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SOME EXISTENCE AND UNIQUENESS RESULTS IN ELASTOSTATICS WITH CONSTRAINTS FOR DISPLACEMENTS AND STRESSES

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The aim of the paper is the investigation of some class of problems of elastostatics with constraints for displacements and stresses. There are considered such restrictions which lead to the problems with generalized displacements and generalized stresses. Some existence and uniqueness results are proved.

1. Basic concepts and assumptions of elastostatics with constraints

Throughout the paper $\{\mathbf{e}_k\}_{k=1,2,3}$ denotes a fixed ortonormal basis in Euclidean 3-space $\mathfrak{E}$. The dual basis is given by $\{\mathbf{e}^k\}_{k=1,2,3}$, $e = (\delta_{ij})$, where δ_{ij}, $i,j = 1, 2, 3$ is the Kronecker delta, is the metric tensor in $\mathfrak{E}$. If $\mathbf{x} \in \mathfrak{E}$ then by (x^k) we denote the ortogonal coordinates of $\mathbf{x}$ relative to the basis $\{\mathbf{e}_k\}$, $\mathbf{x} = \mathbf{e}_k x^k$, and analogously (x_k) are coordinates of $\mathbf{x}$ relative to the basis $\{\mathbf{e}^k\}$, $\mathbf{x} = x_k \mathbf{e}^k$ (In the following the summation convention holds and the indices $i, j, k, \ldots$ run from 1 to 3).

Let B be a bounded region in $\mathfrak{E}$ with the regular boundary ∂B, c.f. [1], occupied by the body in its undeformed state. The problem will be analyzed under the basic assumption of the infinitesimal theory.

Let us denote by $\mathfrak{D}$ the space of all vector functions components of which relative to the basis $\{\mathbf{e}^k\}$ are square integrable together with their first partial derivatives in B, i.e.

$$\mathfrak{D} = \{\mathbf{u} = (u_k(\mathbf{x})), \quad \mathbf{x} \in B, \quad u_k \in H^1(B), \quad k = 1, 2, 3\},$$

equipped with the norm

$$||\mathbf{u}||_1^2 = \int_B [\mathbf{u}^2 + h^2 \mathrm{tr}(\nabla\mathbf{u}\nabla\mathbf{u}^T)]\,dv = \int_B (u_k u^k + h^2 u^k_{,l} u_{k,m} \delta^{lm})\,dv,$$

where $\nabla\mathbf{u} = (u_{k,l})$ is the gradient of u with respect to $\mathbf{x}$, h is a positive constant suitable choosen in the problem under consideration. The space $\mathfrak{D}$ will be interpreted as the displacement space and its elements as displacement fields.

External forces acting at the body will be represented by linear continuous functionals on $\mathfrak{D}$, i.e. by elements of $\mathfrak{D}^*$, $\mathfrak{D}^*$ being the dual of $\mathfrak{D}$. To the system of external forces will be assigned such element $\mathbf{f}^*$ of $\mathfrak{D}^*$ that the value of work done by these forces on arbitrary field $\mathbf{v} \in \mathfrak{D}$ is equal to the value of the functional $\mathbf{f}^*$ at $\mathbf{v}$. Assuming that the body is subjected to the body forces $\mathbf{b} = (b^k(\mathbf{x}))$, $\mathbf{x} \in B$, $b^k \in L^2(B)$, $k = 1, 2, 3$, and to the surface traction $\mathbf{p} = (p^k(\mathbf{x}))$, $\mathbf{x} \in \partial B$, $p^k \in L^2(\partial B)$, $k = 1, 2, 3$, we have the following representation for the corresponding functional $\mathbf{f}^* \in \mathfrak{D}^*$:

$$\langle \mathbf{f}^*, \mathbf{v} \rangle_1 = \int_B \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial B} \mathbf{p} \cdot \mathbf{v} ds, \qquad \mathbf{v} \in \mathfrak{D}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle_1$ being the pairing between $\mathfrak{D}^*$ and $\mathfrak{D}$.

Let us introduce $\mathfrak{S}$ as the space of all symmetric tensor functions, components of which relative to the basis $\{\mathbf{e}^k \otimes \mathbf{e}^l\}_{k,l=1,2,3}$, are square integrable in B, i.e.

$$\mathfrak{S} = \{\mathbf{C} = (C_{kl}(\mathbf{x})), \quad \mathbf{x} \in B, \quad C_{kl} = C_{lk}, \quad C_{kl} \in L^2(B)\}.$$

The space $\mathfrak{S}$ is assumed to be equipped with the norm

$$||\mathbf{C}||_2^2 = \int_B \mathrm{tr}(\mathbf{C}\mathbf{C}) dv = \int_B C_{kl} C^{kl} dv, \qquad \mathbf{C} \in \mathfrak{S}.$$

The displacement-stran relations will be described by the linear continuous operator $\mathbf{E}: \mathfrak{D} \to \mathfrak{S}$, which assignes to any $\mathbf{v} \in \mathfrak{D}$ the symmetric part of the displacement gradient $\nabla \mathbf{u}$, i.e.

$$\mathbf{E}(\mathbf{v}) = 1/2(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \qquad \mathbf{v} \in \mathfrak{D},$$

or equivalently

$$\mathbf{E} = (E_{kl}), \quad E_{kl}(\mathbf{v}) = 1/2(v_{k,l} + v_{l,k}), \qquad \mathbf{v} \in \mathfrak{D}.$$

The subspace of $\mathfrak{S}$ being the image of $\mathfrak{D}$ by the mapping $\mathbf{E}$ is said to be the space of all strain fields. The whole space $\mathfrak{S}$ will be called the strain space.

Independently of $\mathfrak{S}$ we introduce the space $\mathfrak{T}$ of all symmetric tensor functions $\mathbf{T}$, components of which relative to the basis $\{\mathbf{e}_k \otimes \mathbf{e}_l\}_{k=1,2,3}$, are square integrable in B, i.e.

$$\mathfrak{T} = \{\mathbf{T} = (T^{kl}(\mathbf{x})), \quad \mathbf{x} \in B, \quad T^{kl} = T^{lk}, \quad T^{kl} \in L^2(B), \quad k, l = 1, 2, 3\},$$

the space $\mathfrak{T}$ being equipped with the norm

$$||\mathbf{T}||_2^2 = \int_B \mathrm{tr}(\mathbf{T}\mathbf{T}) dv = \int_B T^{kl} T_{kl} dv, \qquad \mathbf{T} \in \mathfrak{T}.$$

$\mathfrak{T}$ will be treated as the stress space and its elements as stress fields. It is easy to see that from the mathematical point of view the spaces $\mathfrak{S}$ and $\mathfrak{T}$ coincide. The main difference between them follows from the physical interpretation.

For any $\mathbf{T} \in \mathfrak{T}$ and any $C \in \mathfrak{S}$ let

$$\langle \mathbf{T}, \mathbf{C} \rangle_2 \stackrel{\mathrm{df}}{=} \int_B \mathrm{tr}(\mathbf{T}\mathbf{C}) dv = \int_B T^{kl} C_{kl} dv. \tag{1.2}$$

Under the above denotation the value of virtual work done by the internal forces corresponding to the stress field $\mathbf{T} \in \mathfrak{T}$ over any strain field $\mathbf{E}(\mathbf{v})$, $\mathbf{v} \in \mathfrak{D}$, can be written as

$$\langle \mathbf{T}, \mathbf{E}(\mathbf{v}) \rangle_2 = \int_B \mathrm{tr}(\mathbf{T}\mathbf{E}(\mathbf{v})) dv.$$

The material properties of the body will be determined by the operator $\mathbf{K}:\mathfrak{T} \to \mathfrak{S}$, which is assumed to satisfy the following monotonicity condition

$$\langle \mathbf{KT}-\mathbf{K}\boldsymbol{\sigma}, \mathbf{T}-\boldsymbol{\sigma}\rangle_2 \geqslant c\|\mathbf{T}-\boldsymbol{\sigma}\|_2^2 \quad \forall \mathbf{T}, \boldsymbol{\sigma}\in\mathfrak{T},$$

where c is a positive constant.

