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FUZZY SETS IN ADEQUACY DESCRIPTION OF MATHEMATICAL MODELS OF MECHANICAL SYSTEMS

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In this paper an attempt towards elaborating adequacy description of mathematical models of technical mechanical systems considering vague adequacy problems has been presented. For this purpose the notions of L. A. Zadeh's fuzzy sets [2], fuzzy measure and fuzzy integral [18] have been applied. A hierarchical arrangement of fuzzy sets on the level of physical variables, sets of physical variables, system relations and sequences of relations has been formed. The method presented here enables carrying out adequacy estimations of mathematical models both comparative and aiming at a goal in a formal way.

1. Introduction

Mathematical modelling technical mechanical systems (t.m.s.) requires adjustment of formal mathematical apparatus to the description of actual reality on one hand, and to the replacement of which would be possible describe in a formal way on the other. Throught the last twenty years strong tendencies to search for mathematical formalism would be useful in the description of some vague (not precise) properties of physical reality can be noted. One should mention here, among others, the theory of fuzzy sets [2÷7], and among conceptions of great importance in mechanics, the conception of tolerance spaces [8, 9, 10]. The theory of fuzzy sets initiated by L. A. ZADEH [2] in 1965 undergoes intense development [11÷15]. This has been expressed, among other things, in a large number of publications of basic and applicable significance since the time of the first publication, the bibliographic specifications for the ten year period recorded 1150 items which was incomplete list [14]. Fuller specification from a latter period (1979) contains 1799 items [15]. At the present stage periodicals to dealing exclusively with theory and applications of fuzzy sets are edited, for example: „An International Journal Fuzzy Sets and Systems" and „Fuzzy Mathematics".

The idea of fuzzy sets comprises many fields of knowledge and technique. Here are some of them:

— technique: control engineering (power plants, beilers, heat exchangers), inexact measurement, fuzzy control algorithms, fuzzy robots,

— physics: fuzzy spin spaces, scattering particles in fuzzy phase space, localizability of relativistic particles in fuzzy phase space, measurement in quantum mechanics as stochastic processes on spaces of fuzzy events,
— mathematics and cybernetics: topology, algebra, graphs, functions, differential equations, groups, relations, category theory, fuzzy information theory, fuzzy simulation of processes, classification theory, fuzzy sets, pattern recognition, clustering,
— philosophy and logic of imprecision and vagueness,
— linguistic and language sings,
— social sciences, e.g. psychology of human behaviour,
— biology and medicine, e.g. model of brain tissue,
— behavioral geography,
— air pollution, e.g. fuzzy programming to air pollution regulation problem.

The paper refers to the interpretation and application of the fuzzy sets theory notions and methods for the mathematical models adequacy description of t.m.s. with modelled mechanical object while taking into account vague problems of this adequacy. Justification of this problem from the pragmatic point of view has been analysed in the paper [1].

2. Mathematical modelling of technical mechanical systems

2.1. Real system. We notice the investigated technical mechanical object as consisting of certain parts which are „essential elements". The way of division and the number of separated elements depends on the aim of investigating an object, kinds of examined properties as well as the level of thoroughness of the analysis. In order to formalise these facts let's introduce the following assumptions and definitions. Let's have:

(i) finite set $\boldsymbol{X}$, $\boldsymbol{x}_k \in \boldsymbol{X}$, $k \in \boldsymbol{K}$, $\boldsymbol{K}$ is a finite set of indices,

(ii) mapping α: $X \ni \boldsymbol{x}_k \rightarrow \alpha(\boldsymbol{x}_k) \ni (X_{1k}, X_{2k}, \ldots, X_{j_k k})$, where $x_{jk} \subset \boldsymbol{R}$, $j \in J_k$, J_k is a finite set of indices, and $\boldsymbol{R}$ is a set of real numbers,

(iii) set of relations $\bar{\bar{R}} \equiv \{R_{x2}, R_{x3}, \ldots, R_{xp}\}$ on Cartesian products: $R_{x2} \subset X_{jk} \times X_{j'k'}$, $R_{x3} \subset X_{jk} \times X_{j'k'} \times X_{j''k''}$, ..., $R_{xp} \subset X_{jk} \times \ldots \times X_{\hat{j}\hat{k}}$, where $k, k', k'', \ldots, \hat{k} \in K$, $j \in J_k$, $j' \in J_{k'}$, ..., $\hat{j} \in J_{\hat{k}}$. R_{xp} is *a* p^{th} product, whereas $p = \sum_{k \in K} \text{card } J_k$

(IV) sequence of relations $R_x = (R_{xn_i})$ where $i \in I_x$, I_x is a finite set of indices, and $R_{xn_i} \ni (x_1, x_2, \ldots, x_{n_i})$; $x_1, x_2, \ldots, x_{n_i} \in X = \cup X_{jk}$ for $k \in K$ and $j \in J_k$.

Definition 1. The pair

$$S_x = \langle X, R_x \rangle, \tag{2.1}$$

we shall call a real system.

In α mapping we attribute to elements („essential elements") sequences of sets of real numbers which have been given physical meaning (physical quantities).

Relation R_x constitute all kinds of actual (real) relationship among physical quantities. In mechanical objects these are spatial relationships, relationships of hierarchy (constituting a part) and of interactions of a mechanical nature (acting of forces, mass and energy flows).

