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FINITE ELEMENT ANALYSIS OF BUCKLING OF STRUCTURES AT SPECIAL PREBUCKLING STATES

Herbert A. Mang, Xin Jia

Vienna University of Technology, Institute for Mechanics of Materials and Structures, Vienna, Austria

e-mail: herbert.mang@tuwien.ac.at

Gerhard Höfinger

BPM Bauprozess Management GmbH, STRABAG AG, Austria

The consistently linearized eigenproblem is used to derive mathematical conditions in the frame of the Finite Element Method (FEM) for loss of static stability of elastic structures at prebuckling states characterized by a constant percentage bending energy of the strain energy in the prebuckling regime. Special cases of such prebuckling states are membrane stress states and pure bending. Buckling from a membrane stress state as a special case within sensitivity analysis of buckling at a constant non-zero percentage bending energy in the prebuckling regime is one of two examples serving the purpose to verify the existence of hitherto unknown subsidiary conditions of buckling in the context of the FEM.

Key words: special prebuckling stress states, consistently linearized eigenproblem, finite element method

1. Introduction

Special prebuckling states are defined as **states with a constant percentage bending energy of the strain energy in the prebuckling regime**. Special cases of the constant percentage bending energy are:

- zero percentage bending energy (membrane stress state) and
- zero percentage membrane energy (pure bending).

The consistently linearized eigenproblem (Helnwein, 1997) will be used to derive mathematical conditions for loss of static stability of elastic structures at special prebuckling states in the frame of the Finite Element Method (FEM). The difference between buckling from a membrane stress state in the frame of sensitivity analysis restricted to such stress states and buckling from a membrane stress state, representing a special case of loss of stability at states with a constant percentage bending energy of the strain energy in the prebuckling regime, will be brought out theoretically and verified numerically.

The paper is organized as follows: In Section 2, the consistently linearized eigenproblem will be used for derivation of mathematical relations for general prebuckling states. These relations will then be specialized for the stability limit. In Section 3, the relations derived in Section 2 will be specialized for the initially mentioned special prebuckling states. In Section 4, results from a numerical investigation will be presented. Section 5 contains the conclusions drawn from this work.

2. General prebuckling states

The consistently linearized eigenproblem for FE analysis of a conservative system with N degrees of freedom is defined as (Helnwein, 1997)

$$[\widetilde{\mathbf{K}}_T + (\lambda^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda}]\mathbf{v}^* = \mathbf{0} \tag{2.1}$$

where

$$\widetilde{\mathbf{K}}_T(\lambda) := \widetilde{\mathbf{K}}_T(\mathbf{q}(\lambda)) \tag{2.2}$$

is the tangent stiffness matrix and

$$\widetilde{\mathbf{K}}_{T,\lambda}(\lambda) := \widetilde{\mathbf{K}}_{T,\lambda}(\mathbf{q}(\lambda), \lambda) \tag{2.3}$$

indicates differentiation of $\widetilde{\mathbf{K}}_T$ with respect to the load multiplier λ along a direction parallel to the primary path $\mathbf{q}(\lambda)$ (Schranz *et al.*, 2006). In (2.1), $\lambda^* - \lambda$ is the eigenvalue corresponding to the eigenvector $\mathbf{v}^*$. λ^* and $\mathbf{v}^*$ are functions of λ. Equation (2.1) represents a set of N implicit equations defining N curves in the $(\lambda^* - \lambda)$-space. Thus, it has got N solutions $(\lambda_j^*, \mathbf{v}_j^*)$, $j \in \{1, 2, \dots, N\}$.

Writing (2.1) for the first eigenpair gives

$$[\widetilde{\mathbf{K}}_T + (\lambda_1^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda}]\mathbf{v}_1^* = \mathbf{0} \tag{2.4}$$

Hence, the following orthogonality relations must hold

$$\mathbf{v}_k^*\widetilde{\mathbf{K}}_T\mathbf{v}_1^* = 0 \qquad \mathbf{v}_k^*\widetilde{\mathbf{K}}_{T,\lambda}\mathbf{v}_1^* = 0 \qquad k \in \{2, 3, \dots, N\} \tag{2.5}$$

Derivation of (2.4) with respect to λ gives

$$[\lambda_{1,\lambda}^*\widetilde{\mathbf{K}}_{T,\lambda} + (\lambda_1^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda\lambda}]\mathbf{v}_1^* + [\widetilde{\mathbf{K}}_T + (\lambda_1^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda}]\mathbf{v}_{1,\lambda}^* = \mathbf{0} \tag{2.6}$$

Premultiplication of (2.6) by $\mathbf{v}_1^*$ and consideration of (2.4) yields

$$\lambda_{1,\lambda}^* = -(\lambda_1^* - \lambda)\frac{\mathbf{v}_1^*\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{v}_1^*}{\mathbf{v}_1^*\widetilde{\mathbf{K}}_{T,\lambda}\mathbf{v}_1^*} \tag{2.7}$$

