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CONSISTENT THEORIES OF ISOTROPIC AND ANISOTROPIC PLATES

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In this paper, the uniform-approximation technique in combination with the pseudo-reduction technique is applied in order to derive consistent theories for isotropic and anisotropic plates. The approach is used to assess and validate the plate theories already established in the literature. Further lines of research are indicated.

Key words: linear elasticity, consistent plate theories, uniform-approximation technique, pseudo-reduction technique

1. Introduction

Plates belong to the class of thin plane structures. The two in-plane dimensions, characterized by a scaling length parameter a, are much larger than the out-of-plane dimension, characterized by a thickness parameter h

$$\frac{h}{a} \ll 1 \tag{1.1}$$

"Much smaller" than unity means in an engineering sense $h/a < 1/10$. In aircraft engineering $1/100$ is usual; and in aerospace engineering $1/1000$ is not unusual. But also structural elements with a thickness-to-length ratio of $1/5$ may still be considered as plates. Commonly, thin plane structures loaded by in-plane forces are designated as discs, those loaded by out-of-plane forces as plates. Generally, the disc and the plate problem are coupled.

It is convenient to introduce a plane middle surface, which bisects the disc or plate continuum transversely. If the material under consideration exhibits at least a symmetry with respect to one plane, i.e., monotropic material behavior, and if the material symmetry plane coincides with the geometrical symmetry plane of the plate, i.e., the middle surface, the plate and the disc problem are decoupled within the linear theory of elasticity and can, therefore, be treated separately. In this contribution, we will concentrate on this most general case in linear elasticity: a plate theory for monotropic materials, which are of increasing interest for, e.g., devices used in microelectronics (generally, "smart materials") built of single crystals.

Plate theories have, of course, a long and rich history. Interesting accounts on this subject and further references may be found, e.g., in Todhunter and Pearson (1960); Timoshenko (1983); Szabó (1987).

Roughly speaking, a plate theory is the attempt to model the three-dimensional behavior of a plate continuum by quantities that "live" on a plane surface. Thus plate theories are inherently approximative. The development of plate theories may be classified into three branches. The "engineering" branch relies on the application of the both admired and feared a priori assumptions. It depends on the intuition of engineers to construct a sound theory based on a set of reasonable a priori assumptions. It is needless to say that this approach can not be systemized and is prone to errors.

The second branch is the so-called "direct approach". Based on the Cosserat theory, the plate is modeled as a deformable initially plane surface with a set of deformable directors attached to each point of the plane. In the special case with one director, the unknown functions are the vector of displacements of the middle surface and the vector describing the deformations of the director. Thus, such a theory contains six degrees of freedom at every point in the plane, and as a consequence, six boundary conditions have to be imposed. The main problem in the application of the direct approach consists in the establishment as well as in the identification of the constitutive equations. A recent review of the direct approach with an extended bibliography is given in Altenbach *et al.* (2010).

The third branch, which will be followed here, is the "consistent approach". All quantities of interest, i.e., deformations, strains, stresses and loads are developed into series in the thickness direction with respect to a suitable basis. The basis might be monomials, scaled Legendre polynomials or trigonometric functions. The series expansions can be truncated at different orders, giving rise to "hierarchical" plate theories. In the following, we will outline the "uniform-approximation technique" in combination with a "pseudo reduction" of the resulting partial differential equation system. Within the paper, we restrict ourselves for reasons of simplicity and clarity to plates with constant thickness and homogeneous materials. The resulting plate theories of different orders will be compared with already established theories.

Finally, the "asymptotic" method should be mentioned, which can be used either within the direct approach or within the consistent approach (cf., e.g., Goldenveizer *et al.* (1993)). Two sets of governing plate equations are developed, one for the "interior" of the plate and the other one for the "boundary layer" in combination with an asymptotic matching technique. Especially for dynamic problems, where the characteristic in-plane dimension, i.e., the wave length λ, is much smaller than the plane extension a, this method supplies accurate results and allows for reliable error estimations (see, e.g., several contributions in Kienzler *et al.*, 2004).

2. Uniform-approximation technique

Let us consider a plate continuum embedded in a cartesian coordinate system x_i as depicted in Fig. 1.

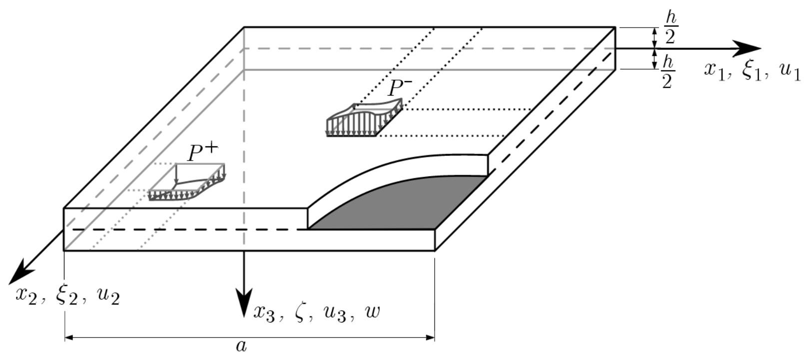

Fig. 1. Plate continuum

The plane $x_3 = 0$ coincides with the plate middle surface, whereas $x_3 = -h/2$ and $x_3 = +h/2$ are the upper and lower faces of the plate, respectively. The plate continuum may be loaded by vertical tractions on the upper and lower face (represented by P^- and P^+ in Fig. 1, respectively) and by vertical body forces f (not shown in Fig. 1).

