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CONTACT BETWEEN A RIGID INDENTER AND A TRANSVERSELY ISOTROPIC LAYER

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The indentation of a transversely isotropic layer by a rigid indenter is investigated. The lower plane of the layer is elastically supported. On the upper surface of the layer certain normal displacement is prescribed inside a circular region with an unknown radius, outside of which certain arbitrary normal stresses are given in an annular region and the normal displacement is zero on a remaining boundary, while the shear stresses vanish all over the boundary.

The author formulates the problem as the solution of a set of triple integral equations. To this end, the differential, integral and series representation of the unknown function is devised, which satisfies two of the three equations exactly, while the third one leads to three infinite sets of elgebraic equations with respect to the coefficients introduced in the representation. The physical quantities which characterize the contact and the stress intensity factors are obtained by means of these coefficients.

Some punch, inclusion and crack problems in a transversely isotropic layer are considered.

1. Introduction

A number of hexagonal crystals are characterized as being transversely isotropic. Many fiber-reinforced composite materials and platelet systems were also characterized as transversely isotropic media, which have five elastic constants [1]. According to effective modulus theory [2] the gross elastic behaviour of the laminated medium is transversely isotropic and homogeneous elastic material with the normal to the layers as the axis of symmetry; the effective elastic constants of such a medium are given by Achenbach ([2], p. 33),

The present work studies the indentation of a transversely isotropic layer by a smooth indenter. Only the circular part of one surface is subjected to the indentation of the indenter, while the outer annulár region is subjected to normal, symmetrical in r, pressure and the normal displacement is zero on a remaining boundary. These displacements cause in the layer by a rigid punch, inclusion which exists in a penny-shaped crack, obstacle which lies between two the same materials, which are pressed together by a pressure.

The method of Hankel transforms is used to satisfy the equilibium equations and the boundary conditions, which have three different parts. The solutions are obtained using the technique of triple integral equations, which are reduced to three infinite systems of simultaneous linear algebraic equations.

Recently Mastrojanis, Mura and Keer [3] studied the mixed boundary-value problem for an isotropic half-space with the following boundary conditions: the constant normal displacement is prescribed inside a circle, outside of which the normal stress vanish in an annular region and the normal displacement is zero on the remaining surface, while the shear stresses vanish all over the boundary. The more general problem, pointed out in the summary is considered in this paper.

2. Formulation

Consider a transversely isotropic elastic layer $0 \leqslant z \leqslant h$, with the planes of isotropy parallel to the boundaries. The stress-strain relationships of such a medium can be written in cylindrical coordinates (r, Θ, z) as follows:

$$\begin{aligned} \sigma_r &= c_{11} e_r + c_{12} e_\Theta + c_{13} e_z, \\ \sigma_\Theta &= c_{12} e_r + c_{11} e_\Theta + c_{13} e_z, \\ \sigma_z &= c_{13} e_r + c_{13} e_\Theta + c_{33} e_z, \\ \sigma_{rz} &= c_{44} e_{rz}, \\ \sigma_{\Theta z} &= c_{44} e_{\Theta z}, \\ \sigma_{r\Theta} &= \frac{1}{2}(c_{11} - c_{12}) e_{r\Theta}. \end{aligned} \tag{2.1}$$

Here c_{ij}'s are the elastic constants.

The foregoing strain e_{ij} can be first written in terms of the displacements and then substituted into the preceding equations to obtain the stress-displacement relationships. The relationships are finally used in the equilibrium equations to form a system of partial differential equations for the displacements. In the problem with axial symmetry the displacements $(u_r, 0, w_z)$ satisfied the equations

$$\begin{aligned} c_{11} \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right] + (c_{13}+c_{44})\frac{\partial^2 w_z}{\partial r\,\partial z} + c_{44}\frac{\partial^2 u_r}{\partial z^2} &= 0, \\ c_{44}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_z}{\partial r}\right) + (c_{13}+c_{44})\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right] + c_{33}\frac{\partial^2 w_z}{\partial z^2} &= 0. \end{aligned} \tag{2.2}$$

The solution of the equilibrium equations is given by two displacement potentials $\varphi_1(r, z)$ and $\varphi_2(r, z)$, and the components of the displacement and stress can be expressed in terms of those potentials as follows:

$$u_r = \frac{\partial}{\partial r}(k\varphi_1 + \varphi_2), \qquad w_z = \frac{\partial}{\partial z}(\varphi_1 + k\varphi_2), \tag{2.3}$$

$$\sigma_r = -c_{44}(k+1)\frac{\partial^2}{\partial z^2}(\varphi_1+\varphi_2)-(c_{11}-c_{12})r^{-1}u_r,$$

$$\sigma_\Theta = -c_{44}(k+1)\frac{\partial^2}{\partial z^2}(\varphi_1+\varphi_2)-(c_{11}-c_{12})\frac{\partial u_r}{\partial r}, \tag{2.4}$$

$$\sigma_z = c_{44}(k+1)\frac{\partial^2}{\partial z^2}(s_1^{-2}\varphi_1+s_2^{-2}\varphi_2),$$

$$\sigma_{rz} = c_{44}(k+1)\frac{\partial^2}{\partial r\,\partial z}(\varphi_1+\varphi_2),$$

provided that the potentials $\varphi_1(r, z)$ and $\varphi_2(r, z)$ are the solution of the differential equations

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{s_i^2}\frac{\partial^2}{\partial z^2}\right)\varphi_i(r, z) = 0, \quad (i = 1, 2), \tag{2.5}$$

and if the parameters s_1^2 and s_2^2 are the roots of the following quadratic equation for s^2

$$c_{33}c_{44}s^4-[c_{11}c_{33}-c_{13}(2c_{44}+c_{13})]s^2+c_{11}c_{44} = 0, \tag{2.6}$$

and the material parameter k is a function of the elastic constants and the characteristic root s_1^2

$$k = (c_{33}s_1^2-c_{44})/(c_{13}+c_{44}). \tag{2.7}$$

The roots s_i^2 are either both real or a pair of complex conjugates, depending on the values of the material constants. Both types of root give physically meaningful results.