Normalization of the eigenvector such that

$$|\mathbf{v}_1^*| = 1 \tag{2.8}$$

results in

$$\mathbf{v}_1^*\mathbf{v}_{1,\lambda}^* = 0 \tag{2.9}$$

Since the eigenvectors $\mathbf{v}_j^*$, $j \in \{1, 2, \dots, N\}$, are a basis of $\mathbb{R}^N$, $\mathbf{v}_{1,\lambda}^*$ can be expressed as

$$\mathbf{v}_{1,\lambda}^* = \sum_{j=1}^{N} c_{1j}\mathbf{v}_j^* \tag{2.10}$$

Substitution of (2.10) into (2.9) and consideration of

$$\mathbf{v}_k^*\mathbf{v}_1^* = 0 \qquad k \in \{2, 3, \dots, N\} \tag{2.11}$$

gives

$$c_{11} = 0 \tag{2.12}$$

Premultiplication of (2.6) by $\mathbf{v}_k^*$ and consideration of (2.5), (2.10), and

$$[\widetilde{\mathbf{K}}_T + (\lambda_k^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda}]\mathbf{v}_k^* = \mathbf{0} \tag{2.13}$$

yields

$$c_{1k} = -\frac{\lambda_1^* - \lambda}{\lambda_1^* - \lambda_k^*} \frac{\mathbf{v}_k^* \widetilde{\mathbf{K}}_{T,\lambda\lambda} \mathbf{v}_1^*}{\mathbf{v}_k^* \widetilde{\mathbf{K}}_{T,\lambda} \mathbf{v}_k^*} \qquad k \in \{2, 3, \dots, N\} \tag{2.14}$$

Introducing the abbreviation

$$\mathbf{A} = \widetilde{\mathbf{K}}_T + (\lambda_1^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda} \tag{2.15}$$

into (2.4), gives

$$\mathbf{A}\mathbf{v}_1^* = \mathbf{0} \tag{2.16}$$

The first, second, and third derivative of (2.16) with respect to λ are obtained as

$$\begin{aligned} &\mathbf{A}_{,\lambda}\mathbf{v}_1^* + \mathbf{A}\mathbf{v}_{1,\lambda}^* = \mathbf{0} \\ &\mathbf{A}_{,\lambda\lambda}\mathbf{v}_1^* + 2\mathbf{A}_{,\lambda}\mathbf{v}_{1,\lambda}^* + \mathbf{A}\mathbf{v}_{1,\lambda\lambda}^* = \mathbf{0} \\ &\mathbf{A}_{,\lambda\lambda\lambda}\mathbf{v}_1^* + 3\mathbf{A}_{,\lambda\lambda}\mathbf{v}_{1,\lambda}^* + 3\mathbf{A}_{,\lambda}\mathbf{v}_{1,\lambda\lambda}^* + \mathbf{A}\mathbf{v}_{1,\lambda\lambda\lambda}^* = \mathbf{0} \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} \mathbf{A}_{,\lambda} &= \lambda_{1,\lambda}^* \widetilde{\mathbf{K}}_{T,\lambda} + (\lambda_1^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda\lambda} \\ \mathbf{A}_{,\lambda\lambda} &= \lambda_{1,\lambda\lambda}^* \widetilde{\mathbf{K}}_{T,\lambda} + (2\lambda_{1,\lambda}^* - 1)\widetilde{\mathbf{K}}_{T,\lambda\lambda} + (\lambda_1^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda} \\ \mathbf{A}_{,\lambda\lambda\lambda} &= \lambda_{1,\lambda\lambda\lambda}^* \widetilde{\mathbf{K}}_{T,\lambda} + 3\lambda_{1,\lambda\lambda}^* \widetilde{\mathbf{K}}_{T,\lambda\lambda} + (3\lambda_{1,\lambda}^* - 2)\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda} + (\lambda_1^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda\lambda} \end{aligned} \tag{2.18}$$

The focus of the present work is on the influence of special prebuckling states on loss of stability. Apart from the fact that buckling in the form of snap-through is impossible for some of these states, it is irrelevant to this work whether loss of stability occurs in the form of bifurcation buckling or snap-through. For the latter mode of buckling

$$d\lambda(\lambda_S) = 0 \tag{2.19}$$

where $\lambda = \lambda_S$ refers to the stability limit. Hence, λ would not be a good choice for parameterizing the equilibrium path in the vicinity of the snap-through point. A detailed account of treating snap-through by means of the consistently linearized eigenproblem is given in Steinboeck *et al.* (2008). For the aforementioned reasons and for the sake of simplicity, it is assumed that loss of stability occurs in the form of bifurcation buckling.