With the characteristic in-plane dimension a, we introduce dimensionless coordinates as

$$\xi_\alpha = \frac{x_\alpha}{a} \qquad \zeta = \frac{x_3}{a} \qquad \alpha = 1,2 \tag{2.1}$$

and partial derivatives as

$$\begin{aligned}\frac{\partial(\cdot)}{\partial x_1} &= \frac{1}{a}\frac{\partial(\cdot)}{\partial \xi_1} = \frac{1}{a}(\cdot)_{,1} = \frac{1}{a}(\cdot)' \\ \frac{\partial(\cdot)}{\partial x_2} &= \frac{1}{a}\frac{\partial(\cdot)}{\partial \xi_2} = \frac{1}{a}(\cdot)_{,2} = \frac{1}{a}(\cdot)^{\bullet}\end{aligned} \tag{2.2}$$

The displacements $u_i = u_i(\xi^\alpha, \zeta)$ are expanded into series in the thickness direction ζ with respect to a suitable basis. Within this contribution, we use a power series expansion, i.e., a basis of monomic polynominals. One also could use unscaled or scaled Legendre polynomials (Vekua, 1982; Rodionova *et al.*, 1996; Schneider, 2010; Schneider *et al.*, 2012), or trigonometric functions. Thus

$$u_i(\xi_\alpha, \zeta) = a\sum_{\ell=0}^{\infty} {}^{\ell}u_i(\xi^\alpha)\zeta^\ell \tag{2.3}$$

The index on the upper left-hand side of the generic symbol, here the dimensionless displacement quantities ${}^{\ell}u_i$, indicates the counting order of the series expansion. The displacement Ansatz is inserted in the kinematical relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.4}$$

and by comparing equal coefficients of ζ^ℓ, strain coefficients are introduced by

$$\varepsilon_{ij}(\xi_\alpha, \zeta) = \sum_{\ell=0}^{\infty} {}^{\ell}\varepsilon_{ij}(\xi_\alpha)\zeta^\ell \tag{2.5}$$

In turn, we insert the strains into the strain-energy density W

$$W = \frac{1}{2}E_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell} \tag{2.6}$$

(fourth-rank elasticity tensor $E_{ijk\ell}$) and introduce the average of the strain energy $\overline{W}$ over the height of the plate

$$\overline{W} = \int_{-\frac{h}{2a}}^{+\frac{h}{2a}} W(\xi_\alpha, \zeta)\, d\zeta = \overline{W}({}^{\ell}\varepsilon_{ij}(\xi_\alpha)) \tag{2.7}$$

By means of the constitutive equations, we introduce stress resultants ${}^{\ell}m_{ij}$ as

$${}^{\ell}m_{ij} = a^{\ell+1}\frac{\partial \overline{W}}{\partial\, {}^{\ell}\varepsilon_{ij}} \tag{2.8}$$

For conservative forces we introduce the average of the potential $\overline{V}$ of external forces as

$$\overline{V} = -\int_{-\frac{h}{2a}}^{+\frac{h}{2a}} f u_3\, d\zeta - P^+ u_3\Big(+\frac{h}{2a}\Big) + P^- u_3\Big(-\frac{h}{2a}\Big) \tag{2.9}$$

and the load resultants ${}^{\ell}P$ as

$${}^{\ell}P = -\frac{\partial \overline{V}}{\partial\, {}^{\ell}u_3} \tag{2.10}$$

Finally, application of the virtual work theorem leads to the Euler-Lagrange equations and the boundary conditions either written in stress resultants ${}^{\ell}m_{ij}$ (equilibrium conditions) or in displacement quantities (Navier-Lamé equations). Based on Weierstrass's approximation theorem, it can be shown (Schneider, 2010) that the complete set of equations is an exact representation of the equations of the three-dimensional linear theory of elasticity.

The question arises, however, where to truncate the infinite series in order to obtain a tractable set of partial differential equations (PDEs). When considering (2.7) and (2.9), integration over the thickness generates more or less automatically a plate parameter c^2 defined as

$$c^2 = \frac{h^2}{12a^2} \ll 1 \tag{2.11}$$

such that the average of the strain-energy density and the stress resultants can schematically be ordered by the powers of c^2 as

$$\begin{aligned} \overline{W} &= Gh\Big\{(c^2)^0(\cdot) + (c^2)^1(\cdot) + (c^2)^2(\cdot) + \ldots + c^{2n}(\cdot) + \ldots\Big\} \\ {}^{\ell}m_{ij} &= Gha^{\ell}\Big\{(c^2)^0(\cdot) + (c^2)^1(\cdot) + (c^2)^2(\cdot) + \ldots + c^{2n}(\cdot) + \ldots\Big\} \end{aligned} \tag{2.12}$$

(G is a characteristic measure of the stiffness, e.g., a shear modulus).

It turns out, that the terms in brackets do not differ in the order of magnitude. In order to obtain a consistent plate theory it is demanded (see, e.g., Naghdi, 1972; Koiter and Simmonds, 1973; Lo *et al.* 1977; Kienzler, 1982; Krätzig, 1989) that all equations are approximated uniformly, i.e., in a n-th order theory, all terms multiplied by $c^{2\ell}$, $\ell \leqslant n$, have to be retained in all governing equations and all terms multiplied by c^{2m}, $m > n$ have to be neglected. This procedure always leads to countable many partial differential equations for countable many unknowns ${}^{\ell}u_i$. The resulting PDE system is symmetric and the number of equations coincides with the number of unknowns. Since there might be easier equivalent PDE systems and since the classical plate theories are also usually systems of at most two PDEs in two variables, we try to reduce the number of unknowns and PDEs by an elimination process seeking for the easiest equivalent PDE system (pseudo reduction).