2.2. Ideal system. A mathematical model of a technical mechanical object is created by simplifications and idealization of mental „picture" of an object [1]. Simplifications result from complexity of object on one hand, and from possibilities of mathematical physics methods on the other and are compromise between the destiny of the model, possibilities of mathematical formalism anp the complexity of reality, Simplifications consist in leaving out components of quantities attributed to them, weak couplings, weak reactions, etc. The simplified object is replaced by an ideal substitue (physical model) consisting of ideal components such as particles, undeformable bodies, deformable mass zones, massless connecting links: rigid, elastic, viscous, etc.

In order to formalise these facts let's have:

(i) set $Y = \bigcup_{q\in Q} Y_q$, $Y_q \subset \boldsymbol{R}$, Q is a finite set of indices,

(ii) set of relations $\bar{R}_y \equiv \{R_{y2}, R_{y3}, \ldots, R_{yr}\}$ on Cartesian products: $R_{y2} \subset Y_q \times Y_{q'}$; $R_{y3} \subset Y_q \times Y_{q'} \times Y_{q''}$; ... $R_{yr} \subset Y_q \times \ldots \times Y_{\hat{q}}$, where $q, q', q'', \ldots, \hat{q} \in Q$; R_{yr} is an r^{th} product, r is a maximum element of Q,

(iii) sequence of relations $R_y = (R_{ym_i})$, $i \in I_y$, where $R_{ym_i} \ni (y_1, y_2, \ldots, y_{m_i})$; $y_1, y_2, \ldots$ $\ldots, y_{m_i} \in Y$, I_y is a finite set of indices.

Definition 2. The pair

$$S_y = \langle Y, R_y\rangle \tag{2.2}$$

we shall call an ideal system.

Definition 3. An ideal system S_y will be called a mathematical model of a real system S_x if there exists a mapping $\Gamma \subset X \times Y$ such that when $R_{xn_{i'}} \ni (x_1, x_2, \ldots, x_{n_i}) \& ((x_1, y_1) \in \Gamma)$ $\& ((x_2, y_2) \in \Gamma) \ldots \& ((x_{n_{i'}}, y_{m_{i''}}) \in \Gamma) \Rightarrow R_{ym_{i''}} \ni (y_1, y_2, \ldots, y_{m_i})$, where $x_1, x_2, \ldots, x_{n_{i'}} \in X$; $y_1, y_2, \ldots, y_{m_{i''}} \in Y$; $i' \in I_x$, $i'' \in I_y$.

The formed ideal system S_y (mathematical model) is a mathematical description of the real system S_x. Forming of an ideal system S_y may be carried out in many ways, not only by means of a mathematical description of the so-called physical model; it is an activity a priori in character and it is chracterized by an action of diverse meaning. The diversity of meaning asserts itself in the possibility of constructing many reasonable models describing the investigated properties. The formed a priori system S_y to be a model the mapping Γ between S_x and S_y must take place. This mapping means that a few relations R_{xn} may by „modelled" by the relation R_{ym} or that one relation R_{xn} may by „modelled" by a few relations R_{ym}. Besides, the relations R_{xn} and R_{ym} may differ from each other in the number of variables and usually $n \geqslant m$.

3. Adequacy of mathematical model

3.1. Physical aspects of the adequacy of a mathematical model. By the adequacy of a mathematical model S_y in relation to the examined object S_x we shall understand measurable property of mapping Γ („degree" of mapping) which we shall assign from comparing R_x and R_y.

An ideal case of adequacy of the object S_x and the model S_y takes place when the mapping $\bar{\Gamma}$ becomes isomorphous mapping Γ. This means there exists a bijection $\bar{\Gamma}$: $S_x \leftrightarrow S_y$

such that: $R_{xn_i} \ni (x_1, x_2, \ldots, x_{n_i}) \Rightarrow R_{yn_i} \ni (\bar{\Gamma}(x_1), \bar{\Gamma}(x_2), \ldots, \bar{\Gamma}(x_{n_i}))$, $R_{ym_i} \ni (y_1, y_2, \ldots \ldots y_{m_i}) \Rightarrow R_{xm_i} \ni (\bar{\Gamma}^{-1}(y_1), \bar{\Gamma}^{-1}(y_2) \ldots \bar{\Gamma}^{-1}(y_{m_i}))$, where $m_i = n_i$ and $x_1, x_2, \ldots, x_{n_i} \in X$; $y_1, y_2, \ldots, y_{m_i} \in Y$. In the practice of technical mechanical systems modelling the mathematical model is not an isomorphous mapping of the investigated object and the problem of adequacy becomes more complex. Physical quantities, determined by measurement on the object „in concreto", forming set $\tilde{X}$, differ from X, the difference resulting from measurement errors possible or impossible to determine ($\tilde{R}_x$ different from R_x, $\tilde{R}_x$-sequence of relations determined within the same spaces as R_x). Numerical sets $\tilde{X}$ are discrete sets and can be replaced by a continuous „representation". Incomplete measurement investigation of a technical object (a finite number of point of measurements and readings) requires „extending" of the empirically established compatibility between R_y and $\tilde{R}_x$ to R_y and R_x which, in turn, requires the usage of unreliable empirical inference [1].