It must be stressed, that in this approach the constitutive operator $\mathbf{K}$ may be non-linear. It means that we can also deal with problems in which some physical non-linearities are taken into account. In the linear case $\mathbf{K}$ coincides with a compliance field tensor $\mathbf{K} = (K_{ijkl}(\mathbf{x}))$, $\mathbf{x} \in B$. Then the above monotonicity condition can be rewritten in the form

$$\langle \mathbf{KT}, \mathbf{T}\rangle_2 \geqslant c\|\mathbf{T}\|_2^2, \quad \forall \mathbf{T} \in \mathfrak{T}.$$

In the paper we shall deal with such problems of elastostatics in which admissible are only certain distingushed subsets of the displacement speace $\mathfrak{D}$ and the stress space $\mathfrak{S}$. It means that on displacements and stresses are imposed some restrictions which will be called displacement constraints and stress constraints, respectively. To precise these concepts we shall assume that in every problem under consideration there are given a priori two subsets $\mathfrak{U} \subset \mathfrak{D}$ and $\boldsymbol{\Sigma} \subset \mathfrak{T}$ of all admissible displacement fields and stress fields, respectively. In particular, if $\mathfrak{U} = \mathfrak{D}$ and $\boldsymbol{\Sigma} = \mathfrak{T}$ then we deal with the unconstrained body.

Throghout the considerations we confine ourselves to some class of constraints, namely it will be assumed that $\mathfrak{U}$ and $\boldsymbol{\Sigma}$ are proper convex and closed subsets of $\mathfrak{D}$ and $\mathfrak{T}$, respectively[1].

The equations of equilibrium in problems with constraints will be assumed in the form of the following condition

$$\int_B \operatorname{tr}(\mathbf{TE}(\mathbf{v}))dv - \int_B \mathbf{b}\cdot\mathbf{v}dv - \int_{\partial B} \mathbf{p}\cdot\mathbf{v}ds - \langle \mathbf{r}^*, \mathbf{v}\rangle_1 = 0, \tag{1.4}$$

which has to be satisfied for any $\mathbf{v} \in \mathfrak{D}$, where $\mathbf{T} \in \boldsymbol{\Sigma}$ is the stress field, $\mathbf{b}$ and $\mathbf{p}$ have the same meaning as in (1.1), $\mathbf{r}^* \in \mathfrak{D}^*$ is the functional which represents the work of reaction forces due to the displacement constraints, c.f. [6]. The condition (1.4) states that the value of work done by the external forces and by the reaction forces over any $\mathbf{v} \in \mathfrak{D}$ is equal to the value of work done by the internal forces over corresponding strain field $\mathbf{E}(\mathbf{v})$.

In this approach together with the displacement constraints we have introduced reaction forces which have to maintain the constraints. As far as the mathematical aspects of the problem is concerned it will be assumed that the functional $\mathbf{r}^* \in \mathfrak{D}^*$ representing the work of reaction forces (contrary to the functional $\mathbf{f}^* \in \mathfrak{D}^*$ which represents the work of external forces) can be an arbitrary element of $\mathfrak{D}^*$. This requirement is due to the fact that in the considerations we are to deal with a wide class of constraints and therefore we ought to introduce suitable wide class of reaction forces in order to maintain these constraints. Thus we take into account not only functionals r^* of the form

$$\langle \mathbf{r}^*, \mathbf{v}\rangle_1 = \int_B \mathbf{r}\cdot\mathbf{v}dv + \int_{\partial B} \mathbf{s}\cdot\mathbf{v}ds, \quad \forall \mathbf{v} \in \mathfrak{D},$$

[1] These assumptions are due the mathematical tool which will be used later.

where $\mathbf{r} \in L^2(B)^3$ and $\mathbf{s} \in L^2(\partial B)^3$ are interpreted as body reaction forces and surface reaction tractions, respectively, but also functionals having the most general representation, namely,[2],

$$\langle \mathbf{r}^*, \mathbf{v} \rangle_l = \int_B (\mathbf{w} \cdot \mathbf{v} + \mathrm{tr}(\nabla \mathbf{w} \nabla \mathbf{v}^T))\, dv, \qquad \forall \mathbf{v} \in \mathfrak{D}, \tag{1.5}$$

for some $\mathbf{w} \in \mathfrak{D}$.

For every displacement field u admissible by the constraints, i.e. $u \in \mathfrak{U}$, we have to determine the set of all reactions which can act at the body. In order to determine this set we shall assume that the displacement constraints are ideal, c.f. [6], i.e.

$$\langle \mathbf{r}^*, \mathbf{v} - \mathbf{u} \rangle_1 \geqslant 0, \qquad \forall \mathbf{v} \in \mathfrak{U}. \tag{1.6}$$

The above condition is said to be the principle of ideal displacement constraints [6 - 8]. It states that the value of work done by the reaction forces over any virtual displacement field $\mathbf{v} - \mathbf{u}$, $\mathbf{v} \in \mathfrak{U}$, is always non-negative.

The constutive relations for stresses will be given in the form of the following condition

$$\int_B \mathrm{tr}[\boldsymbol{\sigma}(\mathbf{KT} - \mathbf{E}(\mathbf{u}))]\, dv - \int_B \mathrm{tr}[\boldsymbol{\sigma}\mathbf{G}]\, dv = 0, \tag{1.7}$$

which has to be satisfied for any $\boldsymbol{\sigma} \in \mathfrak{T}$, where u is the displacement field, $\mathbf{T}$ is the stress field, $\mathbf{G} \in \mathfrak{G}$ is said to be the strain field incompatibilities due to the stress constraints, [8]. Together with the stress constraints we have introduced the strain field incompatibilities which have to maintain these constraints. As in the case of reaction forces, for every stress field $\mathbf{T}$ admissible by the constraints, i.e. $\mathbf{T} \in \mathbf{\Sigma}$, we have to determine a set of all strain incompatibilities. In order to determine this set we shall assume that the stress constraints are ideal, [8], i.e.

$$\int_B \mathrm{tr}[(\boldsymbol{\sigma} - \mathbf{T})\mathbf{G}]\, dv \geqslant 0, \quad \forall \boldsymbol{\sigma} \in \mathbf{\Sigma}. \tag{1.8}$$

The above condition is said to be the principle of ideal stress constraints. It states that the value of work done by internal forces corresponding to any virtual stress $\boldsymbol{\sigma} - \mathbf{T}$, $\boldsymbol{\sigma} \in \mathbf{\Sigma}$, over the strain field incompatibilities $\mathbf{G}$ is always non-negative. In particular, if the stress constraints are absent, i.e. $\mathbf{\Sigma} = \mathrm{T}$, then from (1.8) it follows that $\mathbf{G} = \mathbf{0}$ and (1.7) leads to the well known form of the constitutive equations for elastic body, namely

$$\mathbf{KT} - \mathbf{E}(\mathbf{u}) = \mathbf{0}.$$

Ideal constraints are the special case of so called quasi-ideal constraints which have been discussed in [8]. Such constraints and their realization are defined by means of the

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[2] Since, as it is known, $\mathfrak{D}$ is a Hilbert space with the inner product given by

$$(\mathbf{v}, \mathbf{w}) = \int_B (\mathbf{wv} + \mathrm{tr}(\nabla \mathbf{w} \nabla \mathbf{v}^T))\, \mathrm{d}v, \qquad \mathbf{v}, \mathbf{w} \in \mathfrak{D},$$

so, by Riesz Representation Theorem it follows that an arbitrary element of $\mathfrak{D}^*$ can be represented in the form (1.5).

known proper convex lower semicontinuous functions $\beta: \mathfrak{D} \to \bar{\mathbf{R}}$ and $\gamma: \mathfrak{T} \to \bar{\mathbf{R}}$,[3], suitable choosen in every problem under consideration. Principles of quasi-ideal constraints are assumed to have the following forms:

$$\langle \mathbf{r}^*, \mathbf{v}-\mathbf{u}\rangle_1 + \beta(\mathbf{v}) - \beta(\mathbf{u}) \geqslant 0, \quad \forall \mathbf{v} \in \mathfrak{D}, \tag{1.6$'$}$$

for the displacement constraints, and

$$\int_B \mathrm{tr}[(\sigma - \mathbf{T})\mathbf{G}]dv + \gamma(\sigma) - \gamma(\mathbf{T}) \geqslant 0, \quad \forall \sigma \in \mathfrak{T}, \tag{1.8$'$}$$

for the stress constraints, respectively. In particular, if β and γ are the indicator functions of $\mathfrak{U}$ and $\mathbf{\Sigma}$ then (1.6)′ and (1.8)′ reduce to (1.6) and (1.8), respectively, and we are to deal with ideal constraints.