3. Pseudo-reduction technique

The uniform-approximation argument, which is proposed for the *derivation* of the plate equations, was extended (Kienzler, 1982; 2002; 2004) to the requirement that is must be applied also to all intermediate equations occurring during the elimination process. To this extent we have to find a set of *main variables*, as few as possible, preferably one, and a *main differential equation system* (the same number of equations as the number of main variables), which are entirely formulated in main variables (for more details see Schneider and Kienzler, 2011). In addition, one has to find a set of *reduction differential equations*, in which all non-main variables are expressed in terms of the main variables. The original PDE system must be identically solved by inserting the reduction differential equations, if the main variables are a solution to the main differential equation system, i.e., the reduced PDE system and the original one are equivalent.

Since the PDEs are actually truncated power series in the characteristic parameter c^2, multiplication or division by c^2 would change the accuracy of the given equations. Therefore, every product of different powers of the characteristic parameter with the same displacement coefficient has to be treated formally as an independent variable. This leads to an underdetermined system of PDEs. The missing equations are generated by multiplying the original PDEs by c^2 and neglecting again ensuing terms of c^{2m}, $m > n$.

This procedure can be followed "by hand". For convenience, an algorithm for the automatisation of the pseudo reduction for PDE-systems arising from the uniform-approximation technique has been implemented (Schneider and Kienzler, 2011).

4. Isotropic plates

Since the plate and the disc problem are decoupled (for at least monotropic plates), it is sufficient to consider plate-displacement quantities only. Thus (2.3) becomes

$$\begin{aligned} u_\alpha &= a(\psi_\alpha + {}^3\psi_\alpha\zeta^3 + {}^5\psi_\alpha\zeta^5 + \ldots) \\ u_3 &= a(w + {}^2w\zeta^2 + {}^4w\zeta^4 + \ldots) \end{aligned} \tag{4.1}$$

The classical deformation quantities, the displacement of the middle surface w and its rotations ψ_α are not indexed by the counting index ℓ.

Working through the algebra indicated in paragraph 2, we arrive at a PDE system formally sketched in Fig. 2. The symbolic operators D_n, which will be used also in the following, are of the form

$$D_n = \sum_{i=0}^{n} b_i \frac{\partial^n}{\partial\xi_1^i \partial\xi_2^{n-i}} \tag{4.2}$$

with coefficients b_i which depend on elastic constants.

In an n-th order approximation we have to be sure, that all terms multiplied by $c^{2\ell}$, $\ell \leqslant n$ are retained. For the series expansion of the displacements, it is important that *enough* coefficients are considered. This is indicated in Fig. 2 by the bold lines for the zeroth-, first- and second-order approximation. For a consistent second-order approximation, e.g., we have to take nine displacement coefficients into account. From Fig. 2 it becomes obvious that it does not make sense to add terms in the displacement series expansion in an arbitrary manner (Wang *et al.*, 2000). Either one uses $3(n+1)$ coefficients in the expansion or one does not get a consistent n-th order approximation.

In the next step, we neglect in the $3(n+1) \times 3(n+1)$ PDE system all terms that are multiplied with c^{2m}, $m > n$. The resulting system reveals a triangular structure.

In order to be a little more specific, the PDE operators for a second-order approximation, for *isotropic* material behavior, are given explicitly in Fig. 3, with the following abbreviations

$$\begin{aligned} a_1 &= 2\frac{1-\nu}{1-2\nu} & a_2 &= \frac{2\nu}{1-2\nu} & a_3 &= \frac{1}{1-2\nu} \\ a_4 &= \frac{1-6\nu}{1-2\nu} & a_5 &= \frac{1-10\nu}{1-2\nu} & a_6 &= \frac{3-10\nu}{1-2\nu} \\ \varDelta &= (\cdot)'' + (\cdot)^{\bullet\bullet} & \varDelta_1 &= a_1(\cdot)'' + (\cdot)^{\bullet\bullet} & \varDelta_2 &= (\cdot)'' + a_1(\cdot)^{\bullet\bullet} \end{aligned} \tag{4.3}$$

The right-hand sides are given for the special case where the external forces are applied only through tractions P^+ and P^- at the lower and upper plate, respectively (see Fig. 1). With this assumption, the theory will not distinguish between upper- and lower-face load applications. We introduce the resultant load per unit of area as

$$P = P^+ + P^- \tag{4.4}$$

and the load resultants defined in (2.10) follow to be ${}^0P = P$, ${}^2P = 3P$, ${}^4P = 9P$.

The symbol G is now the shear modulus and is given for isotropic materials in terms of Young's modulus E and Poisson's ratio ν as