The idea of the presented concept is an attempt at such an adequacy description based on the notions and methods of the fuzzy set theory which would be „extending compatibility measures $\tilde{R}_x$ and R_y" to „the reality R_x and R_y".

3.2. Adequacy of a mathematical model of the level of physical variables. By a physical variable we shall understand each set Y_q, $y_q \in Y_q \subset R$, $q \in Q$, if Y_q has physical meaning (physical dimension has been attributed).

Lets have:

(i) metric space (Z, ϱ) where $Z \equiv X \cup Y$; $\varrho: Z \times Z \to \bar{R}_+$,

(ii) subset M of Cartesian product $M \subset X \times Y$ such that $(x, y) \in M \Rightarrow (x, y) \in \Gamma$; $M = \Gamma$ when $n_i = m_i$, (see Def. 3),

(iii) function $\varrho_\Gamma: X \times Y \to \bar{R}_+$ such that for $z_1, z_2 \in Z$, $[\varrho_\Gamma(z_1, z_2) = \varrho(z_1, z_2)] \Leftrightarrow [(z_1, z_2) \in \Gamma \subset X \times Y]$,

(IV) function $f_A: \Gamma \ni (x, y) \to f_A(x, y) \in [0, 1]$ such that:

1° for $(x, y) \in M$, $[f_A(x, y) = 1] \Leftrightarrow x = y$,

2° for $(x', y'), (x'', y'') \in M$, $[\varrho_\Gamma(x', y') \leqslant \varrho_\Gamma(x'', y'')] \Leftrightarrow [f_A(x', y') \geqslant f_A(x, y)]$,

3° for $(x, y) \in \Gamma \setminus M$, $[f_A(x, y) = 0]$.

Let A_f denote a fuzzy set:

$$A_f = \mathrm{graph} f_A \ni \{(\gamma, f_A(\gamma)) | \gamma \in \Gamma\}, \tag{3.1}$$

where $\gamma \equiv (x, y) \in \Gamma$, function $f_A(\gamma)$ is a membership function of the fuzzy set A_f.

The requirements put forward for f_A are general and determine membership function family f_A:

$$\mathscr{F}_A(\Gamma) = \{f_A(x, y) | f_A: \Gamma \to [0, 1)]\}$$

An example of the f_A function meeting the above requirements may be $f_A = \dfrac{A(x, y)}{V(x, y)}$ when $(x, y) \in R_+$ or $(x, y) \in R_-$, $A(x, y)$ and $V(x, y)$ denote a choice of a smaller or a larger element respectively. This function resambles proportional determining of deviations. Membership degree of pairs (x, y) in a fuzzy set, determined by values of the so selected membership function f_A, may be interpreted as an indicator of what we understand by the notion of adequacy of a mathematical model.

It can easily be noticed that if for $\forall (x, y) \in \Gamma$ takes place $f_A(x, y) = 1$ then from the condition 1° and 3° for f_A results that $\Gamma = M$, $(n_i = m_i)$ and for $\forall (x, y) \in M$ holds $x = y$ and $R_{xm_i} \ni (x_1, x_2, \ldots, x_{m_i}) \Leftrightarrow R_{ym_i} \ni (y_1, y_2, \ldots, y_{m_i})$, for $\forall i \in I_x$ and $\forall i \in I_y$. Hence isomorphous mapping S_x to S_y takes place, relations R_{ym_i} are „fully adequate".

From the condition 2° for f_A results that with the increase in value of $\varrho_\Gamma(x, y)$, the values of membership function f_A decrease which we interpret as diminishing „adequacy degree" of a mathematical model. The fuzzy set A_f is a „picture of adequacy" of a mathematical model on the level physical variables.

3.3. Adequacy of a mathematical model on the level of sets of physical variables Y_u. Le'ts pick out for $R_{ym_i} \ni (y_1, y_2, \ldots, y_{m_i})$ in set Γ subsets Γ_u such that $(x_u, y_u) \in \Gamma_u$ and $x_u \in X_u$, $y_u \in Y_u$, where $u = 1, 2, \ldots, m_i$. Let's denote fuzzy set A_f on Γ_u by:

$$A_u = \text{graph} f_A \ni \{(\gamma, f_A(\gamma)) | \gamma \in \Gamma_u\}$$

Let's make cuts $\Gamma_{u\alpha}$ of set Γ_u:

$$\Gamma_{u\alpha} = \{\gamma \in \Gamma_u | f_A(\gamma) \geq \alpha\} \quad \text{for} \quad \forall \alpha \in [0, 1].$$

Let distance specification be given: $\varrho_y : Y \times Y \to \bar{R}_+$

Let's denote diameter of set Γ_u as $D_u : D_u = \sup_{Y_u} \varrho_Y(y', y'')$ while diameter of cut $\Gamma_{u\alpha}$ as $D_{u\alpha} = \sup_{Y_u\alpha} \varrho_Y(y', y'')$.