The conditions specified on $z = 0$ inside and outside the annulus $\lambda < \varrho < 1$ are

$$w_z(\varrho, 0) = \begin{cases} \delta-\varkappa b^2\varrho^2; & 0 \leqslant \varrho \leqslant \lambda, \\ 0; & 1 \leqslant \varrho, \end{cases} \tag{2.8}$$

$$\sigma_z(\varrho, 0) = -p_0 f(\varrho); \quad \lambda < \varrho < 1, \tag{2.9}$$

$$\sigma_{zr}(\varrho, 0) = 0; \quad \varrho \geqslant 0 \tag{2.10}$$

and the displacements and stresses vanish at infinity. The layer at $z = h$ is elastically supported such that

$$\sigma_{zr}(\varrho, \eta) = 0; \quad \varrho \geqslant 0, \tag{2.11}$$

$$\sigma_z(\varrho, \eta) = -c_0 w_z(\varrho, \eta); \quad \varrho \geqslant 0. \tag{2.12}$$

In above equations, nondimensional variables and parameters are as follows: $\varrho = r/b$, $\zeta = z/b$ and $\lambda = a/b$, $\eta = h/b$, where a and b are the inner and outer radii of the annulus, respectively. Inside the annulus normal stress $p_0 f(\varrho)$ is arbitrary, but assumed to be symmetrical about the z-axis. The parameter c_0 is the spring of stiffness of the foundation.

In the boundary conditions (2.8), which have three different parts, only constant or quadratic with respect to r normal displacement were prescribed within the circle. The conditions (2.8) corresponds to the displacement distribution produced by the indentation of a surface of the layer by an indenter, when its shape is specified. If the contact surface of the rigid indenter is spherical in shape with radius R, the shape of the indentation can be written as $g(r) = \delta-r^2/2R$. The condition required to this equation is that the radius a of the contact area is small compared to the radius of the contact surface of the indenter. The condition is indeed satisfied in usual stress ranges. This equation also applies for an oblate spheroid with semiaxes d_z and d_r. In this case, the radius of curvature of the spheroid at the center of the contact area is $R = d_r^2/d_z$, d_z being the minor semiaxis along the z-axis. The displacement-shape function $w_z(\varrho, 0)$ in equation (2.8) is identical to the shape of the rigid indenter inside the contact area with unknown radius equal to $\lambda = a/b$, but is unknown outside the contact area $\lambda < \varrho < 1$. These displacements cause in the layer by a rigid punch, inclusion which exists in a penny-shaped crack, obstacle which lies between two the same materials, which are pressed together by a pressure. In the last case there is the compatibility condition $\{dw_z(r, 0)/dr\}_{r=b} = 0$. For infinitesimal elasticity theory, there is no loss of generality if the profiles of the inclusion and obstacle or of the base of the punch assume that $g(r) = \delta - r^2/2R$, where $1/R = 2\varkappa$ is the curvature at the center of the contact area and δ is semiaxis or the (prescribed) vertical penetration depth of the punch. From a physical consideration, the contact stress should be finite for a smooth indenter whose contact surface does not have any abrupt change in slope. The unknown contact radius a is to be determined later using this condition.

Using the method of Hankel transforms, the condition of vanishing shear stress in equation (2.10) and (2.11) and the boundedness conditions at infinity the displacement functions $\varphi_i(r, z)$ are found to be

$$\varphi_i(\varrho, \zeta) = (-1)^{i+1} \frac{s_{i\pm 1}}{G_1(k+1)(s_1 - s_2)} \int\limits_0^\infty \left\{ \frac{p(x)}{x} \left[e^{-s_i x \zeta} + \right.\right.$$

$$\left. + \frac{1}{\sinh s_i x \eta} \left(e^{-s_i x \eta} \cosh s_i x \zeta - g_1(x\eta) \cosh s_i x (\eta - \zeta) \right) \right] + \tag{2.13}$$

$$\left. + \frac{\omega(x)}{x \sinh s_i x \eta} \left[\cosh s_i x \zeta - g_2(x\eta) \cosh s_i x (\eta - \zeta) \right] \right\} J_0(x\varrho)\, dx; \qquad i = 1, 2,$$

where $G_1 = c_{44}$ is the shear modulus in the z-direction and $J_0(x\varrho)$ is the Bessel function of the first kind and zero order. The unknown functions $p(x)$ and $\omega(x)$ and the unknown contact radius λ are to be determined using the remaining boundary conditions (2.8), (2.9) and (2.12) and the finiteness of the contact stresses between the indenter and the layer surface. The functions $g_1(x\eta)$ and $g_2(x\eta)$ are known and defined as follows:

$$g_i(x) = \frac{1}{\sinh \alpha x + \alpha\beta^{-1} \sinh \beta x} \cdot \begin{cases} \cosh \beta x + \alpha\beta^{-1} \sinh \beta x - e^{-\alpha x}; & i = 1, \\ 2\beta^{-1}(s_1 \sinh s_1 x - s_2 \sinh s_2 x); & i = 2, \end{cases} \tag{2.14}$$

$$\{\alpha, \beta\} = s_1 \pm s_2.$$

The material parameter α is allways real and β is either real or imaginer.

We can easily obtain the displacements and stresses by substitution of the displacement potentials (2.13) into the expressions (2.3) and (2.4). In particular, the displacement w_z and the stresses σ_z and σ_{zr} on the surfaces of the layer $\zeta = 0$ and $\zeta = \eta$ are given as

$$CG_1 b w_z(\varrho, 0) = \int\limits_0^\infty \{p(x)[1 - g_1(x\eta)] - \omega(x) g_2(x\eta)\} J_0(x\varrho)\, dx, \tag{2.15}$$

$$CG_1 bw_z(\varrho, \eta) = -\int_0^\infty \omega(x) J_0(x\varrho)\, dx,$$

$$b^2\sigma_z(\varrho, 0) = -\int_0^\infty xp(x) J_0(x\varrho)\, dx, \tag{2.15}$$

[cont.]

$$b^2\sigma_z(\varrho, \eta) = -\int_0^\infty x\{p(x)g_2(x\eta) + \omega(x)[1 - g_3(x\eta)]\} J_0(x\varrho)\, dx,$$

$$\sigma_{zr}(\varrho, 0) = \sigma_{zr}(\varrho, \eta) = 0,$$

where

$$g_3(x) = 1 - \frac{[1 - g_2(x)][1 + g_2(x)]}{1 - g_1(x)} = 1 - \frac{\cosh \alpha x - 1 - \alpha^2\beta^{-2}(\cosh \beta x - 1)}{\sinh \alpha x + \alpha\beta^{-1} \sinh \beta x}, \tag{2.16}$$

and

$$C = (k+1)(k-1)^{-1}(s_2^{-1} - s_1^{-1}), \tag{2.17}$$

is a real-valued function of the elastic constants and the characteristics roots.

