first and second order in Eqs. (6.1) and (7.1). Hence, we have

$$[(\mu+\alpha)\nabla^2+(\lambda+\mu-\alpha)\partial_1^2]u+(\lambda+\mu-\alpha)\partial_1\partial_2 v+2\alpha\partial_2\varphi+\mathring{p}^1 = 0,$$
$$[(\lambda+\mu-\alpha)\partial_1\partial_2]u+[(\mu+\alpha)\nabla^2+(\lambda+\mu-\alpha)\partial_2^2]v-2\alpha\partial_1\varphi+\mathring{p}^2 = 0, \qquad (8.1)$$
$$-2\alpha\partial_2 u+2\alpha\partial_1 v-4\alpha\cdot\varphi+\mathring{Y}^3 = 0,$$

where

$$\mathring{p}^\alpha = p^\alpha+\overset{*}{p}+\frac{3\eta}{3\eta+\bar{\eta}}\in^{\alpha\beta}\partial_\beta\overset{*}{Y}^3,$$
$$\mathring{Y}^3 = Y^3+\frac{-3\eta+\bar{\eta}}{3\eta+\bar{\eta}}\cdot\overset{*}{Y}^3. \qquad (8.2)$$

The last equilibrium equation can be rearranged to the form

$$\varphi = \frac{1}{2}(\partial_1 v-\partial_2 u)+\frac{1}{4\alpha}\cdot\mathring{Y}^3. \qquad (8.3)$$

Making use of the above formula the function φ can be eliminated from Eqs. $(8.1)_{1,2}$, and, the classical equations (involving u and v only) of isotropic plate in a plane-stress state occur. They can be associated with the name of Horvay to honour of his pioneer achievements concerning effective moduli (cf. remarks in Sec. 6)

$$[(2\mu+\lambda)\partial_1^2+\mu\partial_2^2]u+(\lambda+\mu)\partial_1\partial_2 v+\bar{p}^1 = 0,$$
$$(\lambda+\mu)\partial_1\partial_2 u+[(2\mu+\lambda)\partial_2^2+\mu\partial_1^2]v+\bar{p}^2 = 0, \tag{8.4}$$
$$\bar{p}^\alpha = \mathring{p}^\alpha+\frac{1}{2}\epsilon^{\alpha\beta}\partial_\beta \hat{Y}^3.$$

The system $(8.4)_{1,2}$ is stable, provided

$$2\mu+\lambda > 0, \quad \mu > 0. \tag{8.5}$$

By inserting the definitions $(6.2)_{1,2}$ into above inequalities it is clear that by virtue of positiveness of Young modulus and slenderness ratio η the conditions (8.5) are fulfilled for all real hexagonal-type lattices.

Note that $\bar{p}^\alpha$ do not depend of η. Substituting (8.2) into $(8.4)_3$ one obtains

$$\bar{p}^\alpha = (p^\alpha+\overset{*}{p}{}^\alpha)+\frac{1}{2}\epsilon^{\alpha\beta}\partial_\beta(Y^3+\overset{*}{Y}{}^3). \tag{8.6}$$

It is worth emphasising a fact that external: main as well as intermediate loads affect in (8.6) in an equal manner. Thus the zero-order approximation does not distinguish between main and intermediate nodes: both Eqs. (8.4) as well as (8.6) retain their forms if one choose a family of main nodes by an opposite way to the way previously assumed. The lack of centrosymmetry of neighbourhoods of nodes is „a priori" ignored.

9. Final remarks

It has been shown that only one zero-order version leads to a stable, well established mathematical model, which makes it feasible to examine boundary value problems of the hexagonal-type grid plates. The other models can be applied to analysis of local effects, for instance.

The unstable differential equations can be transformed into stable ones. In the subsequent paper [13] a derivation of such a model of a mathematical structure analogous to that known from the micropolar plane-stress theory will be proposed. On the other hand such models have been considered by Woźniak, [3]. Thus there are two ways of constructing Cosserats'-type approximations: the first due to Woźniak, obtained via variational calculus, and the second one resulting from Rogula-Kunin's methods. As it will be shown in [13], it is difficult to indicate the best version satisfying both conditions of stability and approximation.

In the present paper our attention has been focused on the specific plate of honeycomb layout. Nevertheless, the presented procedure does not lose its value for all dense regular grid plates; in particular it is not diffucult to examine by the same method lattices constructed of two families of orthogonal bars or of three families of bars intersecting at an angle 60°. The mentioned structures belong to the class of simple layout grids, the centrosymmetry of the vicinities of nodes being fulfilled. It can be proved, that an essential difference exists between the lattices of simple geometry and the considered hexagonal structure, namely, an effective modulus, γ (cf. $(6.2)_6$), which is positive in the latter case, and takes a negative value in case of simple layout structures. This fact is of significant interest, because in the Cosserats'-type approximation the modulus γ determines a fluxural stiffness corresponding to polar couples. Specific problems concerning Cosserats' continuum models of hexagonal-type grids will be a subject of the prepared paper [13].

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Резюме

ДИФФЕРЕНЦИАЛЬНЫЕ МОДЕЛИ ГЕКСАГОНАЛЬНЫХ СЕТЧАТЫХ ПЛАСТИНОК

В работе выводятся и анализируются дифференциальные модели аппроксимирующие поведение густых, упругих, гексагональных стержневых пластинок. Дифференциальные аппроксимации разностных уравнений равновесия стержневой решетки получены методом Рогули и Кунина, используя аналогию между этими уравнениями и уравнениями теории кристаллических решеток. Примененный подход дает возможность представить консеквентный анализ точности формулированных математических моделей, получить уравнения в смещениях путем элиминации углов поворота узлов и, кроме того, позволяет выявить физический смысл приближений в к-репрезентации.

В работе доказывается, что среди обсуждаемых приближенных версий, только один вариант нулевой аппроксимации дает стабильные уравнения и потому только в том случае могут быть корректно поставленны краевые задачи для ограниченных решеток. Остальные модели могут быть полезны при анализе локальных эффектов.

Представленные исследования можно использовать для анализа физической корректности моделей типа Коссера (которые были приспособлены Возняком в его монографии посвященной сетчатым поверхностным конструкциям).

Streszczenie

RÓŻNICZKOWE MODELE HEKSAGONALNYCH TARCZ PRĘTOWYCH

W pracy wyprowadzono i przeanalizowano modele różniczkowe aproksymujące deformację gęstych, sprężystych, heksagonalnych tarcz prętowych. Różniczkowe przybliżenia dyskretnych równań równowagi siatki prętowej otrzymano metodą Roguli i Kunina wykorzystując analogię między w/w równaniami i równaniami teorii siatek krystalicznych. Zastosowane podejście zezwala na: konsekwentną analizę dokładności formułowanych modeli, modyfikację równań polegającą na eliminacji przemieszczeń kątowych i umożliwia ponadto fizyczną interpretację przybliżeń dokonywanych na równaniach w k-reprezentacji.

W pracy wykazano, że spośród omawianych wersji jedynie wariant zerowego przybliżenia prowadzi do równań stabilnych. Zatem tylko w tym przypadku można poprawnie formułować zagadnienia brzegowe dla tarcz ograniczonych. Pozostałe modele mogą służyć do badania zjawisk lokalnych.

Przedstawione w pracy wywody zezwalają na analizę fizycznej poprawności modeli typu Cosseratów wykorzystanych przez Woźniaka w jego monografii [3] dotyczącej dźwigarów siatkowych.

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