Let's form fuzzy measure [16, 17] of set $\Gamma_{u\alpha}$:

$$g(\Gamma_{u\alpha}) = \frac{\max\{D_{u\alpha}\}}{D_u} \tag{3.2}$$

Additionally, let it satisfy condition: if $D_u = 0$ and $D_{u\alpha} = 0$ then $g(\Gamma_{u\alpha}) = 1$. Let's perform fuzzy integration [18] of membership function $f_A(\gamma)$ over set Γ_u to regard of fuzzy measure $g(\Gamma_{u\alpha})$:

$$\Theta_u = \oint_{\Gamma_u} f_A(\gamma) \circ g(\Gamma_{u\alpha}) = \sup_{\alpha \in [0, 1]} [\alpha \wedge g(\Gamma_{u\alpha})] \tag{3.3}$$

where Λ denotes choice of smaller element. Let's notice, that

$$\Theta_u = \Theta_A(\Gamma_u) : \Gamma \to [0, 1].$$

Fuzzy set A_Θ will be defined as:

$$A_\Theta = \text{graph}\Theta_u \ni \{(\Gamma_u, \Theta_A)(\Gamma_u)) | \Gamma_u \subset 2^\Gamma\}. \tag{3.4}$$

It can easily be seen that the fuzzy mesaure (3.2) has the following properties:

1° $g(\Gamma_{u\alpha}) = 0$ when $\Gamma_{u\alpha} = \phi$ or $D_{u\alpha} = 0$, $D_u \neq 0$

2° $g(\Gamma_{u\alpha}) = 1$ when $\alpha = 0$

Let the cuts: $\Gamma'_{u\alpha} \subset \Gamma_{u\alpha}$, $\Gamma''_{u\alpha} \subset \Gamma_{u\alpha}$ be given.

If $\Gamma'_{u\alpha} \subset \Gamma''_{u\alpha}$ then $g(\Gamma'_{u\alpha}) \leq g(\Gamma''_{u\alpha})$ which may be interpreted that the „range" of mapping of variable Y_u on the level α is for the „model $\Gamma'_{u\alpha}$" smaller than for the „model $\Gamma''_{u\alpha}$". Measure $g(\Gamma_{u\alpha})$ assigns relative mapping range of variable Y_u" on the level α in the considered mathematical model. From the properties of a fuzzy integral results that:

1° If $\Gamma'_u \subset \Gamma''_u$ where $\Gamma'_u \subset \Gamma_u$, $\Gamma''_u \subset \Gamma_u$ then $\Theta_u(\Gamma'_u) \leq \Theta_u(\Gamma''_u)$ which means that together with enlargement of „scope" of physical variable in model the value Θ_u increases.

2° If $f_1(\gamma) \geqslant f_2(\gamma)$ then $\int_\Gamma f_1(\cdot) \circ g(\cdot) \geqslant \int_\Gamma f_2(\cdot) \circ g(\cdot)$, which means that with increase of membership function value, the value Θ_u increases.
3° $\Theta_u = 1$ when [for $\forall \gamma \in \Gamma_u, f_A(\gamma) = 1$],
4° $\Theta_u = 0$ when [for $\forall \gamma \in \Gamma_u, f_A(\gamma) = 0$],
5° $\Theta_u\, (\Gamma'_u \cup \Gamma''_u) \neq \Theta_u(\Gamma'_u) + \Theta_u(\Gamma''_u)$.

Membership function Θ_u of a fuzzy set A_Θ maps to each pair of sets (X_u, Y_u) a number from the interval [0, 1]. This number is the closer to one the greater values the membership function $f_A(x, y)$ assumes and the greater the diameter of the subset $\Gamma_{u\alpha}$ on which $f_A(x, y)$ reaches these values ($\alpha = f_A(x, y)$). So the membership function Θ_u may be interpreted as „adequacy measure" of particular sets of physical variables Y_u whose elements are in the investigated relation R_{ym_i}.

3.4. Mathematical model adequacy on the level of relation R_{ym_i}. Let's assume that we are given a fuzzy set for i^{th} relation R_{ym_i}, whose membership function Θ_u assumes values: $\Theta_1, \Theta_2, \ldots, \Theta_{m_i}$. We attribute for $\Theta_1, \Theta_2, \ldots, \Theta_{m_i}$ weights $\mu_1, \mu_2, \ldots, \mu_{m_i}$ such that $0 \leqslant \mu_u \leqslant 1$ for $u = 1, 2, \ldots, m_i$. Let's form a sequence $\Theta_i = (\mu_u \Theta_u)$ where $u = 1, 2, \ldots$ $\ldots, m_i$, and set

$$\bar{\bar{\Theta}} = \{\bar{\Theta}_i \,|\, i \in I_y\}.$$

Let's form a functional:

$$\phi_A : \bar{\bar{\Theta}} \ni \bar{\Theta}_i \to \phi_A(\Theta_i) \in [0, 1], \tag{3.5}$$

satisfying conditions:
1° $\Phi_A(\bar{\Theta}_i) = 1$ when [for $u = 1, 2, \ldots, m_i, \mu_u \Theta_u = 1$],
2° $\Phi_A(\bar{\Theta}_i) = 0$ when [for $u = 1, 2, \ldots, m_i, \mu_u \Theta_u = 0$],
3° $\Phi_A(\bar{\Theta}'_i) \geqslant \phi_A(\bar{\Theta}''_i)$ when [for $u = 1, 2, \ldots, m_i, \mu_u \Theta'_u \geqslant u_u \Theta''_u$]

Fuzzy set A_Φ is expressed as

$$A_\Phi = \operatorname{graph} \Phi_A \ni \{(\bar{\Theta}_i, \Phi_A(\bar{\Theta}_i)) \,|\, \bar{\Theta}_i \in 2^{\bar{\bar{\Theta}}}\}, \tag{3.6}$$

where $\phi_A(\cdot)$ is a membership function.