Substituting the stress $\sigma_z(\varrho, \eta)$ and the displacement $w_z(\varrho, \eta)$ into the condition (2.12), we get

$$\int_0^\infty \{p(x)g_2(x\eta) + \omega(x)[1 - g_3(x\eta) + c_1(x\eta)^{-1}\} \times J_0(x\varrho)\, dx = 0; \quad \varrho \geqslant 0, \tag{2.18}$$

where the constant

$$c_1 = \frac{c_0 h}{G_1 C}, \tag{2.19}$$

describes the relative rigidity of foundation to layer, it being zero when the lower surface is stress free and infinitely large for a perfectly smooth rigid base.

Thus

$$p(x)g_2(x\eta) + \omega(x)[1 - g_3(x\eta) + c_1(x\eta)^{-1}] = 0. \tag{2.20}$$

Get a new unknown function $t(x)$ and set as follows

$$p(x)[1 - g_1(x\eta)] - \omega(x)g_2(x\eta) = t(x). \tag{2.21}$$

Then

$$p(x) = t(x)[1 - h(x\eta)], \quad \omega(x) = -t(x)h_1(x\eta), \tag{2.22}$$

where

$$h(x) = \frac{g_3(x) - c_1 g_1(x)}{1 + c_1 - c_1 g_1(x)} =$$

$$= 1 - \frac{\cosh \alpha x - 1 - \alpha^2\beta^{-2}(\cosh \beta x - 1) + c_1 x^{-1}(\sinh \alpha x + \alpha\beta^{-1} \sinh \beta x)}{\sinh \alpha x + \alpha\beta^{-1} \sinh \beta x + c_1 x^{-1}(\cosh \alpha x - \cosh \beta x)},$$

$$h_1(x) = \frac{g_2(x)}{1 + c_1 - c_1 g_1(x)} = \tag{2.23}$$

$$= 2\beta^{-1} \frac{s_1 \sinh s_1 x - s_2 \sinh s_2 x}{\sinh \alpha x + \alpha\beta^{-1} \sinh \beta x + c_1 x^{-1}(\cosh \alpha x - \cosh \beta x)}.$$

With the help of the known functions $h(x\eta)$ and $h_1(x\eta)$ and the only one unknown $t(x)$, the boundary values of the normal displacement and stress can be rewritten as follows:

$$CG_1 bw_z(\varrho, 0) = \int_0^\infty t(x) J_0(x\varrho) dx,$$

$$CG_1 bw_z(\varrho, \eta) = \int_0^\infty t(x) h_1(x\eta) J_0(x\varrho) dx, \tag{2.24}$$

$$b^2 \sigma_z(\varrho, 0) = -\int_0^\infty xt(x)[1-h(x\eta)] J_0(x\varrho) dx,$$

$$\sigma_z(\varrho, \eta) = -c_0 w_z(\varrho, \eta).$$

Hence, $t(x)$ is the only unknown which from Eqs. (2.8) and (2.9) can be found from the triple integral equations

$$\int_0^\infty t(x) J_0(x\varrho) dx = \begin{cases} CG_1 b(\delta - \varkappa b^2 \varrho^2); & 0 \leqslant \varrho \leqslant \lambda, \\ 0; & 1 \leqslant \varrho, \end{cases} \tag{2.25}$$

$$\int_0^\infty xt(x)[1-h(x\eta)] J_0(x\varrho) dx = p_0 b^2 f(\varrho); \quad \lambda < \varrho < 1, \tag{2.26}$$

with $h(x\eta)$ being defined by Eq. (2.23). Since it is difficult to solve Eqs. (2.25) and (2.26) directly, these equations are solved in an approximate method to yield the parameters which characterize the contact.

3. The series solution

We assume the function $t(x)$ in the form

$$t(x) = 2CG_1 \varkappa b^3 \frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\partial}{\partial x} J_0(\lambda x)\right] + \int_0^\pi F(\Psi) J_1(xR) d\Psi, \tag{3.1}$$

where $F(\Psi)$ is an arbitrary continuous function in the interval $0 \leqslant \Psi \leqslant \pi$ and

$$2R^2 = 1 + \lambda^2 - (1-\lambda^2)\cos\Psi; \quad \lambda \leqslant R \leqslant 1, \quad 0 \leqslant \Psi \leqslant \pi. \tag{3.2}$$

Using (3.2), the variable R in $\lambda \leqslant R \leqslant 1$ can be exchanged for a new one Ψ, which is $0 \leqslant \Psi \leqslant \pi$, when $R = \lambda$ corresponds to $\Psi = 0$ and $R = 1$ to $\Psi = \pi$.
Substituting Eq. (3.1) into $w_z(\varrho, 0)$ of Eq. $(2.24)_1$, we obtain

$$w_z(\varrho, 0) = (CG_1 b)^{-1} \int_0^\pi \frac{1}{R} F(\Psi) H(R-\varrho) d\Psi + \varkappa b^2 \begin{cases} (\lambda^2 - \varrho^2); & 0 \leqslant \varrho \leqslant \lambda, \\ 0; & \lambda < \varrho \end{cases} \tag{3.3}$$

where $H(R-\varrho)$ is the Heaviside's function.