At $\lambda = \lambda_S$

$$\widetilde{\mathbf{K}}_T \mathbf{v}_1 = \mathbf{0} \tag{2.20}$$

Hence, following from (2.13)

$$\lambda_1^*(\lambda_S) = \lambda_S \qquad \mathbf{v}_1^*(\lambda_S) = \mathbf{v}_1 \tag{2.21}$$

Substitution of (2.21) into (2.7) and (2.14) gives

$$\lambda^*_{1,\lambda}(\lambda_S) = 0 \tag{2.22}$$

and

$$c_{1k}(\lambda) = 0 \qquad k \in \{2, 3, \ldots, N\} \tag{2.23}$$

respectively. Inserting (2.12) and (2.23) into (2.10) yields

$$\mathbf{v}^*_{1,\lambda}(\lambda_S) = \mathbf{0} \tag{2.24}$$

indicating a singular point on the vector curve $\mathbf{v}^*_1(\lambda)$. Specialization of (2.15) and (2.18) for the stability limit results in

$$\begin{aligned} &\mathbf{A} = \widetilde{\mathbf{K}}_T \\ &\mathbf{A}_{,\lambda} = \mathbf{0} \\ &\mathbf{A}_{,\lambda\lambda} = \lambda^*_{1,\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda} - \widetilde{\mathbf{K}}_{T,\lambda\lambda} \\ &\mathbf{A}_{,\lambda\lambda\lambda} = \lambda^*_{1,\lambda\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda} + 3\lambda^*_{1,\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda\lambda} - 2\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda} \end{aligned} \tag{2.25}$$

Because of (2.24) and $(2.25)_2$, $(2.17)_1$ is trivially satisfied for $\lambda = \lambda_S$. Specialization of $(2.17)_{2,3}$ for $\lambda = \lambda_S$ gives

$$[\lambda^*_{1,\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda} - \widetilde{\mathbf{K}}_{T,\lambda\lambda}]\mathbf{v}_1 + \widetilde{\mathbf{K}}_T\mathbf{v}^*_{1,\lambda\lambda} = \mathbf{0} \tag{2.26}$$

and

$$[\lambda^*_{1,\lambda\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda} + 3\lambda^*_{1,\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda\lambda} - 2\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}]\mathbf{v}_1 + \widetilde{\mathbf{K}}_T\mathbf{v}^*_{1,\lambda\lambda\lambda} = \mathbf{0} \tag{2.27}$$

respectively. Elimination of $\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{v}_1$ in (2.27) with the help of (2.26), followed by premultiplication of the result by $\mathbf{v}_1$ and consideration of (2.20), yields

$$\lambda^*_{1,\lambda\lambda\lambda} = -3\lambda^{*2}_{1,\lambda\lambda} + 2\frac{\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1}{\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda}\mathbf{v}_1} \tag{2.28}$$

For buckling at general prebuckling states, for mechanical reasons beyond the scope of this work,

$$\lambda^*_{1,\lambda\lambda}(\lambda_S) < 0 \qquad \lambda^*_{1,\lambda\lambda\lambda}(\lambda_S) < 0 \qquad \frac{\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1}{\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda}\mathbf{v}_1} > 0 \tag{2.29}$$

Because of

$$\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda}\mathbf{v}_1 < 0 \tag{2.30}$$

Mang and Höfinger (2012)

$$\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1 < 0 \tag{2.31}$$

3. Special prebuckling states

As mentioned at the beginning, special prebuckling states are defined as states with a constant percentage bending energy of the strain energy in the prebuckling regime. For such prebuckling states, $(2.17)_3$ disintegrates into (Mang, 2011)

$$\mathbf{A}_{,\lambda\lambda}\mathbf{v}^*_{1,\lambda} = \mathbf{0} \qquad \wedge \qquad \mathbf{A}_{,\lambda\lambda\lambda}\mathbf{v}^*_1 + 3\mathbf{A}_{,\lambda}\mathbf{v}^*_{1,\lambda\lambda} + \mathbf{A}\mathbf{v}^*_{1,\lambda\lambda\lambda} = \mathbf{0} \tag{3.1}$$

At the stability limit, $(3.1)_1$ is trivially satisfied and $(3.1)_2$ degenerates to (2.27), as is the case with $(2.17)_3$ for buckling at general prebuckling states. However, instead of (2.29), for buckling at special prebuckling states

$$\lambda^*_{1,\lambda\lambda\lambda} \geqslant 0 \qquad \Longrightarrow \qquad \frac{\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1}{\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda}\mathbf{v}_1} \geqslant 0 \tag{3.2}$$

In contrast to (2.31), the numerator in $(3.2)_2$ may become zero. For a constant **non-zero** percentage buckling energy of the strain energy in the prebuckling regime

$$\lambda^*_{1,\lambda\lambda}(\lambda_S) < 0 \qquad \lambda^*_{1,\lambda\lambda\lambda}(\lambda_S) = 0 \qquad \frac{\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1}{\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda}\mathbf{v}_1} = \frac{3}{2}\lambda^{*2}_{1,\lambda\lambda}(\lambda_S) \tag{3.3}$$