$$G = \frac{E}{2(1+\nu)} \tag{4.5}$$

	w	ψ_1	ψ_2	${}^2w$	${}^3\psi_1$	${}^3\psi_2$	${}^4w$	${}^5\psi_1$	${}^5\psi_1$	${}^6w$	${}^7\psi_1$	${}^7\psi_2$	$\cdots$	RHS
δw	D_2	D_1	D_1	c^2D_2	c^2D_1	c^2D_1	c^4D_2	c^4D_1	c^4D_1	c^6D_2	c^6D_1	c^6D_1	$\cdots$	${}^0Pc^0$
$\delta\psi_1$		$1+c^2D_2$	c^2D_2	c^2D_1	$c^2+c^4D_2$	c^4D_2	c^4D_1	$c^4+c^6D_2$	c^6D_2	c^6D_1	$c^6+c^8D_2$	c^8D_2	$\cdots$	0
$\delta\psi_2$			$1+c^2D_2$	c^2D_1	c^4D_2	$c^2+c^4D_2$	c^4D_1	c^6D_2	$c^4+c^6D_2$	c^6D_1	c^8D_2	$c^6+c^8D_2$	$\cdots$	0
$\delta\,{}^2w$				$c^2+c^4D_2$	c^4D_1	c^4D_1	$c^4+c^6D_2$	c^6D_1	c^6D_1	$c^6+c^8D_2$	c^8D_1	c^8D_1	$\cdots$	${}^2Pc^2$
$\delta\,{}^3\psi_1$					$c^4+c^6D_2$	c^6D_2	c^6D_1	$c^4+c^6D_2$	c^6D_2	c^6D_1	$c^6+c^8D_2$	c^8D_2	$\cdots$	0
$\delta\,{}^3\psi_2$						$c^4+c^6D_2$	c^6D_1	c^8D_2	$c^6+c^8D_2$	c^8D_1	$c^{10}D_2$	$c^8+c^{10}D_2$	$\cdots$	0
δ^4w							$c^6+c^8D_2$	c^8D_1	c^8D_1	$c^8+c^{10}D_2$	$c^{10}D_1$	$c^{10}D_1$	$\cdots$	${}^4Pc^4$
$\delta^5\psi_1$		symm.						$c^8+c^{10}D_2$	$c^{10}D_2$	$c^{10}D_1$	$c^{10}+c^{12}D_2$	$c^{12}D_2$	$\cdots$	0
$\delta^5\psi_2$									$c^8+c^{10}D_2$	$c^{10}D_1$	$c^{12}D_2$	$c^{10}+c^{12}D_2$	$\cdots$	0
δ^6w										$c^{10}+c^{12}D_2$	$c^{12}D_1$	$c^{12}D^1$	$\cdots$	${}^6Pc^6$
$\delta^7\psi_1$											$c^{12}+c^{14}D_2$	$c^{14}D_2$	$\cdots$	0
$\delta^7\psi_2$												$c^{12}+c^{14}D_2$	$\cdots$	0
$\vdots$													$\cdots$	$\vdots$

Fig. 2. Schematic sketch of the PDE system

	w	ψ_1	ψ_2	${}^2w$	${}^3\psi_1$	${}^3\psi_2$	${}^4w$	${}^5\psi_1$	${}^5\psi_2$	RHS
δw	Δ	$(\cdot)'$	$(\cdot)^\bullet$	$c^2\Delta$	$3c^2(\cdot)'$	$3c^2(\cdot)^\bullet$	$\frac{9}{5}c^4\Delta$	$9c^4(\cdot)'$	$9c^4(\cdot)^\bullet$	$-\frac{a}{Gh}P$
$\delta\psi_1$	$(\cdot)'$	$1-c^2\Delta_1$	$-c^2a_3(\cdot)'^\bullet$	$c^2a_4(\cdot)'$	$3c^2-\frac{9}{5}c^4\Delta_1$	$-\frac{9}{5}c^4a_3(\cdot)'^\bullet$	$\frac{9}{5}c^4a_5(\cdot)'$	$9c^4$	0	0
$\delta\psi_2$	$(\cdot)^\bullet$	$-c^2a_3(\cdot)'^\bullet$	$1-c^2\Delta_2$	$c^2a^4(\cdot)^\bullet$	$-\frac{9}{5}c^4a_3(\cdot)'^\bullet$	$3c^2-\frac{9}{5}c^4\Delta_2$	$\frac{9}{5}c^4a_5(\cdot)^\bullet$	0	$9c^4$	0
δ^2w	$c^2\Delta$	$c^2a_4(\cdot)'$	$c^2a_4(\cdot)^\bullet$	$-c^24a_1+\frac{9}{5}c^4\Delta$	$\frac{9}{5}c^4a_6(\cdot)'$	$\frac{9}{5}c^4a_6(\cdot)^\bullet$	$-\frac{72}{5}c^4a_1$	0	0	$-3c^2\frac{a}{Gh}P$
$\delta\,{}^3\psi_1$	$3c^2(\cdot)'$	$3c^2-\frac{9}{5}c^4\Delta_1$	$-\frac{9}{5}c^4a_3(\cdot)'^\bullet$	$\frac{9}{5}c^4a_6(\cdot)'$	$\frac{81}{5}c^4$	0	0	0	0	0
$\delta\,{}^3\psi_2$	$3c^2(\cdot)^\bullet$	$-\frac{9}{5}c^4a_3(\cdot)'^\bullet$	$3c^2-\frac{9}{5}c^4\Delta_2$	$\frac{9}{5}c^4a_6(\cdot)^\bullet$	0	$\frac{81}{5}c^4$	0	0	0	0
δ^4w	$\frac{9}{5}c^4\Delta$	$\frac{9}{5}c^4a_5(\cdot)'$	$\frac{9}{5}c^4a_5(\cdot)^\bullet$	$-\frac{72}{5}c^4a_1$	0	0	0	0	0	$-9c^4\frac{a}{Gh}P$
$\delta^5\psi_1$	$9c^4(\cdot)'$	$9c^4$	0	0	0	0	0	0	0	0
$\delta^5\psi_2$	$9c^4(\cdot)^\bullet$	0	$9c^4$	0	0	0	0	0	0	0

Fig. 3. PDE system for a consistent second-order approximation

Zeroth-order approximation

Retaining only terms without any c^2-factor, we obtain as the PDE system

$$\begin{aligned} &\Delta w + \psi_1' + \psi_2^{\bullet} + O(c^2) = -\frac{Pa}{Gh} \\ &w' + \psi_1 + O(c^2) = 0 \qquad w^{\bullet} + \psi_2 + O(c^2) = 0 \end{aligned} \tag{4.6}$$