We see that:

(i) The displacement $w_z(\varrho, 0)$ equals $\delta - \varkappa b^2 \varrho^2$ ine the interval $0 \leqslant \varrho \leqslant \lambda$ provided that the compatibility equation

$$\varkappa b^2\lambda^2+(CG_1 b)^{-1}\int\limits_0^{\pi}\frac{1}{R}F(\Psi)\,d\Psi = \delta \tag{3.4}$$

is satisfied.

(ii) The displacement is a function of ϱ in the interval $\lambda \leqslant \varrho \leqslant 1$ i.e.

$$w_z(\varrho, 0) = (G_1 Cb)^{-1}\int\limits_{\Phi}^{\pi}\frac{1}{R}F(\Psi)\,d\Psi, \tag{3.5}$$

where

$$2\varrho^2 = 1+\lambda^2-(1-\lambda^2)\cos\Phi; \quad \lambda \leqslant \varrho \leqslant 1, \quad 0 \leqslant \Phi \leqslant \pi. \tag{3.6}$$

(iii) The displacement equals zero in the remaining interval $\varrho \geqslant 1$ independent on the function $F(\Psi)$.

We now assume a series expansion with unknown parameters $a_0, a_1, a_2, \ldots$ for the function $F(\Psi)$ as

$$F(\Psi) = \frac{1}{\pi}b^2R\sum_{n=0}^{\infty}a_n\cos n\Psi; \quad 0 \leqslant \Psi \leqslant \pi. \tag{3.7}$$

Equations (3.7) and (3.4) lead to

$$a_0 = CG_1 b^{-1}(\delta-\varkappa b^2\lambda^2). \tag{3.8}$$

This equation yields either an unknown radius of the contact region λ for a curved base of the indenter or for a flat indenter yields the parameter a_0, because in this case the extent of the contact is known beforehand. For a curved base the parameter a_0 may have to be found so that the contact stress is finite at the boundary of the contact region.

Thus the displacement $w_z(\varrho, 0)$ is given by

$$w_z(\varrho, 0) = \begin{cases} \delta-\varkappa b^2\varrho^2; & 0 \leqslant \varrho \leqslant \lambda, \\ (\delta-\varkappa b^2\lambda^2)\left\{1-\frac{1}{\pi}\left[\Phi+\frac{1}{a_0}\sum_{n=1}^{\infty}\frac{a_n}{n}\sin n\Phi\right]\right\}; & \lambda \leqslant \varrho \leqslant 1, \quad 0 \leqslant \Phi \leqslant \pi, \\ 0; & 1 \leqslant \varrho. \end{cases} \tag{3.9}$$

The unknown portion of the displacement-shape function in the interval $0 \leqslant \Phi \leqslant \pi$ is in the form of the Fourier series and is determined by substitution Eqs. (3.7), (3.8) into Eq. (3.5) and integration.

Two of the three equations, namely Eqs. (2.25), are satisfied exactly.

In our problem contact is maintained only by compressive stresses; in these unbonded frictionless contact problems the extent of the contact is the primary unknown quantity, and the contact stress is finite at the ends of the contact regions.

The last condition may be replaced by

$$\left.\frac{\mathrm{d}w_z(\varrho, 0)}{\mathrm{d}\varrho}\right|_{\varrho=\lambda} = -2\varkappa b^2\lambda, \tag{3.10}$$

which leads to the condition $F(0) = 0$ or

$$\sum_{n=0}^{\infty} a_n = 0, \tag{3.11}$$

and then the surface of the layer contacts smoothly at the edge of the contact region with the indenter.

By substitution of Eq. (3.7) into Eq. (3.1), we obtain

$$t(x) = 2CG_1\varkappa b^3 \frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\partial}{\partial x}J_0(\lambda x)\right] - b^2 \sum_{n=0}^{\infty} a_n \frac{\partial Z_n(x)}{\partial x}, \tag{3.12}$$

where

$$Z_n(x) = J_n\left[\frac{x}{2}(1-\lambda)\right] J_n\left[\frac{x}{2}(1+\lambda)\right]. \tag{3.13}$$

The stresses in Eq. $(2.24)_3$ which correspond to (3.12) are as follows

$$\sigma_z(\varrho, 0) = \sum_{n=0}^{\infty} a_n \int_0^{\infty} [1-h(x\eta)] \frac{\partial Z_n(x)}{\partial x} xJ_0(x\varrho)\,dx -$$

$$-2CG_1\varkappa b \int_0^{\infty} [1-h(x\eta)] \frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\partial}{\partial x} J_0(\lambda x)\right] xJ_0(x\varrho)\,dx, \tag{3.14}$$

and are bounded at $\varrho = \lambda$ under the condition (3.11); it is clarified bellow (see Eq. (4.10)).

The preceding equation and Eq. (2.9) lead to

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} [1-h(x\eta)] \frac{\partial Z_n(x)}{\partial x} xJ_0(x\varrho)\,dx =$$

$$-2CG_1\varkappa b \int_0^{\infty} [1-h(x\eta)] \frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\partial}{\partial x} J_0(\lambda x)\right] xJ_0(x\varrho)\,dx - p_0 f(\varrho); \quad \lambda < \varrho < 1. \tag{3.15}$$

Multiplying both sides of Eq. (3.15) by ϱ, using the formula $x\varrho J_0(x\varrho) = \partial[\varrho J_1(x\varrho)]/\partial\varrho$ integrating with respect to ϱ and using the formula $\partial[J_0(x\varrho)]/\partial x = -\varrho J_1(x\varrho)$, we obtain

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} [1-h(x\eta)] \frac{\partial Z_n(x)}{\partial x} \frac{\partial}{\partial x} [J_0(x\varrho)]\,dx =$$

$$= 2CG_1\varkappa b \int_0^{\infty} [1-h(x\eta)] \frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\partial}{\partial x} J_0(\lambda x)\right] \frac{\partial}{\partial x} [J_0(x\varrho)]\,dx - c + p_0 f^*(\varrho); \quad \lambda < \varrho < 1, \tag{3.16}$$

where

$$f^*(\varrho) = \int_0^{\varrho} sf(s)\,ds, \tag{3.17}$$

and c is an unknown integral constant.