Weights $\mu_1, \mu_2, \ldots$ attributed to particular physical quantities ocurring in the investigated relation R_{ym_i} enable us to give these quantities subjective meaning in the description of the adequacy of a model. As a functional (3.5) we may take in particular

$$\phi_A(\bar{\Theta}_i) = \mu_1 \Theta_1 + \mu_2 \Theta_2 + \ldots + \mu_{m_i} \Theta_{m_i}, \tag{3.7}$$

where weights satisfay an additional condition $\mu_1 + \mu_2 + \ldots + \mu_{m_i} = 1$. Values of the functional $\phi_A(\Theta_i)$ mapped to particular relations R_{ym_i}, may be interpreted as „adequacy indices" of particular relations of a mathematical model. Fuzzy set A_Φ is then an „adequacy picture" of a model on relation level.

3.5. Adequacy of a mathematical model on the level of relation sequences R_y^w. Let's assume we are given a fuzzy set A_Φ whose membership function assumes for the sequence $R_y =$ $= (R_{ym_i})$, $i \in I_y$ values $\phi_1, \phi_2, \ldots, \phi_{i_0}$. We attribute for $\phi_1, \phi_2, \ldots, \phi_{i_0}$ weights $\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_{i_0}$ such that $0 \leqslant \bar{\mu}_i \leqslant 1$ for $i = 1, 2, \ldots, i_0$.

Let's form a sequence $\Phi_w = (\bar{\mu}_i \phi_i)$, $i = 1, 2, \ldots, i_{0w}$.

Let's assume that a sequence (R_y^w), $w = 1, 2, \ldots, w_0$, is given whose elements are sequences R_y. Let's form set of sequences:

$$\bar{\phi} = \{\Phi_w;\ w = 1, 2, \ldots, w_0\},$$

and a functional:

$$\psi_A : \bar{\phi} \in \Phi_w \rightarrow \psi_A(\Phi_w) \in [0, 1], \tag{3.8}$$

satisfying conditions:

1° $\psi_A(\Phi_w) = 1$ when [for $i = 1, 2, \ldots, i_{0w}, \bar{\mu}_i \phi_i = 1$],

2° $\psi_A(\Phi_w) = 0$ when [for $i = 1, 2, \ldots, i_{0w}, \bar{\mu}_i \phi_i = 0$],

3° $\psi_A(\Phi'_w) \geqslant \psi_A(\Phi''_w)$ when [for $i = 1, 2, \ldots, i_{0w}, \bar{\mu}_i \phi'_i \geqslant \bar{\mu}_i \phi''_i$].

Fuzzy set A_ψ will be defined by:

$$A_\psi = \operatorname{graph} \psi_A \ni \{(\bar{\varphi}_w, \psi_A(\bar{\varphi}_w)) \mid \bar{\varphi}_w \in 2^{\bar{\Phi}}\}, \tag{3.9}$$

where $\psi_A(\cdot)$ is a membership function of set A_ψ.

A fuzzy set A_ψ has been formed in a similar way to the set A_Φ. Weights $\bar{\mu}_i$ attributed to particular relations R_{ym_i} enable us to give them subjective meaning in the description of the adequacy of a model. In particular we can define as a functional:

$$\psi_A(\Phi_w) = \bar{\mu}_1 \phi_1 + \bar{\mu}_2 \phi_2 + \ldots + \bar{\mu}_{w_0} \phi_{w_0}, \tag{3.10}$$

where weights $\bar{\mu}_u$ satisfy the condition $\bar{\mu}_1 + \bar{\mu}_2 + \ldots + \bar{\mu}_{w_0} = 1$

We shall interpret values of the functional $\psi_A(\Phi_w)$ mapped to particular sequence of relations $R_y = (R_{ym_i})$ as „adequacy indices" for particular sequences of relations. In particular, if for a given mathematical model, we present the examined relations as one sequence, ψ_A will attribute for this sequence a number from the interval [0, 1] which be an „index of the adequacy" of the model. Fuzzy set A_ψ is then the „picture of adequacy" of the model on the level of relation sequences (the level of a mathematical model).

4. Example

In order illustrate the presented concept of forming fuzzy sets „a describing adequacy" of a mathematical model we shall consider the problem of inducing vibrations of collecting electrodes of an electrical precipitator [19, 20]. A physical model of the considered system has been presented in fig. 1. A movable rod the lenght l strikes a stationary fixed rod with the speed v_0. The induced wave of stresses in the rod influences particular solids linked with the immobile base by a spring. Let's introduce the following notation:

l – [m] — length of the striking rod,

v_0 – $\left[\frac{\text{m}}{\text{s}}\right]$ — speed of the striking rod,

τ – [s] — duration of the collision,

m – [kg] — mass of harmonic oscillator,

k – $\left[\frac{\text{N}}{\text{m}}\right]$ — rigidity of spring,

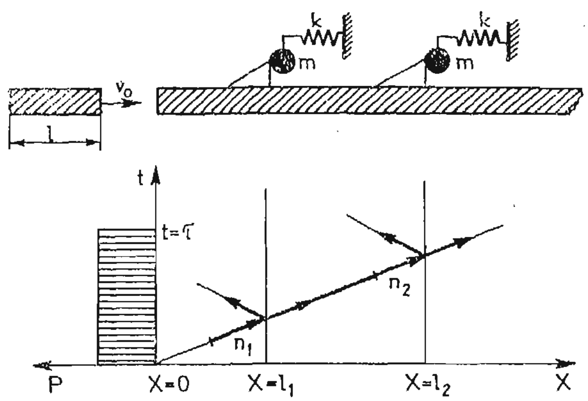

Fig. 1. Model of induction of vibrations to electrostatic precipitator collection electrodes