3.1. Membrane stress state

A membrane stress state represents a special case of a state with a constant percentage bending energy of the strain energy, namely, one with zero percentage bending energy. For such a case, $(2.17)_2$ disintegrates into (Mang, 2011)

$$\mathbf{A}_{,\lambda\lambda}\mathbf{v}^*_1 = 0 \qquad \wedge \qquad 2\mathbf{A}_{,\lambda}\mathbf{v}^*_{1,\lambda} + \mathbf{A}\mathbf{v}^*_{1,\lambda\lambda} = \mathbf{0} \tag{3.4}$$

Derivation of $(3.4)_1$ with respect to λ gives

$$\mathbf{A}_{,\lambda\lambda\lambda}\mathbf{v}^*_1 + \mathbf{A}_{,\lambda\lambda}\mathbf{v}^*_{1,\lambda} = \mathbf{0} \tag{3.5}$$

Substitution of $(3.1)_1$ into (3.5) yields

$$\mathbf{A}_{,\lambda\lambda\lambda}\mathbf{v}^*_1 = \mathbf{0} \tag{3.6}$$

Substitution of (3.6) into $(3.1)_2$ results in

$$3\mathbf{A}_{,\lambda}\mathbf{v}^*_{1,\lambda\lambda} + \mathbf{A}\mathbf{v}^*_{1,\lambda\lambda\lambda} = \mathbf{0} \tag{3.7}$$

At the stability limit, taking (2.24) and $(2.25)_2$, into account

$$\begin{aligned} &\mathbf{A}_{,\lambda\lambda}\mathbf{v}_1 = \mathbf{0} \qquad && \mathbf{A}_{,\lambda\lambda\lambda}\mathbf{v}_1 = \mathbf{0} \\ &\mathbf{A}\mathbf{v}^*_{1,\lambda\lambda} = \mathbf{0} && \mathbf{A}\mathbf{v}^*_{1,\lambda\lambda\lambda} = \mathbf{0} \end{aligned} \tag{3.8}$$

Making use of $(2.25)_{3,4}$, and $(2.25)_1$, gives

$$\begin{aligned} &[\lambda^*_{1,\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda} - \widetilde{\mathbf{K}}_{T,\lambda\lambda}]\mathbf{v}_1 = \mathbf{0} \\ &[\lambda^*_{1,\lambda\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda} + 3\lambda^*_{1,\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda\lambda} - 2\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}]\mathbf{v}_1 = \mathbf{0} \end{aligned} \tag{3.9}$$

and

$$\mathbf{v}^*_{1,\lambda\lambda} = \mathbf{0} \qquad \mathbf{v}^*_{1,\lambda\lambda\lambda} = \mathbf{0} \tag{3.10}$$

respectively, noting that $\mathbf{v}^*_{1,\lambda\lambda}$ and $\mathbf{v}^*_{1,\lambda\lambda\lambda}$ are not eigenvectors of $\mathbf{A}(\lambda_S) = \widetilde{\mathbf{K}}_T$.

Premultiplication of $(3.9)_1$ by $\mathbf{v}^*_k$ and consideration of $(2.5)_2$ yields

$$\mathbf{v}^*_k \widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{v}_1 = 0 \qquad k \in \{2,3,\ldots,N\} \tag{3.11}$$

Premultiplication of $(3.9)_2$ by $\mathbf{v}^*_k$ and consideration of $(2.5)_2$ and (3.11) results in

$$\mathbf{v}^*_k \widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1 = 0 \qquad k \in \{2,3,\ldots,N\} \tag{3.12}$$

In contrast to (2.5), the orthogonality relations (3.11) and (3.12) are restricted to the stability limit.

Buckling from a membrane stress state obeys (3.2) and (3.9)-(3.12). Specialization of $(3.9)_2$ for $\lambda^*_{1,\lambda\lambda\lambda} = 0$, which is a special case of $(3.2)_1$, gives

$$[3\lambda^*_{1,\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda\lambda} - 2\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}]\mathbf{v}_1 = \mathbf{0} \tag{3.13}$$

Elimination of $\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{v}_1$ in (3.13) with the help of $(3.9)_1$ yields

$$[3\lambda^{*2}_{1,\lambda\lambda}\widetilde{\mathbf{K}}_{T,\lambda} - 2\widetilde{\mathbf{K}}_{T,\lambda\lambda}]\mathbf{v}_1 = \mathbf{0} \tag{3.14}$$

An eigenvector of a square matrix cannot correspond to two distinct eigenvalues (Wylie, 1975). Hence, the eigenvalue of (3.14) is obtained as

$$\lambda^*_{1,\lambda\lambda} = 0 \tag{3.15}$$

Substitution of (3.15) into $(3.9)_1$ and (3.13) results in the following remarkable subsidiary buckling conditions (Höfinger, 2010)

$$\boxed{\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{v}_1 = \mathbf{0} \qquad \wedge \qquad \widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1 = \mathbf{0}} \tag{3.16}$$

The mechanical meaning of this special case is buckling from a membrane stress state as a special case in the frame of sensitivity analysis of buckling at a constant non-zero percentage bending energy of the strain energy in the prebuckling regime. Satisfaction of $(3.3)_2$ by (3.15) and $(3.16)_2$ proves this interpretation.