Equation (3.16) is solved under the assumption that the moment $f^*(\varrho)$ of the function $f(\varrho)$ may be expanded in Fourier series, namely

$$\begin{aligned} f^*(\varrho) &= f_0^* + 2\sum_{m=1}^{\infty} f_m^* \cos m\Phi; \\ f_m^* &= \frac{1}{\pi}\int_0^{\pi} f(\Phi)\cos m\Phi\, d\Phi; \end{aligned} \qquad \lambda \leqslant \varrho \leqslant 1, \quad 0 \leqslant \Phi \leqslant \pi. \tag{3.18}$$

By substitution of these equations and the Neumann's formula into Eqs. (3.16) and equation the coefficients of $\cos m\Phi$ in both sides and assumption of the parameters a_n as follows

$$a_n = 2CG_1\varkappa b a_n''' - c a_n'' + p_0 a_n', \tag{3.19}$$

we arrive at the three infinite systems of simultaneous equations for the determination of the parameters a_n', a_n'' and a_n'''

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n' A_{mn} = f_m^*, \\ &\sum_{n=0}^{\infty} a_n'' A_{mn} = \delta_{0m}, \\ &\sum_{n=0}^{\infty} a_n''' A_{mn} = B_m; \quad (m = 0, 1, 2, \ldots) \end{aligned} \tag{3.20}$$

with the Kronecker delta δ_{0m}, the coefficients f_m^* defined by Eq. (3.18) and the matrices

$$A_{mn} = \int_0^{\infty} [1-h(x\eta)]\frac{\partial Z_m(x)}{\partial x}\cdot\frac{\partial Z_n(x)}{\partial x}\,dx,$$

$$B_m = \int_0^{\infty} [1-h(x\eta)]\frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\partial}{\partial x}J_0(\lambda x)\right]\frac{\partial Z_m(x)}{\partial x}\,dx = \tag{3.21}$$

$$= -\frac{\lambda^4}{4}[I_0^m(\lambda)+I_2^m(\lambda)] + \frac{3\lambda^2(1+\lambda)}{2}\left[\frac{\lambda(1-\lambda)}{(1+\lambda)^2}\right]^m \frac{\Gamma[m+(1/2)]}{[\Gamma(m+1)]^2\Gamma[-m+(5/2)]}\times$$

$$\times F_4\left[m-\frac{3}{2},\ m+\frac{1}{2};\ m+1,\ m+1;\ \left(\frac{1-\lambda}{1+\lambda}\right)^2,\ \left(\frac{2\lambda}{1+\lambda}\right)^2\right] - H^m(\lambda,\eta),$$

where $F_4(\cdot\,\cdot;\,\cdot\,\cdot;\,\cdot\,\cdot)$ is hypergeometric series of two variables [4] and $\Gamma(\cdot)$ denotes the Gamma function, $I_0^m(\lambda)$ and $I_2^m(\lambda)$ are presented analytically in the authors paper [5] and the improper integrals

$$H^m(\lambda, \eta) = \lambda^2 \int_0^\infty h(x\eta) \frac{\partial Z_m(x)}{\partial x} \cdot \frac{J_2(\lambda x)}{x} dx; \quad (m = 0, 1, 2, \ldots), \tag{3.22}$$

can be evaluated numerically in finite interval, because those integrand decrease exponentially to zero with the increase of x, are continuous for any $x \in (0, \infty)$ and are bounded for $x \to 0$.

The coefficients f_m^* assume the form

$$f_m^* = \frac{1+\lambda^2}{4} \delta_{0m} - \frac{1-\lambda^2}{8} \delta_{1m}, \quad \text{for } f(\varrho) = 1,$$
$$f_m^* = \frac{1}{\pi} \varrho_0 \cos m\Phi_0, \quad \text{for } f(\varrho) = \delta(\varrho - \varrho_0) = \delta(\Phi - \Phi_0), \tag{3.23}$$

for a constant normal pressure p_0 and concentrated forces P acting at the circumference $\varrho = \varrho_0 \in (\lambda, 1)$, respectively. For $\Phi_0 = \pi/2$, i.e. $\varrho_0 = [(1+\lambda^2)/2]^{1/2}$ the values of the coefficients f_m^* are: ϱ_0/π for $m = 0, 4, 8, \ldots$; 0 for $m = 1, 3, 5, \ldots$ and $-\varrho_0/\pi$ for $m = 2, 6, 10, \ldots$. In the case of the load on the circumference $\varrho = \varrho_0$ the stress p_0 in Eq. (3.19) and subsequent equations would be replaced by P/b.

Notice that the matrix A_{mn} is symmetric and can be evaluated by the similar method as in the autor's paper [5]. The first two systems in Eqs (3.20) correspond to constant displacement in a circular region ($\varkappa = 0$ — cylindrical punch or inclusion). In the last case is $a_n = -ca_n'' + p_0 a_n'$. To evaluate the unknown constant c we make use of the conditions (3.19) and (3.11) which lead to

$$2CG_1 \varkappa b \sum_{n=0}^{\infty} a_n''' - c \sum_{n=0}^{\infty} a_n'' + p_0 \sum_{n=0}^{\infty} a_n' = 0. \tag{3.24}$$

For the constant displacement δ in the circular region $0 \leqslant \varrho \leqslant \lambda$ we have from Eq. (3.8) $a_0 = CG_1 \delta b^{-1}$ and the constant c is determined by equation

$$-ca_0'' + p_0 a_0' = CG_1 b^{-1} \delta. \tag{3.25}$$

Consequently, the presented three-part mixed boundary value problem is reduced to the solution of the simultaneous algebraic equations (3.20). If we determine a_n from Eqs. (3.20), (3.19) and (3.24) the function $t(x)$ will be presented by the equation (3.12). The infinite systems of simultaneous algebraic equations can be solved by truncation [5, 6]. As a result of the above analysis all components of displacements and stresses and the parameters which characterize the contact can be found.