We shall assume that

$$\mathbf{f}^* = \mathbf{f}_0^* + \mathbf{f}_1^*, \tag{1.9}$$

where $\mathbf{f}_0^* \in \mathfrak{D}^*$ represents „dead" load, i.e.

$$\langle \mathbf{f}_1^*, \mathbf{v}\rangle_1 = \int_B \mathbf{b}_0 \cdot \mathbf{v}dv + \int_{\partial B} \mathbf{p}_0 \cdot \mathbf{v}ds, \quad \forall \mathbf{v} \in \mathfrak{D}, \tag{1.10}$$

where $\mathbf{b}_0 \in L^2(B)^3$, $\mathbf{p}_0 \in L^2(\partial B)^3$ are the known vector functions not depending on the displacement field $\mathbf{u}$, and $\mathbf{f}_1^* \in \mathfrak{D}^*$ represents external forces essentially depending on the displacement field by the formula

$$\langle \mathbf{f}_1^*, \mathbf{v}-\mathbf{u}\rangle_1 + \zeta(\mathbf{v}) + \zeta(\mathbf{u}) \geqslant 0, \quad \forall \mathbf{v} \in \mathfrak{D}, \tag{1.11}$$

where $\zeta: \mathfrak{D} \to \bar{\mathbf{R}}$ is the known proper convex lower semicontinuous function such that its effective domain satisfies the condition $D(\zeta) \supset \mathfrak{U}$[4]. For instance, by means of (1.11) we can charecterize the potential forces (ζ being Gateaux differentiable), the forces of friction [1], the forces of mutual interaction between the body and its foundation, [1], e.c.t.[5].

Summing up, foundations of elastostatics with constraints for displacements and stresses are given by equation of equilibrium (1.4), constitutive equation for stresses, (1.7), constitutive relations for external forces (1.9) - (1.11), the principle of ideal (quasi-ideal) constraints (1.6) ((1.6)′) and that of the ideal (quasi-ideal) stress constraints (1.8) ((1.8)′).

2. Generalized displacements and generalized stresses

In this Section the restrictions will be given leading to problems with so called generalized displacement and generalized stresses.

Let us begin the formulation of the displacement constraints. It is supposed that there

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[3] Here and what follows $\bar{R} = R \cup \{+\infty\}$.

[4] For any convex function $\alpha: X \to \bar{R}$ we use the symbol $D(\alpha)$ to denote the effective domain of α, i.e.

$$D(\alpha) = \{x \in X : \alpha(x) < +\infty\}.$$

[5] Eqs. (1.9) - (1.11) can be reffered to as the constitutive relations for external forces $\mathbf{f}^*$.

is given a space $\mathfrak{V}$, being a closed linear subspace of $\mathfrak{D}$, and an element $\mathbf{u}_0 \in \mathfrak{D}$ such that

$$||\mathbf{E}(\mathbf{v})||_2 \geqslant c||\mathbf{v}||_1, \quad \mathbf{v} \in \mathfrak{V}, \tag{2.1}$$

i.e. on $\mathfrak{V}$ Korn's inequality holds, and

$$\mathfrak{U} \subset \mathfrak{V}+\mathbf{u}_0 = \{\mathbf{v}+\mathbf{u}_0 \colon \mathbf{v} \in \mathfrak{V}\}. \tag{2.2}$$

For instance, such situations take a place if $\mathfrak{U}$ is a subset of all displacement fields satisfying on a given part of the body the known displacement boundary conditions.

Moreover, we suppose that there is given the linear continuous operator $\mathbf{\Phi}\colon\mathfrak{Q} \to \mathfrak{V}$ from a reflexive Banach space $\mathfrak{Q}$ into $\mathfrak{V}$ such that

$$||\mathbf{\Phi}(\mathbf{q})||_1 \geqslant c||\mathbf{q}||_{\mathfrak{Q}}, \quad \mathbf{q} \in \mathfrak{Q},\ c > 9,^{(6)}, \tag{2.3}$$

and there is known the non-empty closed convex subset $\tilde{\mathfrak{Q}} \subset \mathfrak{Q}$ for which

$$\mathfrak{U} = \tilde{\mathfrak{U}}+\mathbf{u}_0 = \{\mathbf{v}+\mathbf{u}_0 \colon \mathbf{v} \in \tilde{\mathfrak{U}}\}, \tag{2.4}$$

where

$$\tilde{\mathfrak{U}} = \mathbf{\Phi}(\tilde{\mathfrak{Q}}) = \{\mathbf{v} \in \mathfrak{V} \colon \mathbf{v} = \mathbf{\Phi}(\mathbf{q}) \text{ for some } \mathbf{q} \in \tilde{\mathfrak{Q}}\}. \tag{2.5}$$

$\mathfrak{Q}$ will be called the space of generalized displacements and $\tilde{\mathfrak{Q}}$ the set of all admissible generalized displacements. The set $\mathfrak{U}$ is the image of $\tilde{\mathfrak{Q}}$ by the mapping $\mathbf{\Phi}$, translated by $\mathbf{u}_0$. From Eq. (2.3) it follows that to any displacement field $\mathbf{u}$ admissible by the constraints, i.e. $\mathbf{u} \in \mathfrak{U}$, corresponds exactly one $\mathbf{q} \in \tilde{\mathfrak{Q}}$ such that $\mathbf{u} = \mathfrak{V}(\mathbf{q})+\mathbf{u}_0$.

In order to specify the set of all admissible stress fields we suppose that there is known linear continuous operator $\mathbf{\Psi}\colon\mathbf{\Pi} \to \mathfrak{T}$ from a reflexive Banach space $\mathbf{\Pi}$ into $\mathfrak{T}$ with

$$||\mathbf{\Psi}(\pi)||_2 \geqslant c||\pi||_{\Pi}, \quad \pi \in \mathbf{\Pi},\ c > 0. \tag{2.6}$$

There is also known the non-empty closed convex subset $\tilde{\mathbf{\Pi}} \subset \mathbf{\Pi}$ such that

$$\mathbf{\Sigma} = \mathbf{\Psi}(\tilde{\mathbf{\Pi}}) = \{\mathbf{T} \in \mathfrak{T} \colon \mathbf{T} = \mathbf{\Psi}(\pi) \text{ for some } \pi \in \tilde{\mathbf{\Pi}}\}. \tag{2.7}$$

$\mathbf{\Pi}$ will be called the space of generalized stresses and $\tilde{\mathbf{\Pi}}$ — the set of all admissible generalized stresses. The set $\mathbf{\Sigma}$ is the image of $\tilde{\mathbf{\Pi}}$ by the mapping $\mathbf{\Psi}$. From (2.6) it follows that to any stress field $\mathbf{T}$ admissible by the constraints, i.e. $\mathbf{T} \in \mathbf{\Sigma}$, corresponds exactly one $\pi \in \tilde{\mathbf{\Pi}}$ such that $\mathbf{T} = \mathbf{\Psi}(\pi)$,[7].

Now, let us pass to the governing relations of the problems of elastostatics with constraints defined above.