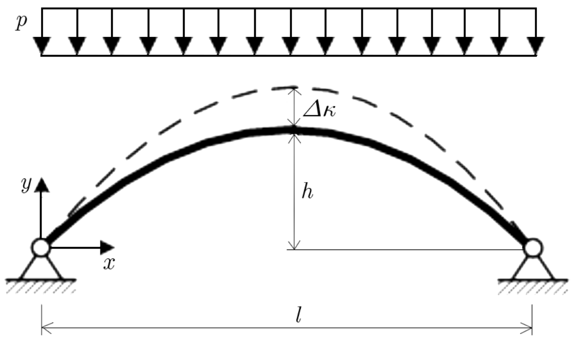

Fig. 1. Two-hinged arches (solid line: thrust line arch, dashed line: modified configuration)

An example for such a sensitivity analysis, in the frame of the FEM, is a parameterized family of two-hinged arches, subjected to a uniformly distributed load p (Mang and Höfinger, 2012). The design parameter $\Delta\kappa$ refers to the deviation of the geometric form of the axis of the arch from a quadratic parabola for which $\Delta\kappa = 0$, representing a thrust-line arch (Fig. 1). Hence, for $\Delta\kappa = 0$, buckling occurs from a membrane stress state. Numerical results from sensitivity analysis of the mentioned family of arches will be presented in Section 4.

The general case of $(3.2)_1$ is characterized by $\lambda^*_{1,\lambda\lambda\lambda} > 0$. It refers to sensitivity analysis restricted to buckling from membrane stress states.

An example for such a sensitivity analysis is a *von Mises* truss with an elastic spring attached to the load point (Fig. 2). The stiffness of the spring is given as κk where k is a constant and κ is the variable design parameter. $\overline{P}$ is the reference load. Numerical results from sensitivity analysis of the *von Mises* truss will be presented in Section 4.

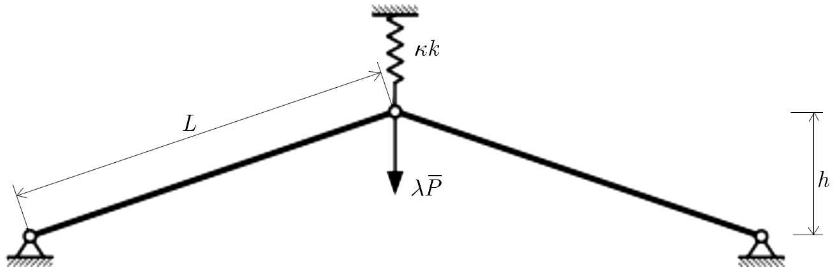

Fig. 2. *Von Mises* truss with an elastic vertical spring attached to the load point)

In Mang and Höfinger (2012) it is shown that

$$\boxed{\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{q}_{,\lambda\lambda} = 0} \tag{3.17}$$

is a necessary and sufficient condition for buckling from a membrane stress state. The general case

$$\widetilde{\mathbf{K}}_{T,\lambda\lambda} \neq \mathbf{0} \qquad \mathbf{q}_{,\lambda\lambda} \neq \mathbf{0} \tag{3.18}$$

represents a nonlinear stability problem with nonlinear prebuckling paths. The two special cases

$$\begin{array}{ll} \widetilde{\mathbf{K}}_{T,\lambda\lambda} \neq \mathbf{0} & \mathbf{q}_{,\lambda\lambda} = \mathbf{0} \\ \widetilde{\mathbf{K}}_{T,\lambda\lambda} = \mathbf{0} & \mathbf{q}_{,\lambda\lambda} \neq \mathbf{0} \end{array} \tag{3.19}$$

show that linear stability problems and linear prebuckling paths need not be mutually conditional. The third special case is obtained as

$$\widetilde{\mathbf{K}}_{T,\lambda\lambda} = \mathbf{0} \qquad \mathbf{q}_{,\lambda\lambda} = \mathbf{0} \tag{3.20}$$

For the special case of a linear stability problem

$$\widetilde{\mathbf{K}}_T = \mathbf{K}_0 + \lambda\overline{\mathbf{K}}_\sigma \tag{3.21}$$

where $\mathbf{K}_0$ is the constant small-displacement stiffness matrix and $\overline{\mathbf{K}}_\sigma$ is the constant initial stress matrix evaluated with the help of the stresses obtained from the first step of the analysis (Zienkiewicz and Taylor, 1989). Substitution of (3.21) and of

$$\widetilde{\mathbf{K}}_{T,\lambda} = \overline{\mathbf{K}}_\sigma \tag{3.22}$$

into (2.4) gives

$$[\mathbf{K}_0 + \lambda^*_1\overline{\mathbf{K}}_\sigma]\mathbf{v}^*_1 = \mathbf{0} \tag{3.23}$$