$\varrho - \left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right]$ — density of material,

$S - [\mathrm{m}^2]$ — cross-section area of the rod,

$P_1, P_2 - [\mathrm{N}]$ — force of acting of the rod on harmonic oscillators, $x = l_1$, $x = l_2$,

$\Pi_1, \Pi_2 - [\mathrm{N}]$ — „pression" in front of the wave reaching the cross-section of the rod $x_l = l_1, x = l_2$,

$a - \left[\frac{\mathrm{m}}{s}\right]$ — speed of longitudinal wave in the rod.

In appropriate assumptions are satisfied [19, 20] together with $\frac{k}{m} \ll h^2 = \varrho^2 a^2 \frac{S^2}{m^2} \Rightarrow$ $\Rightarrow \varrho^2 a^2 S^2 \gg km$ the investigated mechanical system is defined by interrelations:

$$R_{y6_1}: P_0(t) = \varrho a v_0 S \quad \text{for} \quad 0 \leqslant t \leqslant \tau \text{ as well as } P(t_0) = 0 \text{ for } t > \tau,$$

$$R_{y7_2}: P_1(t) = 2P_0(t) - 4h \int_0^t P_0(s) \exp[-2h(t-s)]\, ds,$$

$$R_{y7_3}: P_2(t) = 2P_1(t) - 4h \int_0^t P_1(s) \exp[-2h(t-s)]\, ds,$$

$$R_{y5_4}: \varrho^2 a^2 S^2 \gg km$$

So we investigate the relations R_{ym_i}, $i = 1, 2, 3, 4$:

$$R_{y6_1} \ni (P_0, \varrho, a, v_0, S, t),$$
$$R_{y7_2} \ni (P_1, P_0, \varrho, a, S, m, t),$$
$$R_{y7_3} \ni (P_2, P_1, \varrho, a, S, m, t),$$
$$R_{y5_4} \ni (\varrho, a, S, k, m).$$

Lets assume we have at our disposal empirical data for the modelled object i.e. physical quantities which are in the following relations:

$$\tilde{R}_{x6_1} \ni (\tilde{P}_0, \tilde{\varrho}, \tilde{a}, \tilde{V}_0, \tilde{S}, \tilde{t}),$$
$$\tilde{R}_{x7_2} \ni (\tilde{P}_1, \tilde{P}_0, \tilde{\varrho}, \tilde{a}, \tilde{S}, \tilde{m}, \tilde{t}),$$

$$\tilde{R}_{x7_3} \ni (\tilde{P}_2, \tilde{P}_1, \tilde{\varrho}, \tilde{a}, \tilde{S}, \tilde{m}, \tilde{t}),$$
$$\tilde{R}_{x5_4} \ni (\tilde{\varrho}, \tilde{a}, \tilde{S}, \tilde{k}, \tilde{m}).$$

Relations R_{y6_1}, R_{y7_2}, R_{y7_3}, R_{y5_4} and $\tilde{R}_{x6_1}$, $\tilde{R}_{x7_2}$, $\tilde{R}_{x7_3}$, $\tilde{R}_{x5_4}$ are of course defined within the same spaces but between the physical quantities which are in these relations mapping Γ takes place, and mapping Γ_u takes place in distinguishing particular variables, where $u = P_0, P_1, P_2, \varrho, a, V_0, S, t, m, k$, (see Def. 3). For simplification sake we assume that the subsets Γ_u are the same for the given variable „u" irrelevant of the fact in which of the examined relations this variable appears.

In order to form a set A_f lets denote $\gamma_u = (\tilde{x}, y)$, $\gamma_u \in \Gamma_u$ while $\gamma p_0 = (\tilde{P}_0, P_0)$, ..., $\gamma_k = (\tilde{k}, k)$. Let's define membership function as follow $f_A(\gamma_u) = \dfrac{\Lambda(x, y)}{V(x, y)}$, (see p. 3.2) where $\gamma_u = (\tilde{x}, y) \in R_+$ or $\gamma_u \in R_-$ and for $\tilde{x} = y = 0$ takes place $f_A(\gamma_u) = 1$. We shall denote the fuzzy set A_f over the subsets Γ_u as:

$$A_u = \operatorname{graph} f_A \ni \{(\gamma_u, f_A(\gamma_u)) | \gamma_u \in \Gamma_u\},$$

whereas the set A_f over Γ as:

$$A_u = \operatorname{graph} f_A \ni \{(\gamma, f_A(\gamma)) | \gamma \in \Gamma\},$$

where $\gamma = (\tilde{x}, y)$.