By Riesz Representation Theorem the stress space $\mathbf{T}$ can be identified with the dual of $\mathfrak{S}$. Thus $\langle \cdot, \cdot \rangle_2$, defined by (1.2) can be treated as the paring between $\mathfrak{T}$ and $\mathfrak{S}$. The adjoint of the restriction $\mathbf{E}$ to $\mathfrak{V}$, $\mathbf{E}|_{\mathfrak{V}}$, as the operator from $\mathfrak{T}$ into $\mathfrak{V}^*$, $\mathfrak{V}^*$ being the dual of $\mathfrak{V}$, will be denoted by $\mathbf{E}^*$, $\mathbf{E}^*\colon\mathfrak{T} \to \mathfrak{V}^*$. Recall that $\mathbf{E}^*$ assignes to any $\boldsymbol{\sigma} \in \mathbf{T}$ such element $\mathbf{E}^*\boldsymbol{\sigma} \in \mathfrak{V}^*$ that

$$\langle \mathbf{E}^*\boldsymbol{\sigma}, \mathbf{v}\rangle_1 \overset{\text{df}}{=} \langle \boldsymbol{\sigma}, \mathbf{E}(\mathbf{v})\rangle_2, \quad \mathbf{v} \in \mathfrak{V},$$

(we use the same denotation for the pairing between $\mathfrak{V}^*$ and $\mathfrak{V}$ as that of between $\mathfrak{D}^*$ and $\mathfrak{D}$).

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[6] Throughout this paper c denote generic positive constants, necessarily the same at each occurance.

[7] The physical meaning and selected applications of the introduced constraints can be found in [3.4].

Assuming that the displacement constraints given by (2.4) are ideal, i.e. (1.6) holds, for the restriction of $\mathbf{r}^* \in \mathfrak{D}^*$ to $\mathfrak{B}$, $\mathbf{r}^*|_{\mathfrak{B}} \in \mathfrak{B}^*$, we obtain

$$\begin{aligned} \mathbf{r}^*|_{\mathfrak{B}} &\in \{\mathbf{v}^* \in \mathfrak{B}^* : \langle \mathbf{v}^*, \mathbf{v}-\mathbf{u}\rangle_1 \geqslant 0, \quad \forall \mathbf{v} \in \mathfrak{U}\} = \\ &= \{\mathbf{v}^* \in \mathfrak{B}^* : \langle \mathbf{v}^*, \mathbf{\Phi}(\mathbf{p})+\mathbf{u}_0-(\mathbf{\Phi}(q)+\mathbf{u}_0)\rangle_1 \geqslant 0, \quad \forall \mathbf{p} \in \tilde{\mathfrak{Q}}\} = \\ &= \{\mathbf{v}^* \in \mathfrak{B}^* : \langle \mathbf{\Phi}^*\mathbf{v}^*, \mathbf{p}-\mathbf{q}\rangle_{\mathfrak{Q}} \geqslant 0, \quad \forall \mathbf{p} \in \tilde{\mathfrak{Q}}\}, \quad u = \mathbf{\Phi}(q)+u_0, \end{aligned} \tag{2.8}$$

where $\mathbf{\Phi}^* : \mathfrak{B}^* \to \mathfrak{Q}^*$ is the adjoint of $\mathbf{\Phi}$, $\mathfrak{Q}^*$ denotes the dual of $\mathfrak{Q}$, $\langle \cdot , \cdot \rangle_{\mathfrak{Q}}$ is the pairing between $\mathfrak{Q}^*$ and $\mathfrak{Q}$. Let $\mathrm{ind}_{\mathfrak{Q}} : \mathfrak{Q} \to \bar{\mathbf{R}}$ be the indicator function of $\tilde{\mathfrak{Q}}$ and $\partial \mathrm{ind}_{\tilde{\mathfrak{Q}}} : \mathfrak{Q} \to 2^{\mathfrak{Q}^*}$ be its subdifferential,[8]. Then (2.8) can be written as

$$\mathbf{\Phi}^*\mathbf{r}^*|_{\mathfrak{B}} \in -\partial \mathrm{ind}_{\tilde{\mathfrak{Q}}}(\mathbf{q}), \quad \mathbf{q} \in \tilde{\mathfrak{Q}}. \tag{2.9}$$

Analogously, assuming that the stress constraints (2.7) are ideal, i.e. (1.8) holds, for the strain field incompatibilities we obtain the similar results to that given by (2.9) namely

$$\mathbf{\Psi}^*\mathbf{G} \in -\partial \mathrm{ind}_{\tilde{\Pi}}(\pi), \quad \pi \in \tilde{\mathbf{\Pi}}, \quad T = \mathbf{\Psi}(\pi), \tag{2.10}$$

where $\mathbf{\Psi}^* : \mathfrak{S} \to \mathbf{\Pi}^*$ is the adjoint to $\mathbf{\Psi}$, $\mathbf{\Pi}^*$ being the dual of $\mathbf{\Pi}$, $\partial \mathrm{ind}_{\tilde{\Pi}} : \mathbf{\Pi} \to 2^{\Pi^*}$ being the subdifferential of the indicator function of $\tilde{\mathbf{\Pi}}$, $\mathrm{ind}_{\tilde{\Pi}} : \mathbf{\Pi} \to \bar{\mathbf{R}}$.
By virtue of (1.10) and (1.11) for the external forces we have

$$\mathbf{\Phi}^*\mathbf{f}^*|_{\mathfrak{B}} \in \mathbf{\Phi}^*\mathbf{f}_0^*|_{\mathfrak{B}} - \partial\bar{\zeta}(\mathbf{q}), \tag{2.11}$$

where $\mathbf{f}^*|_{\mathfrak{B}}$ and $\mathbf{f}_0^*|_{\mathfrak{B}}$ are the restrictions of $\mathbf{f}^*$ and $\mathbf{f}_0^*$ to $\mathfrak{B}$ (treated as elements of $\mathfrak{B}^*$) and $\bar{\zeta} : \mathfrak{Q} \to \bar{\mathbf{R}}$ is the function defined by

$$\bar{\zeta}(\mathbf{p}) \stackrel{\mathrm{df}}{=} \zeta(\mathbf{\Phi}(\mathbf{p})+\mathbf{u}_0), \quad \mathbf{p} \in \mathfrak{Q},$$

$\partial\bar{\zeta} : Q \to 2^{\mathfrak{Q}^*}$ is the subdifferential.

Combining equations of equilibrium (1.4) and constitutive relation (1.7) with the reaction force relation (2.9), the strain incompatibility relation (2.10) and the external force relation (2.11) we arrive at the following system of two variational inequalities

$$\begin{cases} \mathbf{\Phi}^*\mathbf{E}^*\mathbf{\Psi}(\pi) - \mathbf{\Phi}^*\mathbf{f}_0^*|_{\mathfrak{B}} \in -\partial\xi(\mathbf{q}) \\ \mathbf{\Psi}^*\bar{\mathbf{K}}\mathbf{\Psi}(\pi) - \mathbf{\Psi}^*\mathbf{E}\mathbf{\Phi}(q) \in -\partial \mathrm{ind}_{\tilde{\Pi}}(\pi) \end{cases} \tag{2.12}$$

for the basic unknown $(\mathbf{q}, \pi) \in \tilde{\mathfrak{Q}} \times \tilde{\mathbf{\Pi}}$, provided that a function $\xi : \mathfrak{Q} \to \bar{\mathbf{R}}$, defined by $\xi \stackrel{\mathrm{df}}{=} \mathrm{ind}_{\tilde{\mathfrak{Q}}} + \bar{\zeta}$, satisfies the condition $\partial\xi = \partial \mathrm{ind}_{\tilde{\mathfrak{Q}}} + \partial\bar{\zeta}$. In the foregoing system $\bar{\mathbf{K}} : \mathfrak{T} \to \mathfrak{S}$ stands for the operator given by

$$\bar{\mathbf{K}}(\cdot) \stackrel{\mathrm{df}}{=} \mathbf{K}(\cdot) + \mathbf{E}(\mathbf{u}_0). \tag{2.13}$$

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[8] Let α be an arbitrary proper convex function defined on a Banach space X. Following [10] we are the notation

$$\partial\alpha(x) = \{x^* \in X^* : \alpha(y) - \alpha(x)) \geqslant \langle x^*, y-x\rangle_X, \forall y \in X\} \in 2_X{}^*,$$

where X^* is the duál of X, $\langle \cdot , \cdot \rangle_X$ is the pairing between X^* and X, 2^{X^*} stands for the family of all subsets of X^*. The mapping

$$X \in x \to \partial\alpha(x) \in 2_X{}^*$$

is said to be the subdifferential of α.