Since $\mathbf{K}_0$ and $\overline{\mathbf{K}}_\sigma$ are constant matrices (Fig. 3)

$$\lambda^*_1 = \text{const} \qquad \wedge \qquad \mathbf{v}^*_1 = \mathbf{v}_1 = \mathbf{const} \tag{3.24}$$

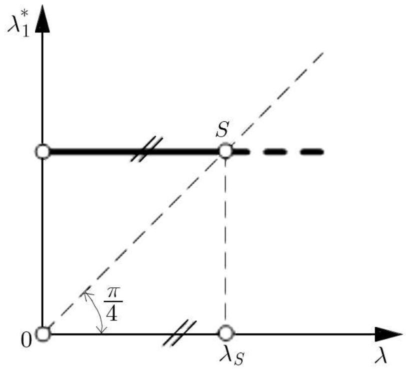

Fig. 3. $\lambda_1^* - \lambda$ diagram for a linear stability problem

3.2. Pure bending

Pure bending represents the second special case of a state with a constant percentage bending energy of the strain energy, namely, one with zero percentage membrane energy. For such a case, $(2.17)_1$ disintegrates into Mang (2011)

$$\mathbf{A}_{,\lambda}\mathbf{v}_1^* = \mathbf{0} \qquad \wedge \qquad \mathbf{A}\mathbf{v}_{1,\lambda}^* = \mathbf{0} \tag{3.25}$$

Since $\mathbf{v}_{1,\lambda}^*$ is not an eigenvector of $\mathbf{A}$

$$\mathbf{v}_{1,\lambda}^*(\lambda) = \mathbf{0} \qquad \forall\lambda \tag{3.26}$$

Thus,

$$\mathbf{v}_1^*(\lambda) = \mathbf{v}_1 = \mathbf{const} \tag{3.27}$$

Substitution of $(2.18)_1$ and (3.27) into $(3.25)_1$ gives

$$[\lambda_{1,\lambda}^*\widetilde{\mathbf{K}}_{T,\lambda} + (\lambda_1^* - \lambda)\widetilde{\mathbf{K}}_{T,\lambda\lambda}]\mathbf{v}_1 = \mathbf{0} \tag{3.28}$$

Premultiplication of (3.28) by

$$\mathbf{v}_k^*(\lambda) = \mathbf{v}_k = \mathbf{const} \tag{3.29}$$

and consideration of $(2.5)_2$ yields

$$\mathbf{v}_k\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{v}_1 = 0 \qquad \forall\lambda \qquad k \in \{2, 3, \ldots, N\} \tag{3.30}$$

In contrast to (3.11), (3.30) is not restricted to the stability limit. In Aminbaghai and Mang (2012) it is shown that

$$\widetilde{\mathbf{K}}_T = \mathbf{K}_0 + \lambda\overline{\mathbf{K}}_\sigma + \mathbf{K}_L \tag{3.31}$$

where $\mathbf{K}_L(\mathbf{q}(\lambda))$ denotes the large-displacement stiffness matrix (Zienkiewicz and Taylor, 1989).

For $\lambda = 0$

$$\mathbf{K}_L = \mathbf{0} \tag{3.32}$$

Specialization of (2.4) for $\lambda = 0$, considering (3.31) and (3.32), gives

$$\mathbf{K}_0 + \lambda_1^*(\overline{\mathbf{K}}_\sigma + \mathbf{K}_{L,\lambda})\mathbf{v}_1 = \mathbf{0} \tag{3.33}$$

where

$$\boxed{(\overline{\mathbf{K}}_\sigma + \mathbf{K}_{L,\lambda})\mathbf{v}_1 = \mathbf{0}} \tag{3.34}$$

(Aminbaghai and Mang, 2012), which requires

$$\lambda_1^* = \infty \tag{3.35}$$

(Fig. 4).

At the stability limit

$$\lambda^*_{1,\lambda\lambda\lambda}(\lambda_S) = 0 \qquad \lambda^*_{1,\lambda\lambda}(\lambda_S) > 0 \tag{3.36}$$

as follows from $(3.3)_2$ and Fig. 4. Hence, the curvature of the curve $\lambda_1^*(\lambda)$ becomes a minimum at S.

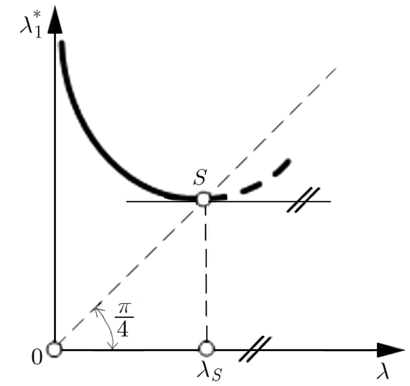

Fig. 4. $\lambda_1^* - \lambda$ diagram for the buckling from a pure bending stress state (lateral torsional buckling)

For all other cases of the buckling at prebuckling states characterized by a constant non-zero percentage buckling energy of the total strain

$$\lambda^*_{1,\lambda\lambda\lambda}(\lambda_S) = 0 \qquad \lambda^*_{1,\lambda\lambda}(\lambda_S) < 0 \tag{3.37}$$

Hence, lateral torsional buckling is not a special case of these cases.