It becomes obvious that the PDE system is only satisfied, if P itself is of the order c^2

$$P = O(c^2) \tag{4.7}$$

and the rigid body transformations

$$\begin{aligned} &w = \alpha_0 + \alpha_1\xi_1 + a_2\xi_2 \\ &\psi_1 = -\alpha_1 \qquad \psi_2 = -\alpha_2 \qquad \alpha_0, \alpha_1, \alpha_2 = \text{ const} \end{aligned} \tag{4.8}$$

are solutions to the system (4.6). If the plate is rigidly supported, we have the trivial solution

$$w \equiv 0 \tag{4.9}$$

First-order approximation

We neglect all terms multiplied by c^4 in the table in Fig. 3, choose w as the main variable and

$$\Delta w + \psi_1' + \psi_2^{\bullet} + c^2(\Delta^2 w + 3\,{}^3\psi_1' + 3\,{}^3\psi_2^{\bullet}) + O(c^4) = -\frac{Pa}{Gh} \tag{4.10}$$

as the main PDE. The five additional equations of Fig. 3 have to be used for the reduction

$$\begin{aligned} &w' + \psi_1 + c^2(-a_1\psi_1'' - \psi_2^{\bullet\bullet} - a_3\psi_2'^{\bullet} + a_4\,{}^2w' + 3\,{}^3\psi_1) + O(c^4) = 0 && \text{(a)} \\ &w^{\bullet} + \psi_2 + c^2(-a_3\psi_1'^{\bullet} - \psi_2'' - a_1\psi_2^{\bullet\bullet} + a_4\,{}^2w^{\bullet} + 3\,{}^3\psi_2) + O(c^4) = 0 && \text{(b)} \\ &c^2(\Delta w + a_4(\psi_1' + \psi_2^{\bullet}) - 4a_1\,{}^2w) + 0(c^4) = O(c^4) && \text{(c)} \\ &c^2(3w' + 3\psi_1) + O(c^4) = 0 && \text{(d)} \\ &c^2(3w^{\bullet} + 3\psi_2) + O(c^4) = 0 && \text{(e)} \end{aligned} \tag{4.11}$$

Starting the pseudo reduction, we immediately obtain from $(4.11)_{\text{(d)},\text{(e)}}$

$$\begin{aligned} &\text{(d)}: \qquad c^2\psi_1 = -c^2 w' + O(c^4) \\ &\text{(e)}: \qquad c^2\psi_2 = -c^2 w^{\bullet} + O(c^4) \end{aligned} \tag{4.12}$$

It follows that the deviation from Kirchhoff's normal hypothesis is of $O(c^4)$. Inserting (4.12) into $(4.11)_{\text{(c)}}$ leads to

$$\text{(c)}: \qquad c^2\,{}^2w = c^2\frac{\nu}{2(1-\nu)}\Delta w + O(c^4) \tag{4.13}$$

Hence, also the deviation from plane stress is of $O(c^4)$, i.e., the influence of σ_{33} on the equations of the first-order approximation is of $O(c^4)$ and thus negligible. Note, however, that the transverse displacement u_3 (see Eq. (4.1)) is *not* constant over the plate thickness

$$u_3 = a\Big(w + \zeta^2\frac{\nu}{2(1-\nu)}\Delta w\Big) \neq u_3(\xi_1, \xi_2) \qquad \varepsilon_{33} \neq 0 \tag{4.14}$$

Proceeding further, we obtain by insertion of (4.12) and (4.13) into $(4.11)_{(a),(b)}$

$$\begin{aligned} &\text{(a)}: \qquad \psi_1 + 3c^2\, {}^3\psi_1 = -w' - c^2 \frac{4+v}{2(1-\nu)} \Delta w' + O(c^4) \\ &\text{(b)}: \qquad \psi_2 + 3c^2\, {}^3\psi_2 = -w^\bullet - c^2 \frac{4+\nu}{2(1-\nu)} \Delta w^\bullet + O(c^4) \end{aligned} \tag{4.15}$$

Inserting (4.13) and (4.15) into the main PDE (4.10) results in the classical Kirchhoff-plate PDE

$$\Delta\Delta w = \frac{Pa^3}{K} \qquad \text{with} \qquad K = \frac{Eh^3}{12(1-\nu^2)} \tag{4.16}$$

On inspection (Kienzler, 2002), all stress resultants and also the boundary conditions involving Kirchhoff's "Ersatz-shear forces" are recovered. Thus Kirchhoff's plate theory turns out to be a fully consistent first-order approximation of the three-dimensional linear theory of elasticity! In our approach, *a priori hypotheses* are not necessary since the plane stress and normal hypotheses are *a posteriori results*. It may further be noted that the variables ${}^3\psi_1$ and ${}^3\psi_2$ are not fixed by the pseudo-reduction procedure. Neither the result for the governing PDE nor the stress resultants and boundary conditions are effected by the specific choice of ${}^3\psi_\alpha$. They may arbitrarily be set equal to zero. If we choose instead

$${}^3\psi_\alpha = \frac{1}{2}\frac{2-\nu}{1-\nu} \Delta w_{,\alpha} \tag{4.17}$$

a parabolic shear-stress distribution can be embedded into the theory a posteriori, e.g.

$$\tau_{\alpha 3} = \frac{G\Delta w_{,\alpha}}{1-\nu}(\zeta^2 - 3c^2) = \frac{3}{2}\frac{Q_\alpha}{h}\Big(1 - \frac{4x_3^2}{h^2}\Big) \tag{4.18}$$

(transverse shear resultant Q_α). This is the classical "Dübel formula".

Thus, assuming a priori distributions of shear stresses $\tau_{\alpha 3}$ or normal stresses σ_{33} over the thickness of the plate are not necessary for the establishment of a consistent theory, they might even be contra productive.

Finally, we may note that a linear representation of the displacements with respect to the thickness direction involving merely w and ψ_α cannot not lead to a theory which is in accordance with the consistent approximation without posing additionally a priori assumptions. A linear approximation would lead to the following result

$$\Delta\Delta w = \frac{Pa^3}{K}\frac{1-2\nu}{(1-\nu)^2} \tag{4.19}$$

Equation (4.19) coincides with Vekua's (Vekua, 1982) approximation of the order $N = 1$. Vekua has established a hierarchical structure of plate (and shell) theories, counting the order of the approximation differently as the order of the polynomials considered in the series expansion. His second-order approximation leads to the correct result (4.16). Higher-order theories are only sketched and are not worked out explicitly.