In particular, if constraints for generalized stresses are absent, i.e. $\tilde{\mathbf{\Pi}} = \mathbf{\Pi}$, then (2.12) reduces to the following system of relations

$$\begin{cases} \mathbf{\Phi}^*\mathbf{E}^*\mathbf{\Psi}(\boldsymbol{\pi}) - \mathbf{\Phi}^*\mathbf{f}_0^*|_{\mathfrak{V}} \in -\partial\xi(\mathbf{q}) \\ \mathbf{\Phi}^*\bar{\mathbf{K}}\mathbf{\Psi}(\boldsymbol{\pi}) - \mathbf{\Psi}^*\mathbf{E}\mathbf{\Phi}(\mathbf{q}) = \mathbf{0}, \end{cases} \tag{2.14}$$

for the basic unknown $(\mathbf{q}, \boldsymbol{\pi}) \in \tilde{\mathbf{\Omega}} \times \mathbf{\Pi}$. If only „dead" loads act at the body then $\xi = \mathrm{ind}_{\tilde{\Omega}}$.

The system (2.12) can be represented in equivalent form as follows

$$\begin{cases} \int\limits_B \mathrm{tr}[\mathbf{\Psi}(\boldsymbol{\pi})(\mathbf{E}\mathbf{\Phi}(\mathbf{p}) - \mathbf{E}\mathbf{\Phi}(\mathbf{q}))]dv - \int\limits_B \mathbf{b}_0 \cdot (\mathbf{\Phi}(\mathbf{p}) - \mathbf{\Phi}(\mathbf{q}))dv \\ - \int\limits_{\partial B} \mathbf{p}_0 \cdot (\mathbf{\Phi}(\mathbf{p}) - \mathbf{\Phi}(\mathbf{q}))ds + \xi(\mathbf{p}) - \xi(\mathbf{q}) \geqslant 0, \quad \forall \mathbf{p} \in \mathbf{\Omega}, \\ \int\limits_B \mathrm{tr}[(\mathbf{\Psi}(\boldsymbol{\sigma}) - \mathbf{\Psi}(\boldsymbol{\pi}))(\bar{\mathbf{K}}\mathbf{\Psi}(\boldsymbol{\pi}) - \mathbf{E}\mathbf{\Phi}(\mathbf{q}))]dv \geqslant 0, \quad \forall \boldsymbol{\sigma} \in \mathbf{\Pi}. \end{cases} \tag{2.12'}$$

Now, let us assume that the realization of the constraints are quasi-ideal, i.e. (1.6)′ and (1.8)′ hold. In this case the governing relations take the form of two following variational inequalities

$$\begin{cases} \mathbf{\Phi}^*\mathbf{E}^*\mathbf{\Psi}(\boldsymbol{\pi}) - \mathbf{\Phi}^*\mathbf{f}_0^*|_{\mathfrak{V}} \in -\partial\alpha(\mathbf{q}) \\ \mathbf{\Psi}^*\bar{\mathbf{K}}\mathbf{\Psi}(\boldsymbol{\pi}) - \mathbf{\Psi}^*\mathbf{E}\mathbf{\Phi}(\mathbf{q}) \in -\partial\bar{\gamma}(\boldsymbol{\pi}), \end{cases} \tag{2.15}$$

in which $(\mathbf{q}, \boldsymbol{\pi}) \in \mathbf{\Omega} \times \mathbf{\Pi}$ is the basic unknow, provided that the function $\boldsymbol{\alpha}: \mathbf{\Omega} \to \bar{\mathbf{R}}$, defined by $\boldsymbol{\alpha} \overset{\mathrm{df}}{=} \bar{\boldsymbol{\beta}} + \bar{\boldsymbol{\zeta}}$, satisfies the condition $\partial\boldsymbol{\alpha} = \partial\bar{\boldsymbol{\beta}} + \partial\bar{\boldsymbol{\zeta}}$, where $\bar{\boldsymbol{\beta}}(\mathbf{p}) \overset{\mathrm{df}}{=} \boldsymbol{\beta}(\mathbf{\Phi}(\mathbf{p}) + \mathbf{u}_0)$, $\forall \mathbf{p} \in \mathbf{\Omega}$ and $\bar{\boldsymbol{\gamma}}(\boldsymbol{\rho}) \overset{\mathrm{df}}{=} \boldsymbol{\gamma}(\mathbf{\Psi}(\boldsymbol{\rho}))$, $\forall \boldsymbol{\rho} \in \mathbf{\Pi}$, and $\partial\boldsymbol{\alpha}$, $\partial\bar{\boldsymbol{\gamma}}$ stand for the subdifferentials of $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{\gamma}}$, respectively.

3. System of variational inequalities

The general form of the governing relations for problems of elastostatics with constraints for displacements and stresses given by (2.4) and (2.7) takes the form of two variational inequalities (2.12) — for the ideal constraints, and (2.15) — for the quasi-ideal constraints.

In this Section we shall consider more general abstract problem which can be stated as follows: find $(u, \sigma) \in V \times Y^*$ such that

$$\begin{cases} L^*\sigma - f \in -\partial\varphi(u) \\ K\sigma - Lu \in -\partial\psi(\sigma), \end{cases} \tag{3.1}$$

where $L: V - Y$ is a linear continuous operator from reflexive Banach space V into a reflexive Banach space Y with domain $D(L) = V$, V^* and Y^* are the duals of V and Y, respectively, $K: Y^* \to Y$ is a maximal monotone operator from Y^* into Y with the domain $D(K) = Y^*$, $\varphi: V \to \bar{R}$ and $\psi: Y^* \to \bar{R}$ are proper lower semicontinuous functions on, V and Y^*, $\partial\varphi: V \to 2^{V^*}$ and $\partial\Psi: Y^* \to 2^Y$ are subdifferentials of φ and ψ, respectively f is a given fixed element of V^*. The norms on V and Y will be denoted by $\|\cdot\|_V$ and $\|\cdot\|_Y$, respectively.

Note, that putting in (3.1) $V = \mathbf{\Omega}$, $Y = \mathbf{\Pi}^*$, $L = \mathbf{\Psi}^*\mathbf{E}\mathbf{\Phi}$, $\varphi = \alpha$, $\psi = \bar{\gamma}$, $f = \mathbf{\Phi}^*\mathbf{f}_0^*|_{\mathfrak{V}}$ we obtain the system (2.15).

Systems of variational inequalities of the form (3.1) have been analyzed in [2] under the assumption that Ker $L = 0$ (Ker L denotes the null space of L). In our considerations this condition in general is not satisfied.