4. Numerical investigation

4.1. Sensitivity analysis of two-hinged arches subjected to a uniformly distributed load (Fig. 1)

The span of the arches l is chosen as 6 m, the rise of the thrust-line arch h as 2.4 m, and the side length of the constant square cross-section as 0.07 m. The geometric form of the axis of the arch is given as (Mang and Höfinger, 2012)

$$x \in [0, l] \qquad y = \frac{4h}{l^2}x(l-x) + \Delta\kappa \sin\Big(\frac{l-x}{l}\pi\Big) \tag{4.1}$$

The modulus of elasticity is assumed as $2.1 \cdot 10^{11}\,\mathrm{N/m^2}$. FEAP (Taylor, 2001) was used for sensitivity analysis of bifurcation buckling of the arches by means of beam elements. The system was discretized, using 100 beam elements available in the FEAP version 7.5. This discretization was sufficient to obtain numerically stable results for the load-displacement relations. For the chosen configuration of arches, bifurcation buckling with an antisymmetric buckling mode is relevant (Mang and Höfinger, 2012). Figure 5 shows the Euclidean norms $\|\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{v}_1\|_2$ and $\|\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1\|_2$ as functions of the design parameter $\Delta\kappa$. They were computed, employing a scheme for numerical differentiation of higher order of the global tangent stiffness matrix $\widetilde{\mathbf{K}}_T(\lambda)$ by using function values at five interpolation nodes. As soon as the discretization was fine enough to get reliable data for the load-displacement relations, no significant dependency of the calculated values of the norms on the number of elements was observed. For the special case of a thrust-line arch

$$\|\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{v}_1\|_2 = 0 \qquad \|\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1\|_2 = 0 \tag{4.2}$$

which confirms (3.16).

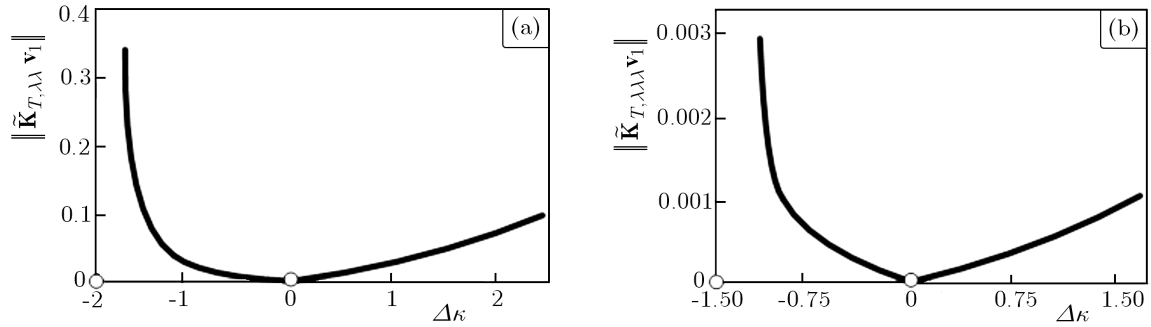

Fig. 5. Sensitivity analysis of bifurcation buckling of a family of two-hinged arches: (a) $\|\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{v}_1\|_2$ and (b) $\|\widetilde{\mathbf{K}}_{T,\lambda\lambda\lambda}\mathbf{v}_1\|_2$ as functions of $\Delta\kappa$ representing the deviation from a thrust-line arch (Mang and Höfinger, 2012)

4.2. Sensitivity analysis of a *von Mises* truss with an elastic spring attached to the load point (Fig. 2)

The length of the two bars in the undeformed configuration L is chosen as 100 cm, the corresponding rise h as 30.9 cm, the side length of the square cross-section as 17 cm, the elastic modulus as $2.8 \cdot 10^{11}\,\mathrm{kN/cm^2}$, and the vertical reference load $\overline{P}$ as 1 N. The value of k in the expression for the spring constant κk, where $\kappa \in \mathbb{R}$ is a scaling parameter, was taken as 1 N/cm. To avoid a multiple bifurcation point, only one half of the truss is analyzed. A detailed analytical treatment of similar structures can be found in Schranz *et al.* (2006) and Steinboeck *et al.* (2008). The truss was designed such that for $\kappa = 0$ the bifurcation point is relatively close to the snap-through point (Höfinger, 2010). With increasing spring stiffness, the distance of the snap-through point from the bifurcation point is increasing. The structure was discretized by means of 30 FEAP beam elements for finite displacements. Figure 6 serves the purpose of verification of (3.17) for the general case of a nonlinear stability problem with nonlinear prebuckling paths, characterized by

$$\|\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{q}_{,\lambda\lambda}\|_2 \neq 0 \tag{4.3}$$

(Fig. 6a). However, apart from numerical noise for relatively small values of κ, the bilinear form $\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{q}_{,\lambda\lambda}$ vanishes (Fig. 6b), which proves (3.17).