4. Displacement and stress fields

The normal displacement on the upper surface of the layer is given by Eq. (3.9) and on the lower one is

$$w_z(\varrho, \eta) = 2\varkappa b^2\lambda^2 \int_0^\infty h_1(x\eta)x^{-1}J_2(x\lambda)J_0(x\varrho)dx -$$

$$-(G_1 C)^{-1} b \sum_{n=0}^{\infty} a_n \int_0^\infty h_1(x\eta)\frac{\partial Z_n(x)}{\partial x} J_0(x\varrho)dx \tag{4.1}$$

and can be rewritten to the form

$$w_z(\varrho, \eta) = \delta \frac{4\alpha}{4\alpha + c_1(\alpha^2-\beta^2)} + 2\varkappa b^2\lambda^2 \int_0^\infty (x\lambda)^{-1}J_1(x\lambda)[J_0(x\varrho)h_1'(x\eta) -$$

$$-\varrho J_1(x\varrho)h_1(x\eta)]dx + \frac{\delta - \varkappa b^2\lambda^2}{a_0}\sum_{n=0}^{\infty} a_n \int_0^\infty Z_n(x)[J_0(x\varrho)h_1'(x\eta) - \varrho J_1(x\varrho)h_1(x\eta)]dx, \tag{4.2}$$

where $h_1'(x\eta)$ is the x-derivative of the function $h_1(x\eta)$. The normal stress on the surface $\zeta = \eta$ is proportional to these displacement.

Making use of the identity

$$\int_0^\infty xJ_0(x\varrho)\frac{\partial Z_n(x)}{\partial x}dx = -\left(I_0^n + \varrho\frac{\partial}{\partial\varrho}I_0^n\right), \tag{4.3}$$

where the improper integrals

$$I_0^n = \int_0^\infty J_0(x\varrho)Z_n(x)dx \tag{4.4}$$

are presented analytically in the author's paper [5], the stresses $\sigma_z(\varrho, 0)$ in Eq. (3.14) can be rewritten as follows

$$\sigma_z(\varrho, 0) = -\sum_{n=0}^{\infty} a_n\left[I_0^n + \varrho\frac{\partial}{\partial\varrho}I_0^n + h^n(\varrho; \lambda, \eta)\right] +$$

$$+2CG_1\varkappa b\lambda\left[\lambda h(\varrho; \lambda, \eta) + \begin{cases} -F\left(\frac{3}{2}, -\frac{1}{2}; 1; \frac{\varrho^2}{\lambda^2}\right); & 0 \leqslant \varrho \leqslant \lambda, \\ \frac{1}{\pi}; & \lambda = \varrho, \\ \frac{1}{8}\left(\frac{\lambda}{\varrho}\right)^3 F\left(\frac{3}{2}, \frac{3}{2}; 3; \frac{\lambda^2}{\varrho^2}\right); & \lambda \leqslant \varrho \end{cases}\right] \tag{4.5}$$

The symbols $F(\cdot, \cdot; \cdot; \cdot)$ denote the common Gaussian hypergeometric series and $h^n(\varrho; \lambda, \eta)$, $h(\varrho; \lambda, \eta)$ are the improper integrals defined as follows

$$h^n(\varrho;\ \lambda,\ \eta) = \int_0^\infty xh(x\eta)J_0(x\varrho)\frac{\partial Z_n(x)}{\partial x}dx,$$

$$h(\varrho;\ \lambda,\ \eta) = \int_0^\infty h(x\eta)J_2(x\lambda)J_0(x\varrho)\,dx, \tag{4.6}$$

which can be numerically evaluated at the finite interval, because the function $h(x\eta)$ tends exponentially to zero as x tends to infinity. The maximum value of the contact stress at the center $\varrho = 0$ is

$$[\sigma_z(\varrho,0)]_{max} = \sigma_z(0,0) = -\frac{2}{\pi}\frac{1}{1+\lambda}\sum_{n=0}^{\infty} a_n\left(\frac{1-\lambda}{1+\lambda}\right)^n \frac{\Gamma[n+(1/2)]}{\Gamma(n+1)}\times$$

$$\times F\left(\frac{1}{2},\ n+\frac{1}{2};\ n+1;\ \left(\frac{1-\lambda}{1+\lambda}\right)^2\right) - \sum_{n=0}^{\infty} a_n h^n(0;\ \lambda,\ \eta)+$$

$$+2CG_1\varkappa b\lambda[\lambda h(0;\ \lambda,\ \eta)-1], \tag{4.7}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

On the other hand, making use of the asymptotic expansion of $J_n(x\varrho)$ with large value of x, we obtain

$$x\frac{\partial Z_n(x)}{\partial x} \doteqdot -\frac{2}{\pi\sqrt{1-\lambda^2}}[\lambda\sin x\lambda-(-1)^n\cos x]+0(x^{-1}) \tag{4.8}$$

and

$$\int_0^\infty x\frac{\partial Z_n(x)}{\partial x}J_0(x\varrho)\,dx = \int_0^\infty \left\{x\frac{\partial Z_n(x)}{\partial x}+\frac{2}{\pi\sqrt{1-\lambda^2}}[\lambda\sin x\lambda-(-1)^n\cos x]\right\}J_0(x\varrho)\,dx-$$

$$-\frac{2}{\pi\sqrt{1-\lambda^2}}\left[\frac{\lambda H(\lambda-\varrho)}{\sqrt{\lambda^2-\varrho^2}}-(-1)^n\frac{H(\varrho-1)}{\sqrt{\varrho^2-1}}\right] \tag{4.9}$$

where the values of the Weber-Schafheitlin integrals were employed.