For the following, we observe that the quantity

$$c^2\psi := c^2(\psi_2' - \psi_1^\bullet) = c^2 \operatorname{rot} \vec{\psi} = O(c^4) \tag{4.20}$$

is of the order c^4. ψ may be regarded as a measure of the transverse-shear deformation.

Second-order approximation

For the second-order approximation, we choose w and ψ (4.20) as independent variables. The pseudo-reduction procedure of the full PDE-system shown in Fig. 3 has been dealt with in detail in Kienzler (2002, 2004), Bose and Kienzler (2006) and will not be repeated here. We restrict ourselves to the discussion of the resulting main PDEs after pseudo reduction

$$K\Delta\Delta w = a^3\Big(P - \frac{3}{10}\frac{8-3\nu}{1-\nu}c^2\Delta P\Big) \qquad c^2\Big(\psi - \frac{3}{2}c^2\Delta\psi\Big) = 0 \tag{4.21}$$

Note that no shear correction factor has been introduced. The stress resultants and boundary conditions are given explicitly in, e.g., Kienzler (2004). As in the first-order approximation, the second-order approach contains displacement parameters which are not fixed by the reduced governing equations $({}^4w,\ {}^5\psi_\alpha, \ldots)$. They provide enough freedom to fulfill the boundary conditions at the upper and lower faces of the plate a posteriori

$$\tau_{\alpha 3}\Big(\zeta = \pm\frac{h}{2a}\Big) = 0 \qquad \sigma_{33}\Big(\zeta = \pm\frac{h}{2a}\Big) = \begin{cases} +P^+ \\ -P^- \end{cases} \tag{4.22}$$

The existing theories for shear deformable plates deliver similar PDEs. For instance, Reissner (1945) developed a theory based on several a priori assumptions and arrived finally at the PDE

$$\Delta\Delta w_R = \frac{a^3}{K}\Big(P - \frac{1}{\Gamma_R}\frac{2-\nu}{1-\nu}c^2\Delta P\Big) \qquad \Gamma_R = \frac{5}{6} \tag{4.23}$$

which is quite similar to (4.21) (shear correction factor Γ_R introduced by Reissner). It has to be observed, however, that the transverse displacement w_R is differently defined than w used here, namely by

$$Q_\alpha w_R = \int\limits_{-\frac{h}{2}}^{+\frac{h}{2}} \tau^{\alpha 3} u_3\, dx_3 \tag{4.24}$$

It has been shown (Kienzler, 2002) that w_R as given in (4.24) is related to w by

$$w_R = w + c^2\frac{3\nu}{10(1-\nu)}\Delta w + O(c^4) \tag{4.25}$$

Inserting (4.25) and $(4.23)_{(\mathrm{b})}$ into $(4.23)_{(\mathrm{a})}$, equation $(4.21)_{(\mathrm{a})}$ is recovered. Thus Reissner's plate equation (4.23) is equivalent to (4.21) within the second-order approximation.

Zhilin (1992) advanced a similar plate equation as

$$\Delta\Delta w_Z = \frac{a^3}{K}\Big[P - \frac{2}{1-\nu}\Big(\frac{1}{\Gamma_Z} - \frac{\nu}{2}\Big)c^2\Delta P\Big) \qquad \Gamma_Z = \frac{5}{6-\nu} \tag{4.26}$$

(shear correction factor Γ_Z introduced, by Zhilin), whereas his transverse displacement is defined as

$$h\, w_Z = \frac{1}{a}\int\limits_{-\frac{h}{2}}^{+\frac{h}{2}} u_3\, dx_3 \tag{4.27}$$

It follows with (Kienzler, 2004) that

$$w_Z = w + c^2\frac{\nu}{2(1-\nu)}\Delta w + O(c^4) \tag{4.28}$$

and again, Zhilin's theory is equivalent to (4.21) and is thus a consistent second-order plate theory. Reissner's and Zhilin's engineering intuition is admirable by all means.

Ambartsumyan (1970) concentrates in his work on anisotropic plates. Using his equations within the framework of the uniform-approximation technique and introducing the special case of isotropy we arrive at (see a forthcoming report)

$$\Delta\Delta w_A = \frac{a^3}{K}\Big(P - \frac{3}{10}\frac{8-3\nu}{1-\nu}\frac{12-6\nu}{12-7\nu}c^2\Delta P\Big) \qquad w_A = w \tag{4.29}$$

The difference in comparison to (4.21) is quite small since the factor $\alpha = (12-6\nu)/(12-7\nu)$ is not far from unity ($\nu = 0: \ \alpha = 1$; $\nu = 0.5: \ \alpha = 1.06$), at least for isotropy. For anisotropic plates, the difference might be more pronounced.

The reason for Ambartsumyan's approach being not consistent is that his modeling does not pay respect to all equations of linear elasticity. His displacement Ansatz, which is reversely constructed from a priori assumptions concerning the distribution of shear stresses, results in an intrinsic over determination of the system of the equations of linear elasticity. This is compensated very tricky by a semi-inversion of Hooke's law (Ambartsumyan, 1970, Eq. 2.2.3--2.2.7), which assumes σ_{33} to be given. In fact, since Hooke's law for the stresses σ_{13} and σ_{23} is decoupled from the other stresses even for a monotropic material (cf. Eq. (5.2)), these two stresses can be computed in accordance with the semi-inversion without the knowledge of σ_{33}. This determines σ_{33} afterwards by the corresponding equilibrium condition, which in turn determines all other stresses by the semi-inverted Hooke's law. However, Hooke's law for σ_{33} is not at all taken into account by this approach and is violated by terms of order $O(c^4)$.