If Ker L is not trivial subspace of V, i.e. Ker $L \not\equiv 0$, we pass to the quotient with Ker L. Thus, we introduce

$$V^{\cdot} = V/_{\mathrm{Ker} L}$$

and define $L^{\cdot}: V^{\cdot} \to Y$ setting

$$L^{\cdot} u^{\cdot} \stackrel{\mathrm{df}}{=} Lu, \quad u \in u^{\cdot}, \quad u^{\cdot} \in V^{\cdot}.$$

As it is known, $V^{\cdot}$ equipped with the norm

$$\|u^{\cdot}\|^{\cdot} \stackrel{\mathrm{df}}{=} \inf_{\varrho \in \mathrm{Ker} L} \|u+\varrho\|_V, \quad u \in u^{\cdot}, \quad u^{\cdot} \in V^{\cdot},$$

is a reflexive Banach space, [9]. Assuming that

$$\langle f, \varrho \rangle_V = 0 \quad \forall \varrho \in \mathrm{Ker} L,$$

$$\varphi(v+\varrho) = \varphi(v) \quad \forall \varrho \in \mathrm{Ker} L, \quad v \in V,$$

we can define $f^{\cdot} \in (V^{\cdot})^*$ and $\varphi^{\cdot}: V^{\cdot} \to \bar{R}$ setting

$$\langle f^{\cdot}, u^{\cdot} \rangle_{\cdot} \stackrel{\mathrm{df}}{=} \langle f, u \rangle_V, \quad u \in u^{\cdot}, \quad u^{\cdot} \in V^{\cdot},$$

$$\varphi^{\cdot}(u^{\cdot}) = \varphi(u), \quad u \in u^{\cdot}, \quad u^{\cdot} \in V^{\cdot},$$

where $(V^{\cdot})^*$ stands for the dual of $V^{\cdot}$. As a results we obtain the following system of variational inequalities

$$\begin{cases} (L^{\cdot})^* \sigma - f^{\cdot} \in -\partial \varphi^{\cdot}(u^{\cdot}) \\ K\sigma - L^{\cdot} u^{\cdot} \in -\partial \psi(\sigma) \end{cases} \tag{3.2}$$

for the basic unknown $(u^{\cdot}, \sigma) \in V^{\cdot} \times Y^*$. Above, $(L^{\cdot})^*: Y^* \to (V^{\cdot})^*$ and $\partial \varphi^{\cdot}: V^{\cdot} \to 2^{(V^{\cdot})^*}$ stand for the adjoint of $L^{\cdot}$ and the subdifferential of $\varphi^{\cdot}$, respectively.

It must be stressed that if $(u^{\cdot}, \sigma) \in V^{\cdot} \times Y^*$ is a solution of (3.2) then any element $(u, \sigma) \in V \times Y^*$, where $u \in u^{\cdot}$, is a solution of (3.1).

The formal structure of (3.2) is the same as that of (3.1) and furthermore $\mathrm{Ker} L^{\cdot} = 0$. Thus for the formulation of existence resulsts we can use the argument given in [2]. To this aid let us define the function $\alpha^{\cdot}: Y^* \to \bar{R}$ putting

$$\alpha^{\cdot}(\eta) \stackrel{\mathrm{df}}{=} (\varphi^{\cdot})^*(-(L^{\cdot})^* \eta + f^{\cdot}), \quad \forall \eta \in Y^*, \tag{3.3}$$

where $(\varphi^{\cdot})^*: (V^{\cdot})^* \to \bar{\mathbf{R}}$ is the conjugate of $\varphi^{\cdot}$.

Theorem 3.1, [2]. Suppose that

(i) $\|L^{\cdot} u^{\cdot}\|_Y \geqslant \bar{c} \|u^{\cdot}\|^{\cdot}, \quad \forall u^{\cdot} \in V^{\cdot}, \quad \bar{c} > 0;$

(ii) $\langle K\eta - K\sigma, \eta - \sigma \rangle_Y \geqslant c \|\eta - \sigma\|^2_{Y^*}, \quad \forall \eta, \sigma \in Y^*, \quad c > 0;$

(iii) $\partial \psi + \partial \alpha^{\cdot}$ is maximal monotone.

Then (3.2) has at least one solution. Moreover, the solution is unique with respect to σ.

As an immediately consequence of the above Theorem we have

Theorem 3.2. Suppose that

(i) $\varphi(v) = \varphi(v+\varrho) \quad \forall\varrho \in \operatorname{Ker} L, \quad v \in V;$

(ii) $\langle f, \varrho\rangle_V = 0 \quad \forall\varrho \in \operatorname{Ker} L;$

(iii) $\|Lv\|_Y \geqslant \bar{c} \inf_{\varrho \in \operatorname{Ker} L} \|v+\varrho\|_V \quad \forall v \in V;$

(iv) $\langle K\eta - K\sigma, \eta-\sigma\rangle_Y \geqslant c\|\eta-\sigma\|^2_{Y^*} \quad \forall\eta, \sigma \in Y^*,$

(v) $\partial\psi + \partial\alpha^\cdot$ is maximal monotone.

Then (3.1) has at least one solution. The solution is unique with respect to σ. Moreover, *if* (u, σ) is a solution of (3.1) then for any $\varrho \in \operatorname{Ker} L$ the pair $(u+\varrho, \sigma)$ is also a solution of (3.1).

In Theorem 3.2 conditions (i) - (iii) assure the maximal monotonicity of $\partial\alpha^\cdot$. In fact, from (iii) we have the surjectivity of $(L^\cdot)^*$, i.e $\operatorname{Im}(L^\cdot)^* = (V^\cdot)^*$, which together with (i) and (ii) guarantees that $\alpha^\cdot$ is a proper lower semicontinuous convex function. As it is known, subdifferentials of such functions are maximal monotone mappings [5].

In particular, if $\psi \equiv 0$ on Y^*, then $\partial\psi \equiv 0$ and the sum $\partial\psi + \partial\alpha^\cdot$ reduces to $\partial\alpha^\cdot$, which is maximal monotone. Thus the condition (v) in Theorem 3.2 is satisfied immediately. Moreover, the second inequality in (3.1) becomes the equality. It implies that if (u_1, σ), (u_2, σ) are any solutions of (3.1), then $u_1 - u_2 \in \operatorname{Ker} L$. As a results we can formulate the following

Theorem 3.3. Suppose that

(i) $\varphi(v) = \varphi(v+\varrho) \quad \forall\varrho \in \operatorname{Ker} L, \quad v \in V;$

(ii) $\langle f, \varrho\rangle_V = 0 \quad \forall\varrho \in \operatorname{Ker} L;$

(iii) $\|Lv\|_Y \geqslant \bar{c} \inf_{\varrho \in \operatorname{Ker} L} \|v+\varrho\|_Y, \quad v \in V, \quad c > 0;$

(iv) $\psi \equiv 0$ on Y^*;

(v) $\langle K\eta - K\sigma, \eta-\sigma\rangle_Y \geqslant c\|\eta-\sigma\|^2_V, \quad \forall\eta, \sigma \in Y^*.$

Then (3.2) has at least one solution. The solution is unique with respect to σ. Moreover, if (u_1, σ) and (u_2, σ) are any solutions of (3.2) then $u_1 - u_2 \in \operatorname{Ker} L$.

From our considerations it follows that the condition (iii) in Theorems (3.2) and (3.3) plays the fundamental role in the existence of solutions to the problem under consideration. As it is known this condition is equivalent to the closedness of the image of L, $\operatorname{Im} L$ in Y, [9].

Let us consider the case in which $L: V \to Y$ can be represented in the following composition $L = AB$, where $A: X \to Y$ and $B: V \to X$ are linear continuous operators, X is a reflexive Banach space. Note, that setting $A = \mathbf{\Psi}^*$ and $B = \mathbf{E\Phi}$ we obtain $L = \mathbf{\Psi}^*\mathbf{E\Phi}$. The below lemma characterizes in terms of A and B conditions under which $\operatorname{Im} L$ is a closed subspace of Y.

Lemma 3.1. Suppose that A is surjective, i.e. $\operatorname{Im} A = Y$, and that there exists an absolute constant $\tilde{c} > 0$ such that $\|Bv\|_X \geqslant \tilde{c}\|v\|_V$, $\forall v \in V$. Then $\operatorname{Im} L$, where $L = AB$, is a closed linear subspace of Y if and only if $\operatorname{Ker} A + \operatorname{Im} B$ is a closed linear subspace of X.