In order to form the set A_Θ we perform fuzzy integration over the sets Γ_u of appropriate physical variables which are in the relations R_{y6_1}, R_{y7_2}, R_{y7_3}, R_{y5_4}:

$$\Theta_u = \oint_{\Gamma_u} f_A(\gamma_u) \circ g(\Gamma_{u\alpha}) = \delta_u,$$

where: $u = P_0, P_1, P_2, \varrho, a, V_0, S, t, m, k$.
For the shake of physical motivation it seems reasonable for $\delta_\varrho = 1$, $\delta_a = 1$, $\delta_t = 1$, $\delta_m = 1$, $\delta_k = 1$. A detailed example of calculating the fuzzy integral Θ_u has been presented in paper [1]. The sets A_Θ, corresponding to particular relations R_{y6_1}, R_{y7_2}, R_{y7_3}, R_{y5_4}, we shall put down as follows:

$$A_{\Theta_1} = \{(\Gamma_u, \delta_u) | u = P_0, \varrho, a, v_0, S, t\},$$
$$A_{\Theta_2} = \{(\Gamma_u, \delta_u) | u = P_1, P_0, \varrho, a, S, m, t\},$$
$$A_{\Theta_3} = \{(\Gamma_u, \delta_u) | u = P_2, P_1, \varrho, a, S, m, t\},$$
$$A_{\Theta_4} = \{(I_u, \delta_u) | u = \varrho, a, S, k, m\},$$

whereas set A_Θ for all considered relations as follows:

$$A_\Theta = \{\Gamma_u, \delta_u) | u = P_0, P_1, P_2, \varrho, a, v_0, S, t, m, k\}.$$

In order to form set A_Φ let's attribute particular weights μ_u to the physical variables which are in the relations R_{y6_1}, R_{y7_2}, R_{y7_3}, R_{y5_4}. For physical variables in the relation R_{y6_1} let it be: $\mu_{P_0} = 0{,}5$; $\mu_\varrho = 0$, $\mu_a = 0$, $\mu_{v_0} = 0{,}5$; $\mu_t = 0$.

This signifies that it is of vital importance for us to determine correct values of the force P_0 as well as the speed of striking v_0. Weights μ_u satisfy the condition $\mu_{P_0} + \mu_\varrho + + \mu_a + \mu_{V_0} + \mu_s + \mu_t = 1$. For physical variables in the relation R_{y7_2} let's assume in an analogous way that: $\mu_{P_1} = 0{,}5$; $\mu_{P_0} = 0{,}5$; $\mu_Q = 0$, $\mu_a = 0$, $\mu_s = 0$, $\mu_m = 0$, $\mu_t = 0$; for physical variables in relation R_{y7_3}: $\mu_{P_2} = 0{,}5$; $\mu_{P_1} = 0{,}5$; $\mu_Q = 0$, $\mu_a = 0$, $\mu_s = 0$, $\mu_m = 0$, $\mu_t = 0$; for physical variables in relation R_{y5_4}: $\mu_Q = 0{,}2$; $\mu_a = 0{,}2$; $\mu_s = 0{,}2$; $\mu_k = 0{,}2$; $\mu_m = 0{,}2$.

Appropriate sequences $\bar{\Theta}_u$ expressed as:

$$\bar{\Theta}_1 = (0{,}5\delta_{P_0};\, 0\delta_Q;\, 0\delta_a;\, 0{,}5\delta_{V_0};\, 0\delta_s;\, 0\delta_t),$$

$$\bar{\Theta}_2 = (0{,}5\delta_{P_1};\, 0{,}5\delta_{P_0};\, 0\delta_Q;\, 0\delta_a;\, 0\delta_s;\, 0\delta_m;\, 0\delta_t),$$

$$\bar{\Theta}_3 = (0{,}5\delta_{P_2};\, 0{,}5\delta_{P_1};\, 0\delta_Q;\, 0\delta_a;\, 0\delta_s;\, 0\delta_m;\, 0\delta_t),$$

$$\bar{\Theta}_4 = (0{,}2\delta_Q,\, 0{,}2\delta_a;\, 0{,}2\delta_s;\, 0{,}2\delta_k;\, 0{,}2\delta_m),$$

Accepting the membership function in the set A_Φ, given by the proposition (3.7), we shall write down the following:

$$\phi_A(\bar{\Theta}_1) = 0{,}5(\delta_{P_0} + \delta_{V_0}) = \varepsilon_1;\ \phi(\bar{\Theta}_2) = 0{,}5(\delta_{P_1} + \delta_{P_0}) = \varepsilon_2;$$

$$\phi_A(\bar{\Theta}_3) = 0{,}5(\delta_{P_2} + \delta_{P_1}) = \varepsilon_3;\ \phi(\bar{\Theta}_4) = 0{,}2(\delta_Q + \delta_a + \delta_s + \delta_k) = \varepsilon_4.$$

The fuzzy set A_Φ assumes form:

$$A_\Phi \equiv \{(\bar{\Theta}_i, \varepsilon_i) \mid i = 1, 2, 3, 4\}.$$

Values ε_1, ε_2, ε_3, ε_4 of the membership function of the set A_Φ are adequacy indices for particular relations R_{ym_i}.