Then, the normal stress in Eq. (3.14) can be represented as

$$\sigma_z(\varrho,0) = \sum_{n=0}^{\infty} a_n\int_0^\infty \left[x\frac{\partial Z_n(x)}{\partial x}+\frac{2}{\pi\sqrt{1-\lambda^2}}(\lambda\sin x\lambda-(-1)^n\cos x)\right]J_0(x\varrho)\,dx-$$

$$-\frac{2}{\pi\sqrt{1-\lambda^2}}\left[\frac{\lambda H(\lambda-\varrho)}{\sqrt{\lambda^2-\varrho^2}}\sum_{n=0}^{\infty}a_n-\frac{H(\varrho-1)}{\sqrt{\varrho^2-1}}\sum_{n=0}^{\infty}(-1)^n a_n\right]-$$

$$-\sum_{n=0}^{\infty}-a_n h^n(\varrho;\ \lambda,\ \eta)+2G_1C\varkappa b\lambda[\lambda h(\varrho;\ \lambda,\ \eta)+ \tag{4.10}$$

$$+\left\{\begin{array}{ll} -F\left(\frac{3}{2}, -\frac{1}{2}; 1; \frac{\varrho^2}{\lambda^2}\right); & 0 \leqslant \varrho \leqslant \lambda, \\ \frac{1}{\pi}; & \varrho = \lambda \\ \frac{1}{8}\left(\frac{\lambda}{\varrho}\right)^3 F\left(\frac{3}{2}, \frac{3}{2}; 3; \frac{\lambda^2}{\varrho^2}\right); & \lambda \leqslant \varrho \end{array}\right\}\Bigg]\Bigg] \qquad [4.10] \text{ [cont.]}$$

where $H(\cdot)$ is the Heaviside's function.

The first series in equation (4.10) is finite, the second must be zero because the contact stress is finite at $\varrho \to \lambda^-$ for a identer with a smooth curved base (this condition corresponds to Eqs. (3.10), (3.11)) or has a sinularity when $\varrho \to \lambda^-$ for a identer with a flat base or corners, the third series has a singularity when $\varrho \to 1^+$ in both cases and the others terms are nonsingular.

The physical quantity of interest is the stress intensity factor L_b which is defined as

$$L_b = \sqrt{2b} \lim_{\varrho \to 1^+} \sqrt{\varrho - 1}\, \{\sigma_z(\varrho, 0)\}_{\varrho > 1}. \tag{4.11}$$

Using Eq. (4.10) the stress intensity factor can be expressed in terms of a_n as

$$L_b = \frac{2\sqrt{b}}{\pi\sqrt{1-\lambda^2}} \sum_{n=0}^{\infty} (-1)^n a_n. \tag{4.12}$$

The stresses decrease from the maximum value to zero in the interval $\varrho \in \langle 0, \lambda \rangle$, where are always compressive (for a indenter with a smooth curved base), are given by — $p_0 f(\varrho)$ in the interval $\varrho \in (\lambda, 1)$ and decrease from infinity to zero in the remaining interval $\varrho > 1$, where are tensile. Notice that the stress is finite inside the contact region and has the desired square-root singularity at $\varrho = 1^+$.

Use is made of families of above solutions, as described in the following sections.

5. Punch problem

A punch problem is a particular case of the more general case considered in the previous sections and the formulae obtained there can give its immediate solution.

If $\varkappa b^2 \varrho^2$ denotes the shape of the punch, δ is measure of the penetration of the punch, the boundary radius of the contact region λ is given by Eq. (3.8) and the total load on the punch is given by

$$P = \pi a^2 \sum_{n=0}^{\infty} a_n \{I_0^n(\lambda) - I_2^n(\lambda) + H^n(\lambda, \eta)\} + \frac{8}{3} G_1 C \varkappa a^3 \left[1 - \frac{3}{2}\pi H(\lambda, \eta)\right], \tag{5.1}$$

where the improper integrals

$$H^n(\lambda, \eta) = 2\lambda^{-1} \int_0^{\infty} h(x\eta) J_1(x\lambda) \frac{\partial Z_n(x)}{\partial x}\, dx,$$

$$H(\lambda, \eta) = \int_0^{\infty} x^{-1} h(x\eta) J_2(x\lambda) J_1(x\lambda)\, dx, \tag{5.2}$$

are convergent and equal to zero for a half-space problem.

The relevant solutions of the special cases are summarized: (i) Indentation of a layer or a half-space by a cylindrical punch.

The curvature $\varkappa$ is zero for a punch with a flat base. Apart from the displacement and stress there are four parameters which characterize the contact; these relate to total load P, the central displacement $w_z(0,\eta)$ on the lower surface of the layer and the stress singularities at the outer and inner boundaries of the annulus. It may be shown that these are:

$$P = \pi CG_1\delta a\left\{\frac{4}{\pi}\cdot\frac{\lambda}{1+\lambda}\left[F\left(\frac{1}{2},\frac{1}{2};1;\frac{1-\lambda}{1+\lambda}\right)\right]^2+\right.$$
$$+\left(\frac{\lambda}{1+\lambda}\right)^3 F\left(\frac{3}{2},\frac{3}{2};1;\frac{1-\lambda}{1+\lambda}\right)F\left(\frac{3}{2},\frac{3}{2};3;\frac{2\lambda}{1+\lambda}\right)+\lambda H^0(\lambda,\eta)+$$
$$\left.+\lambda\sum_{n=1}^{\infty}a_n[I_0^n(\lambda)-I_2^n(\lambda)+H^n(\lambda,\eta)]\right\}, \tag{5.3}$$

$$\frac{w_z(0,\eta)}{\delta} = \frac{4\alpha}{4\alpha+c_1(\alpha^2-\beta^2)}+\frac{1}{a_0}\sum_{n=0}^{\infty}a_n\int_0^{\infty}h_1'(x\eta)Z_n(x)\,dx, \tag{5.4}$$

$$L_a = -\frac{2}{\pi}\sqrt{\frac{a}{1-\lambda^2}}\left[CG_1\frac{\delta}{b}+\sum_{n=1}^{\infty}a_n\right],$$
$$L_b = \frac{2}{\pi}\sqrt{\frac{b}{1-\lambda^2}}\left[CG_1\frac{\delta}{b}+\sum_{n=1}^{\infty}(-1)^n a_n\right], \tag{5.5}$$

where

$$a_n = CG_1\frac{\delta}{b}\frac{a_n''}{a_0''}-p_0 a_0'\left(\frac{a_n''}{a_0''}-\frac{a_n'}{a_0'}\right) \tag{5.6}$$

and the parameters a_n', a_n'' are the roots of the first and second systems of algebraic equations (3.20).