Proof. (*Sufficiency*). Assume that $\operatorname{Ker} A + \operatorname{Im} B$ is closed and that $\{Lv_n\}$, $v_n \in V$, $n = 1, 2, \dots$, converges to y, i.e. $Lv_n \to y$ as $n \to \infty$. It suffices to prove that $y = Lv$ for some $v \in V$. By the surjectivity of A we can find a constant $c_1 > 0$ with

$$\|Ax\|_Y \geqslant c_1 \inf_{z \in \operatorname{Ker} A} \|x+z\|_X, \quad x \in X. \tag{3.3}$$

The sequence $\{ABv_n\}$ satisfies Cauchy condition. From (3.3) it follows that we can find $\{x_n\}$, $x_n \in \operatorname{Ker} A$ such that $\{Bv_n+x_n\}$ is Cauchy sequence, so it converges. By the assumption the limit of this sequence belongs to $\operatorname{Ker} A+\operatorname{Im} B$, i.e. $Bv_n+x_n \to Bv+x$ as $n \to \infty$ for some $v \in V$ and $x \in \operatorname{Ker} A$. Hence $ABv_n \to ABv$ and consequently we obtain $y = ABv = Lv$.

(*Necessity*). Assume that $Bv_n+x_n \to x$ as $n \to \infty$, where $v_n \in V$, $x_n \in \operatorname{Ker} A$, $n = 1, 2, \ldots$. From the continuity of A we have $ABv_n \to Ax$. But the range of $L = AB$ is closed by the assumption and therefore $Ax = ABv$ for some $v \in V$. Hence $x = Bv+x$ for a certain $x \in \operatorname{Ker} A$. This completes the proof of the lemma.

The above Lemma shows that the closedness of $\operatorname{Im} L$ is equivalent to the closedness of the sum $\operatorname{Ker} A+\operatorname{Im} B$ (under suitable assumptions related to A and B). Now we give some usefull result for the sum of closed lineare subspaces of a Hilbert space to be again closed.

Lemma 3.2. [11] Let X be a Hilbert space and let M and N be closed linear subspaces of X. Let us denote by R the intersection of M and N, i.e. $R = M \cap N$. The ortogonal projection of X onto R will be denoted by P, $P: X \to R$. Suppose that there exists $\varepsilon > 0$ such that

$$\|x-Px+y-Py\|_X \geqslant \varepsilon \tag{*}$$

for any $x \in M$ and $y \in N$ such that $\|x-Px\|_X = 1$ and $\|y-Py\|_X = 1$. Then the sum $M+N$ is a closed subspace of X.

Proof. Let $\{x_n+y_n\}$ be a sequence in $M+N$ ($x_n \in M$, $y_n \in N$, $n+1, 2, \ldots$) converging to x and such that $\|x_n-Px_n\|_X = 1$, $\|y_n-Py_n\|_X = 1$, $n = 1, 2, \ldots$. We have to prove that $x \in M+N$. First, we will show that the sequence $\{x_n-Px_n\}$ satisfies Cauchy condition. Suppose that it is not true. Then there exists $\delta > 0$ and a sequence $\{k_n\}$ of natural numbers that

$$\|x_{n+k_n}-Px_{n+k_n}-x_n+Px_n\|_X \geqslant \delta \tag{3.4}$$

for any natural number n. From the continuity and linearity of P we have

$$P(x_n+y_n) = Px_n+Py_n \to Px, \quad \text{as } n \to \infty.$$

Since the sequence $\{x_n+y_n-P(x_n+y_n)\}$ converges, it has to satisfy Cauchy condition. Hence

$$\|x_{n+k_n}-Px_{n+k_n}+y_{n+k_n}-Py_{n+k_n}-x_n+Px_n-y_n+Py_n\|_X \to 0, \quad n \to \infty. \tag{3.5}$$

For the simplicity of denotations let us put

$$\alpha_m = x_m-Px_m \quad \text{and} \quad \beta_m = y_m-Py_m, \quad m = 1, 2, \ldots$$

Under the above denotations (3.4) and (3.5) take the form

$$\|\alpha_{n+k_n}-\alpha_n\|_X \geqslant \delta, \quad n = 1, 2, \ldots \tag{3.6}$$

and

$$\|\alpha_{n+k_n}-\alpha_n+\beta_{n+k_n}-\beta_n\|_X \to 0, \quad \text{as } n \to \infty. \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$\frac{\alpha_{n+k_n}-\alpha_n}{\|\alpha_{n+k_n}-\alpha_n\|_X} + \frac{\beta_{n+k_n}-\beta_n}{\|\alpha_{n+k_n}-\alpha_n\|_X} \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

Since

$$\left\| \frac{\alpha_{n+k_n} - \alpha_n}{\|\alpha_{n+k_n} - \alpha_n\|_X} \right\|_X = 1$$

for any n, the condition (3.8) yields

$$\left\| \frac{\beta_{n+k_n} - \beta_n}{\|\alpha_{n+k_n} - \alpha_n\|_X} \right\|_X \to 1, \quad \text{as } n \to \infty. \tag{3.9}$$

At the same time we have

$$\left\| \frac{\alpha_{n+k_n} - \alpha_n}{\|\alpha_{n+k_n} - \alpha_n\|_X} + \frac{\beta_{n+k_n} - \beta_n}{\|\beta_{n+k_n} - \beta_n\|_X} \right\|_X \leqslant \left\| \frac{\alpha_{n+k_n} - \alpha_n}{\|\alpha_{n+k_n} - \alpha_n\|_X} + \frac{\beta_{n+k_n} - \beta_n}{\|\alpha_{n+k_n} - \alpha_n\|_X} \right\|_X + \\ + \left| 1 - \left\| \frac{\beta_{n+k_n} - \beta_n}{\|\alpha_{n+k_n} - \alpha_n\|_X} \right\|_X \right|$$

This condition together with (3.8) and (3.9) gives

$$\left\| \frac{\alpha_{n+k_n} - \alpha_n}{\|\alpha_{n+k_n} - \alpha_n\|_X} + \frac{\beta_{n+k_n} - \beta_n}{\|\beta_{n+k_n} - \beta_n\|_X} \right\|_X \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

On nothing that

$$\frac{\alpha_{n+k_n} - \alpha_n}{\|\alpha_{n+k_n} - \alpha_n\|_X} = \frac{x_{n+k_n} - x_n}{\|x_{n+k_n} - x_n - Px_{n+k_n} + Px_n\|_X} - P\left(\frac{x_{n+k_n} - x_n}{\|x_{n+k_n} - x_n - Px_{n+k_n} - Px_n\|_X} \right)$$

$$\frac{x_{n+k_n} - x_n}{\|x_{n+k_n} - x_n - Px_{n+k_n} + Px_n\|_X} \in M, \quad \left\| \frac{\alpha_{n+k_n} - \alpha_n}{\|\alpha_{n+k_n} - \alpha_n\|_X} \right\|_X = 1, \quad n = 1, 2, \ldots$$

$\left(\text{similarly for } \frac{\beta_{n+k_n} - \beta_n}{\|\beta_{n+k_n} - \beta_n\|_X}\right)$ and taking into account (3.10) we arrive at the contradiction with the assumption (*). Thus sequences $\{x_n - Px_n\}$ in M and $\{y_n - Py_n\}$ in N have to satisfy Cauchy condition. From the closedness of M and N it follows that there exists $\bar{x} \in M$ and $\bar{y} \in N$ that $x_n - Px_n \to \bar{x}$ and $y_n - Py_n \to y$. Hence $x_n + y_n = x_n + y_n - Px_n - Py_n + P(x_n + y_n) \to \bar{x} + \bar{y} + Px$ and finally we obtain $x = \bar{x} + \bar{y} + Px \in M + N$. This ends the proof of the lemma.

4. Existence and uniqueness results

Now, on the base of the results given in Section 3 we shall formulate some sufficient conditions for the existence and uniqueness of solutions to problems governed by systems (2.12) and (2.14).