5. Monotropic plates

When treating materials with anisotropic behavior, it is common to introduce a "vector-matrix" notation for the generalized Hooke's law (cf., e.g., Altenbach *et al.*, 1998)

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16}\\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26}\\ & & E_{33} & E_{34} & E_{35} & E_{36}\\ & sym. & & E_{44} & E_{45} & E_{46}\\ & & & & E_{55} & E_{56}\\ & & & & & E_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\end{bmatrix}$$
$$= \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1112} & E_{1123} & E_{1131}\\ & E_{2222} & E_{2233} & E_{2212} & E_{2223} & E_{2231}\\ & & E_{3333} & E_{3312} & E_{3323} & E_{3331}\\ & sym. & & E_{1212} & E_{1223} & E_{1231}\\ & & & & E_{2323} & E_{2331}\\ & & & & & E_{3131}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\end{bmatrix} \tag{5.1}$$

Equation (5.1) provides a unique correlation between the "matrix" coefficients E_{AB} $(A, B = 1, 2, \ldots, 6)$ and the tensor coefficients $E_{ijk\ell}$ $(i, j, k, \ell = 1, 2, 3)$, which is necessary if coordinate transformations have to be carried out. If the plane of material symmetry coincides with the $x_3 = 0$ plane, some of the coefficients have to vanish yielding a reduced matrix

$$[E_{AB}] = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & 0 & 0\\ & E_{22} & E_{23} & E_{24} & 0 & 0\\ & & E_{33} & E_{34} & 0 & 0\\ & sym. & & E_{44} & 0 & 0\\ & & & & E_{55} & E_{56}\\ & & & & & E_{66}\end{bmatrix} \tag{5.2}$$

The consistent first-order approximation leads after pseudo reduction to a Kirchhoff-type PDE (see $(4.16)_{(\mathrm{a})}$) in w as (Schneider, 2010; Schneider *et al.*, 2012)

$$\begin{aligned} c^2\Big\{&(E_{33}E_{11}-E_{13}^2)w^{IV}+4(E_{14}E_{33}-E_{34}E_{13})w'''^{\bullet}\\ &+2[E_{12}E_{33}-E_{13}E_{23}+2(E_{44}E_{33}-E_{34}^2)]w''^{\bullet\bullet}\\ &+4(E_{24}E_{33}-E_{34}E_{23})w'^{\bullet\bullet\bullet}+(E_{33}E_{22}-E_{23}^2)w^{\bullet\bullet\bullet\bullet}\Big\}=E_{33}\frac{a}{h}P \end{aligned} \tag{5.3}$$

This equation is attributed to Huber (1929).

For orthotropic material ($E_{14}=E_{24}=E_{34}=E_{56}=0$), equation (5.3) contains only even derivatives in ξ_1 and ξ_2. If the material is transversally isotropic with x_3 as the axis of rotational symmetry of the material, (5.3) reduces to

$$\Delta\Delta w=\frac{Pa^3}{c^2\Big(E_{11}-\frac{E_{13}^2}{E_{33}}\Big)}=:\frac{Pa^3}{\widehat{K}} \tag{5.4}$$

Assuming isotropic material behaviour we finally arrive at (4.16).

A check of the literature reveals that all anisotropic first-order plate theories (at least known to the authors) are consistent, since a linear displacement Ansatz with the usual set of Kirchhoff-type a priori hypotheses is equivalent to the consistent first-order approximation.

In a symbolic notation (see (4.2)), equation (5.3) may be written as

$$c^2D_4w=\frac{a}{h}P \tag{5.5}$$

Note that the coefficients b_i of the symbolic notation (4.2) are given explicitly in (5.3).

The second-order approximation combined with the pseudo-reduction procedure is complex and involves many algebraic manipulations. Finally, in symbolic notation (4.2), the reduced main PDE system is given as

$$\begin{aligned} &(c^2D_4+c^4D_6)w+c^4\widetilde{D}_4\psi=\frac{a}{h}(1+c^2\widetilde{D}_2)P\\ &c^4\widetilde{D}_4w+(c^2D_0+c^4D_2)\psi=0 \end{aligned} \tag{5.6}$$

The operators with tilde differ from the operators without tilde only in that the constant coefficients b_i and $\widetilde{b}_i$ are different. In Schneider (2010), the coefficients are given in detail. To our best knowledge, (5.6) is the only consistent second-order approximation for monotropic plates. The coefficients of the symbolic operators in (5.6) have a complicated but well defined appearance depending on E_{AB}. As soon as a specific material is chosen they are just "numbers". Within the framework of the consistent second-order approximation, the differential equation problem is of the sixth-order in ξ_1 and ξ_2 giving rise to three boundary conditions at each boundary.

Again, for orthotropic materials, the operators D_n contain only even derivatives in ξ_1 and ξ_2. For transversally isotropic materials, the PDE system (5.6) becomes uncoupled and resembles system (4.21). For isotropic materials, (4.21) is recovered.

6. Concluding remarks

The messages from the consistent n-th order plate theory in combination with the pseudo-reduction technique are as follows:

- If you want to apply a refined theory, i.e., an improved Kirchhoff-plate theory, including shear deformation, warping of the cross section and transverse normal stress effects, use one of the consistent second-order approximations.
- There is no way to improve a consistent plate theory by adding in an unsystematic way single terms in the displacement Ansatz or by assuming specific stress distributions in the thickness direction.
- If you want to improve the consistent second-order theory (this might be useful for dynamic plate problems, where the characteristic in-plane length becomes the wave length $\lambda \ll a$) use a consistent third-order approach with all terms of $O(c^6)$ included.

The uniform-approximation technique was quite helpful to derive material conservation laws within a consistent-plate theory (Bose and Kienzler, 2006). It has also been applied to shells (Kienzler, 1982).