5. Conclusions

- The characteristic feature of special prebuckling states, defined as states with a constant percentage bending energy of the strain energy in the prebuckling regime, is disintegration of the third derivative of the mathematical formulation of the consistently linearized eigenproblem with respect to the load multiplier λ (see (3.1)).

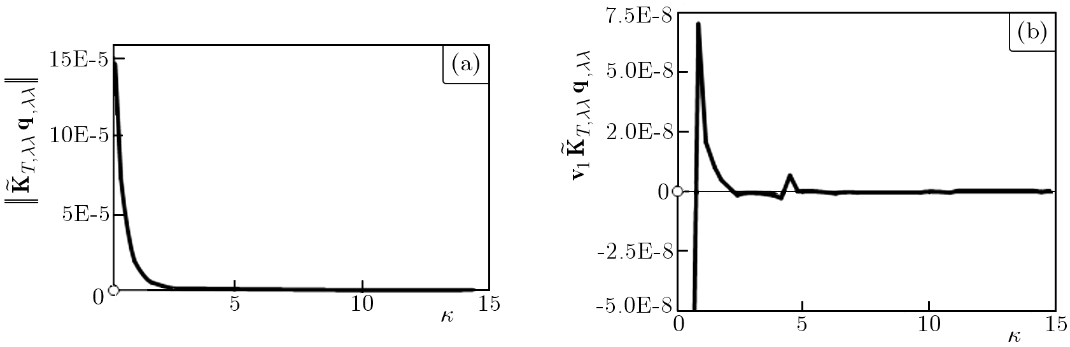

Fig. 6. Sensitivity analysis of bifurcation buckling of a *von Mises* truss with an elastic spring attached to the load point: (a) $\|\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{q}_{,\lambda\lambda}\|_2$ and (b) $\mathbf{v}_1\widetilde{\mathbf{K}}_{T,\lambda\lambda}\mathbf{q}_{,\lambda\lambda}$ as functions of the scaling parameter κ of the spring stiffness (Höfinger, 2010)

- The characteristic feature of buckling from a membrane stress state, representing the special state of zero percentage bending energy of the total strain energy, is disintegration of the second derivative of the mathematical formulation of the mentioned eigenproblem with respect to λ, in addition to disintegration of the third derivative (see (3.4) and (3.1)).
- For buckling from a membrane stress state, obtained as a special case in the frame of sensitivity analysis of buckling from a state of constant percentage bending energy of the strain energy, the buckling mode is also the eigenvector of the second and the third derivative of the tangent stiffness matrix with respect to λ (see (3.16)). This remarkable result was verified numerically by means of sensitivity analysis of two-hinged arches subjected to a uniformly distributed load, containing a thrust-line arch as a special case.
- The difference between such a sensitivity analysis and one that is restricted to the buckling from membrane stress states is reflected by $\lambda^*_{1,\lambda\lambda\lambda} = 0$ (see (3.3)) and $\lambda^*_{1,\lambda\lambda\lambda} > 0$ (see the general case of (3.2)).
- A previously derived necessary and sufficient condition for the buckling from a membrane stress state (see (3.17)) was verified numerically by means of sensitivity analysis of a *von Mises* truss with an elastic spring attached to the load point and the spring stiffness serving as a variable design parameter.
- The characteristic feature of lateral torsional buckling, representing the state of zero percentage membrane energy of the total strain energy, is disintegration of the first derivative of the mathematical formulation of the consistently linearized eigenproblem with respect to λ (see (3.25)). For this special case, for $\lambda = 0$, $\lambda^*_1 = \infty$ (see (3.35)). At the stability limit, $\lambda^*_{1,\lambda\lambda\lambda} = 0$ and $\lambda^*_{1,\lambda\lambda} > 0$ (see (3.37)), indicating a minimum of the curvature of the curve $\lambda^*_1(\lambda)$.

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Analiza wyboczenia konstrukcji w specjalnych przypadkach wyboczenia wstępnego za pomocą metody elementów skończonych

Streszczenie

W pracy przedyskutowano warunki matematyczne w ramach metody elementów skończonych dla niesprzecznie zlinearyzowanego zagadnienia własnego struktur sprężystych w celu określenia granicy statycznej stateczności tych struktur, gdy te poddane zostają wyboczeniu wstępnemu scharakteryzowanemu stałym udziałem energii zginania w stosunku do całkowitej energii odkształcenia. Szczególnym przypadkiem wyboczenia wstępnego jest stan naprężeń powłokowych (brak zginania) oraz czyste zginanie. Wyboczenie przy wstępnych naprężeniach membranowych, jako specjalny przykład analizy wrażliwości wyboczenia na obecność niezerowej energii odkształceń giętych, jest jednym z dwóch przypadków zbadanych dla weryfikacji istnienia nieznanych, uzupełniających warunków wyboczenia w kontekście metody elementów skończonych.

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