Let us begin with problems in which on generalized displacements and generalized stresses are imposed some restrictions. In this case the corresponding system of variational inequalities takes the form (2.12). Putting

$$\mu(\pi) \stackrel{\text{df}}{=} \xi^*(-\mathbf{\Phi}^*\mathbf{E}^*\mathbf{\Psi}(\pi) + \mathbf{\Phi}^*\mathfrak{f}_0^*|_{\mathfrak{V}}), \quad \pi \in \mathbf{\Pi},$$

ξ^* being the conjugate of ξ, denoting by $\mathfrak{N}$ the intersection of Ker $\mathbf{\Psi}^*$ and Im $\mathbf{E\Phi}$, i.e. $\mathfrak{N} = \operatorname{Ker}\mathbf{\Psi}^* \cap \operatorname{Im}\mathbf{E\Phi}$ and by $P_{\mathfrak{N}}$ the ortogonal projection on $\mathfrak{N}$, $P_{\mathfrak{N}}: \mathfrak{S} \to \mathfrak{S}$, we arrive at the following

Theorem 4.1. Suppose that (2.1), (2.3) and (2.6) hold. Moreover, let the following conditions be satisfied:

(i) $\mathbf{K}$ is maximal monotone operator with the domain $D(\mathbf{K}) = \mathfrak{T}$ such that there exists positive constant c, $c > 0$, with

$$\langle \mathbf{K}\boldsymbol{\sigma} - \mathbf{K}\eta, \boldsymbol{\sigma} - \boldsymbol{\eta}\rangle_2 \geqslant C\|\boldsymbol{\eta} - \boldsymbol{\sigma}\|_2^2, \qquad \forall \boldsymbol{\sigma}, \boldsymbol{\eta} \in \mathfrak{T};$$

(ii) There exists positive constant ε, $\varepsilon > 0$, such that for any $\boldsymbol{\sigma} \in \operatorname{Ker} \boldsymbol{\Psi}^*$, $\|\boldsymbol{\sigma} - P_{\mathfrak{N}}\boldsymbol{\sigma}\|_2 = 1$ and any $\boldsymbol{\eta} \in \operatorname{Im}\mathbf{E}\boldsymbol{\Phi}$, $\|\boldsymbol{\eta} - P_{\mathfrak{N}}\boldsymbol{\eta}\|_2 = 1$ the following inequality holds

$$\|\boldsymbol{\sigma} - P_{\mathfrak{N}}\boldsymbol{\sigma} + \boldsymbol{\eta} - P_{\mathfrak{N}}\eta\|_2 \geqslant \varepsilon;$$

(iii) $\langle \mathbf{f}_0^*|_{\mathfrak{B}}, \boldsymbol{\Phi}(\mathbf{p})\rangle_1 = 0, \quad \forall \mathbf{p} \in \operatorname{Ker}\boldsymbol{\Psi}^*\mathbf{E}\boldsymbol{\Phi};$

(iv) $\xi(\mathbf{q}) = \xi(\mathbf{q}+\mathbf{p}) \quad \forall \mathbf{p} \in \operatorname{Ker}\boldsymbol{\Psi}^*\mathbf{E}\boldsymbol{\Phi}, \quad \mathbf{q} \in \mathfrak{Q};$

(v) $\partial\boldsymbol{\mu} + \partial \operatorname{ind}_{\tilde{\Pi}}$ is maximal monotone.

Then (2.12) has at least one solution $(\mathbf{q}, \boldsymbol{\pi}) \in \tilde{\mathfrak{Q}} \times \tilde{\boldsymbol{\Pi}}$. The solution is unique with respect to $\boldsymbol{\pi}$ and for any $\mathbf{p} \in \operatorname{Ker} \boldsymbol{\Psi}^*\mathbf{E}\boldsymbol{\Phi}$ the pair $(\mathbf{q}+\mathbf{p}, \boldsymbol{\pi})$ is also a solution.

Proof of the above theorem follows immediately from Theorem 3.2 and Lemmas 3.6 and 3.7.

An analogous result can be formulated for (2.15) replacing only in the hypotheses (iv) and (v) of Theorem 4.1 the functions ξ and $\operatorname{ind}_{\tilde{\Pi}}$ by α and $\bar{\gamma}$, respectively.

If there are no restrictions on generalized stresses then $\tilde{\boldsymbol{\Pi}} = \boldsymbol{\Pi}$ and the system (2.12) reduces to (2.14). In this case the assumptions (iii) and (iv) imply immediately (v) and therefore we obtain

Remark 4.2. Suppose that (2.1), (2.3) and (2.6) hold. Moreover, let the hypotheses (i), (ii), (iii), (iv) of Theorem 4.1 be satisfied. Then problem (2.14) has at least one solution $(\mathbf{q}, \boldsymbol{\pi}) \in \tilde{\mathfrak{Q}} \times \boldsymbol{\Pi}$. The solution is unique with respect to $\boldsymbol{\pi}$ and for any $\mathbf{p} \in \operatorname{Ker} \boldsymbol{\Psi}^*\mathbf{E}\boldsymbol{\Phi}$ the pair $(\mathbf{q}+\mathbf{p}, \boldsymbol{\pi})$ is also a solution.

Remark 4.3. Let us suppose that $\tilde{\boldsymbol{\Pi}} = \boldsymbol{\Pi}$ and $\tilde{\mathfrak{Q}} = \mathfrak{Q}$ (restrictions on generalized displacements and generalized stresses are absent) and that $\xi \equiv 0$ (the body is subjected to the „dead" load only). Then (2.12) reduces to the system of two equations. In this case if $(\mathbf{q}_1, \boldsymbol{\pi})$ and $(\mathbf{q}_2, \boldsymbol{\pi})$ are any two solutions of (2.12) then $\mathbf{q}_1 - \mathbf{q}_2 \in \operatorname{Ker} \boldsymbol{\Psi}^*\mathbf{E}\boldsymbol{\Phi}$, i.e. we have obtained the uniqueness of generalized displacements with respect to elements of Ker $\boldsymbol{\Psi}^*\mathbf{E}\boldsymbol{\Phi}$.

Remark 4.4. From our considerations it follows that in general the displacement field in problems with stress constraints of the form (2.7) is not uniquely determined. The uniqueness depends on the kernel of the operator $\boldsymbol{\Psi}^*\mathbf{E}\boldsymbol{\Phi}$. If $\operatorname{Ker} \boldsymbol{\Psi}^*\mathbf{E}\boldsymbol{\Phi} = \mathbf{0}$ then the solution is unique, but if $\operatorname{Ker} \boldsymbol{\Psi}^*\mathbf{E}\boldsymbol{\Phi}$ is not trivial subspace of $\mathfrak{Q}$ then the strain field can be determined up to strain fields belonging to the set $\{\mathbf{H} = \mathbf{E}\boldsymbol{\Phi}(\mathbf{p})$ for some $\mathbf{p}$ such that $\int_B \operatorname{tr}[\mathbf{E}\boldsymbol{\Phi}(\mathbf{p})\, \boldsymbol{\Psi}(\boldsymbol{\pi})]dv = 0, \forall \boldsymbol{\pi} \in \boldsymbol{\Pi}\}$.

Some applications of obtained results to problems of plates, shells, membranes and discretization can be found in [3, 4].

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Резюме

НЕКОТОРЫЕ РЕЗУЛЬТАТЫ КОСАЮЩИЕСЯ СУЩЕСТВОВАНИЯ И ОДНОЗНАЧНОСТИ В ЭЛАСТОТАТИКЕ СО СВЯЗЯМИ В ПЕРЕМЕЩЕНИЯХ И НАПРАЖЕНИЯХ

В работе рассмотрен некоторый класс проблем теории эластотатики со связями в перемещениях и напряжениях. Рассмотривается связи приводящие к проблемом с обобщёнными перемещениямх и обобщёнными напряжениями. Доказано теоремы существования и однозначности для такии проблем.

Streszczenie

ISTNIENIE I JEDNOZNACZNOŚĆ ROZWIĄZAŃ W ELASTOSTATYCE Z WIĘZAMI DLA PRZEMIESZCZEŃ I NAPRĘŻEŃ

W pracy rozpatrzono pewną klasę zagadnień elastostatyki z więzami dla przemieszczeń i naprężeń. Zbadano takie ograniczenia, które prowadzą do problemów z uogólnionymi przemieszczeniami i uogólnionymi naprężeniami. Udowodniono twierdzenia o istnieniu i jednoznaczności dla tych zagadnień.

Praca została złożona w Redakcji dnia 21 grudnia 1984 roku