Recently, the same technique has been applied to beams. At first glance, it is astonishing that the beam theory is more advanced than plate theory. The reason is that the series expansion has to be performed in two directions and two parameters have to be considered, a height and a width parameter, which results in a much more complex pseudo-reduction problem. The results of this investigation will be dealt with in a forthcoming paper.

The proposed plate theory may be extended to introduce external moment loadings, variable plate thickness and geometrically non-linear effects. This is the subject of ongoing research.

References

1. Altenbach H., Altenbach J., Naumenko K., 1998, *Ebene Flächentragwerke: Grundlagen der Berechnung von Scheiben und Platten*, Springer, Berlin
2. Altenbach J., Altenbach H., Eremeyev V.A., 2010, On generalized Cosserat-type theories of plates and shells: a short review and bibliography, *Archive of Applied Mechanics*, **80**, 73-92
3. Ambartsumyan S.A., 1970, *Theory of Anisotropic Plates*, Progress in Material Science Series II, Technonic, Stamford, Conn.
4. Bose D.K., Kienzler R., 2006, On material conservation laws for a consistent plate theory, *Archive of Applied Mechanics*, **75**, 606-617
5. Goldenveizer, A.L., Kaplunov J.D., Nolde E.V., 1993, On Timoshenko-Reissner type theories of plates and shells, *International Journal of Solids and Structures*, **30**, 675-694
6. Huber M.T., 1929, *Probleme der Statik technisch wichtiger orthotroper Platten*, Gebetner and Wolff, Warsaw
7. Kienzler R., 1982, Eine Erweiterung der klassischen Schalentheorie; der Einfluss von Dickenverzerrungen und Verwölbungen, *Ingenieur Archiv*, **52**, 311-322; Extended version in: Phd-Thesis, Technische Hochschule Darmstadt, Darmstadt (1980)
8. Kienzler R., 2002, On consistent plate theories, *Archive of Applied Mechanics*, **72**, 229-247
9. Kienzler R., 2004, On consistent second-order plate theories, [In:] Kienzler R., Altenbach H., Ott I. (Eds.), *Theories of Plates and Shells. Critical Review and New Applications*, Lecture Notes in Applied and Computational Mechanics, **16**, Springer, Berlin, 85-96
10. Kienzler R., Altenbach H., Ott I. (Eds.), 2004, *Theories of Plates and Shells; Critical Review and New Applications*, Lecture Notes in Applied and Computational Mechanics, **16**, Springer, Berlin
11. Koiter W.T., Simmonds J.G., 1973, Foundation in shell theories, [In:] Becker E., Mikhailov G.K. (Eds.), *Theoretical and Applied Mechanics*, Proc. 13th Int. Congr. Theor. Appl. Mech., 150-176

12. Krätzig W.B., 1989, On the structure of consistent plate theories, [In:] Koiter W.T., Mikhailov G.K. (Eds.), *Proc. 3rd IUTAM Symp. on Shell Theory*, North-Holland, Amsterdam, 353-368

13. Lo K.H., Christensen R.M., Wu E.M., 1977, A higher-order theory of plate deformation, Part 1: Homogeneous plates, *Journal of Applied Mechanics*, **44**, 663-668

14. Naghdi P.M., 1972, The theory of plates and shells, [In:] Flügge S. (Ed.), *Handbuch der Physik*, Springer, Berlin, VI, A2, 425-646

15. Reissner E., 1945, The effect of transverse shear deformation on the bending of elastic plates, *Journal of Applied Mechanics*, **12**, A69-A77

16. Rodionova V.A., Titaev V.F, Chernykh K.F., 1996, *Applied Theory of Anisotropic Plates and Shells*, St. Petersburg University Press, St. Petersburg [in Russian]

17. Schneider P., 2010, Eine konsistente Plattentheorie zweiter Ordnung für monotropes Material, Diploma Thesis, University of Bremen, Bremen

18. Schneider P., Kienzler, R., 2011, An algorithm for the automatization of pseudo reductions of PDE systems arising from the uniform-approximation technique, [In:] Altenbach H., Eremeyev V.A. (Eds.), *Shell-like Structures; Non-classical Theories and Applications*, Advanced Structured Materials, **15**, Springer, Berlin, 377-390

19. Schneider P., Kienzler, R., Böhm M., 2012, A consistent second-order plate theory for monotropic material, *Zeitschrift für Angewandte Mathematik und Mechanik*, submitted

20. Szabó I., 1987, *Geschichte der mechanischen Prinzipien*, Birkhäuser, Basel

21. Timoshenko S.P., 1983, *History of Strength of Materials*, Dover, New York

22. Todhunter I., Pearson K., 1960, *A History of the Theory of Elasticity and of the Strength of Materials*, Dover, New York

23. Vekua I.N., 1982, *Shell Theory: General Methods of Construction*, Pitman, Boston

24. Wang C.M., Reddy J.N., Lee K.H., 2000, *Shear Deformable Beams and Plates; Relationships with Classical Solutions*, Elsevier, Amsterdam

25. Zhilin P.A., 1992, On the Poisson and Kirchhoff plate theories from the point of view of the modern plate theory, *Izvesta Akademii Nauk Rossii, Mekhanica Averdogotela*, Moscow, **3**, 48-64 [in Russian]

Spójne teorie izotropowych i anizotropowych płyt

Streszczenie

W pracy omówiono metodę jednorodnej aproksymacji połączonej z techniką pseudo-redukcji zastosowaną do sformułowania spójnych teorii płyt izotropowych i anizotropowych. Zaprezentowaną metodologię wykorzystano do oceny i weryfikacji teorii już istniejących i dobrze znanych z literatury. Wskazano także dalsze kierunki badań.

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