In order to form the set A_ψ let's attribute the following weights: $\bar{\mu}_1 = 0{,}25$; $\bar{\mu}_2 = 0{,}25$; $\bar{\mu}_3 = 0{,}25$; $\bar{\mu}_4 = 0{,}25$ to the particular relations R_{y6_1}, R_{y7_2}, R_{y7_3}, R_{y5_4}. This means that each the investigated relations has the same significance in the investigated mathematical model. Let's formulate a sequence

$$\Phi_1 = (\mu_i \phi_i) = (0{,}25\varepsilon_1;\, 0{,}25\varepsilon_2;\, 0{,}25\varepsilon_3;\, 0{,}25\varepsilon_4).$$

Accepting the membership function of the set A_ψ, given by the proposition (3.10), we can put down: $\psi_A(\Phi_1) = 0{,}25\varepsilon_1 + 0{,}25\varepsilon_2 + 0{,}25\varepsilon_3 + 0{,}25\varepsilon_4 = \varphi_1$. The fuzzy set A_ψ will be as follows:

$$A_\psi \equiv \{(\phi_w, \varphi_w) \mid w = 1\}.$$

Value φ_1 of the membership function constiutes a fuzzy index the investigated mathematical model.

5. Final remarks

The considerations presented above constitute a modification of the problem of identification of mathematical models of technical mechanical systems including the problems of vagueness. In order to describe the adequacy of a mathematical model of technical mechanical system a hierarchic arrangement of fuzzy sets has been formed on the level of physical variables, sets of physical variables, system relations and sequences of relations. The presented formalism enables a description of adequacy on various level of minutenes of detail analysis and for models having various degree of mathematical complexity.

This description may serve the purposes of making comparative evaluations as those aiming at a goal of the adequacy of the model when the aim of the model has been presented in the form of a fuzzy set of type 2 [21]. A relevant algorithm of the adequacy evaluation of the model, resolving the problem into inclusion of fuzzy sets has been presented in paper [1].

6. Basic notions and properties concerning fuzzy sets

When fuzzy sets are given:

$$A = \text{graph} f_A \ni \{(x, f_A(x)) | x \in X, f_A(x) \in [0, 1]\},$$
$$B = \text{graph} f_A \ni \{(x, f_B(x)) | x \in X, f_B(x) \in [0, 1]\},$$

hence their sum $A \cup B = C$ is a fuzzy set:

$$C = \text{graph} f_C \ni \{(x, f_C(x)) | x \in X\}, \text{ where,}$$
$$f_C(x) = \max(f_A(x), f_B(x)).$$

The fuzzy set $D = \text{graph} f_D \ni \{(x, f_D(x)) | x \in X\}$ is the product of $A \cap B = D$ when $f_D = \min(f_A(x), f_B(x))$.

Inclusion of fuzzy sets $A \subset B$ means that $f_A(x) \leqslant f_B(x)$ for $\forall x \in X$.

If the set X and Borel field β of set X are given then function $g(\cdot)$ defined on β and satisfying the following three conditions will be the fuzzy measure of the set:

1° $g(\phi) = 0, g(X) = 1$,

2° If $A, B \in \beta$ and $A \subset B$ then $g(A) \leqslant g(B)$,

3° If $A_n \in \beta$ for $1 \leqslant n \leqslant \infty$ and A_n is monotonic in the sense of inclusion then $\lim_{n \to \infty} (A_n) = g(\lim_{n \to \infty} A_n)$.

Given a fuzzy set A, cut F_α of set A: $F_\alpha \equiv \{x | f_A(x) \geqslant \alpha \in [0, 1]\}$ and $F_\alpha \in \beta$, where β is Borel field, by a fuzzy integral of the $f_A(x)$ function over the set $E \subset X$ to regard of the fuzzy measure $g(\cdot)$ we shall understand: $\oint_E f_A(x) \circ g(\cdot) = \sup_{\alpha \in [0, 1]} [\alpha \Lambda g(E \cap F)]$, where Λ denotes the choice of a smaller element.

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Р е з ю м е

РАЗМЫТИЕ МНОЖЕСТВА В ОПИСАНИИ АДЕКВАТНОСТИ МАТЕМАТИЧЕСКИХ МОДЕЛЕЙ ТЕХНИЧЕСКИХ МЕХАНИЧЕСКИХ СИСТЕМ

В работе проведено описание адекватности математических моделей технических механических систем, которое принимает во внимание нечеткие проблемы адекватности. Использовано понятия размытых множеств Л. А. Заде, размытой меры и размытого интеграла. Сообразовано иерархическую систему размытых множеств на уровне физических переменных, множеств физических переменных, отношений системы и кортежей отношений. Представленный метод описания адекватности делает возможным формальное проведение сравнительных и направленых на цель оценок адекватности математических моделей о различной степени математического усложнения.

S t r e s z c z e n i e

ZBIORY ROZMYTE W OPISIE ADEKWATNOŚCI MODELI MATEMATYCZNYCH TECHNICZNYCH SYSTEMÓW MECHANICZNYCH

W pracy przedstawiono próbę opisu adekwatności modeli matematycznych technicznych systemów mechanicznych uwzględniającego nieostre problemy adekwatności. Wykorzystano do tego celu pojęcia zbiorów rozmytych L. A. Zadeha [2], rozmytej miary i rozmytej całki [18]. Utworzono hierarchiczny

układ zbiorów rozmytych na poziomie zmiennych fizycznych, zbiorów zmiennych fizycznych, relacji systemowych i ciągów relacji. Przedstawiona metoda opisu adekwatności umożliwia formalizację dokonywania ocen porównawczych i docelowych adekwatności modeli matematycznych o różnym stopniu złożoności matematycznej.

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