The contact stress is given by series

$$\sigma_z(\varrho,0) = -CG_1\frac{\delta}{a}\lambda\left\{I_0^0+\varrho\frac{\partial}{\partial\varrho}I_0^0+h^0(\varrho;\ \lambda,\eta)\right\}-$$
$$-\sum_{n=1}^{\infty}a_n\left\{I_0^n+\varrho\frac{\partial}{\partial\varrho}I_0^n+h^n(\varrho;\ \lambda,\eta)\right\} \tag{5.7}$$

and has the minimum value

$$\sigma_z(0,0) = -2CG_1\frac{\delta}{a}\left[\frac{\lambda}{1+\lambda}F\left(\frac{1}{2},\frac{1}{2};1;\left(\frac{1-\lambda}{1+\lambda}\right)^2\right)+\lambda h^0(0;\ \lambda,\eta)\right]-$$

$$-\frac{2}{\sqrt{\pi}}\cdot\frac{1}{1+\lambda}\sum_{n=1}^{\infty}a_n\left(\frac{1-\lambda}{1+\lambda}\right)^n\frac{\Gamma[n+(1/2)]}{\Gamma(n+1)}F\left(\frac{1}{2},n+\frac{1}{2};n+1;\left(\frac{1-\lambda}{1+\lambda}\right)^2\right)-$$

$$-\sum_{n=1}^{\infty}a_n h''(0;\lambda,\eta). \tag{5.8}$$

Taking $h \gg b(\eta\to\infty)$, we have $H''(\lambda,\eta) = h''(\varrho;\lambda,\eta) = 0$ and the above solution leads to the one of a half-space problem. In order taking $p_0 = 0$, we obtain the solution of the problem in which the annular region $\lambda < \varrho < 1$ is stress free. Similarly, if $p_0 = 0$ and $b \gg a$ $(\lambda \to 0)$, we can also obtain, by evaluation of the limit under $\lambda \to 0$ the solution corresponding to stress free surface outside the contact region. Notice that if $f(\varrho) = 1$, which corresponds to the constant pressure in the annular region and $b \gg a(\lambda\to 0)$, but b is bounded in the half-space problem the roots of the system $(3.20)_1$ have the form of the set as [6]

$$a_n' = -\frac{4}{\pi(4n^2-1)(1+\delta_{n0})}, \quad (n = 0, 1, 2, \ldots), \tag{5.9}$$

which satisfies

$$\sum_{n=0}^{\infty}(-1)^n a_n' = 1, \quad \sum_{n=0}^{\infty}a_n' = 0, \tag{5.10}$$

and the parameters a_n are

$$a_n = -\frac{2}{\pi}p_0\left[\frac{2}{(4n^2-1)(1+\delta_{n0})}+\frac{a_n''}{a_0''}\right]+CG_1\frac{\delta}{b}\frac{a_n''}{a_0''}. \tag{5.11}$$

In this limiting case the stress intensity factors are

$$L_a = -\frac{2}{\pi}\sqrt{a}\left(CG_1\frac{\delta}{b}-\frac{2}{\pi}p_0\right)\left(1+\frac{1}{a_0''}\sum_{n=1}^{\infty}a_n''\right),$$

$$L_b = \frac{2}{\pi}\sqrt{b}\left[p_0+\left(CG_1\frac{\delta}{b}-\frac{2}{\pi}p_0\right)\left(1+\frac{1}{a_0''}\sum_{n=1}^{\infty}(-1)^n a_n''\right)\right], \tag{5.12}$$

where the parameters a_n'' are the roots of the equations

$$\sum_{n=0}^{\infty}a_n''\int_0^{\infty}\frac{\partial}{\partial x}\left[J_m^2\left(\frac{x}{2}\right)\right]\frac{\partial}{\partial x}\left[J_n^2\left(\frac{x}{2}\right)\right]dx = \delta_{0m}, \quad (m = 0, 1, 2, \ldots). \tag{5.13}$$

It is interesting to note that the presence of compressive outside stresses makes indentation easier while tensile stresses make indentation harder. For example, if $2p_0/\pi = CG_1\delta/b$ the preceding stress intensity factors tend to zero and to the value in the classical penny-shaped crack problem, respectively.

Only in the limiting case of the half-space problem with stress free surface outside the contact region we can obtained from the above mentioned results the closed-form solution. It is

$$P = 4CG_1\,\delta a, \quad \sigma_z(\varrho, 0) = -\frac{2}{\pi}\,CG_1\,\frac{\delta}{a}\,\frac{1}{\sqrt{1-\varrho^2}}\,H(1-\varrho),$$

$$w_z(\varrho, 0) = \delta\left[H(1-\varrho)+\frac{2}{\pi}\arcsin\left(\frac{1}{\varrho}\right)H(\varrho-1)\right]; \quad \varrho = r/a, \tag{5.14}$$

$$L_a = -\frac{2}{\pi}\,CG_1\,\frac{\delta}{a}\sqrt{a}, \quad L_b \to 0.$$

(ii) Two punches, stress free lower surface.

The ratio $c_1 = c_0 h/CG_1$ describes the relative rigidity of foundation to layer; it being zero when the lower surface is stress free and infinitely large for a perfectly rigid base. *Using the functions $h(x\eta)$ and $h_1(x\eta)$, in Eq.* (2.23), which for these limiting cases are reducible, we obtain the solution of the contact problem for a thick plate of height $2h$ by a pair of the same punches, which are pressed onto both surfaces of the plate and the solution corresponding to the stress free lower surface of the layer respectively.

(iii) Concave punch.

If the method is applied to concave punch, then using Eqs. (3.8) and (3.19) the parameters a_n are found to be

$$a_n = CG_1 b^{-1}\delta\frac{a_n''}{a_0''}-p_0 a_0'\left(\frac{a_n''}{a_0''}-\frac{a_n'}{a_0'}\right)+2CG_1 b^{-1}u\lambda^{-2}a_0'''\left(\frac{a_n''}{a_0''}-\frac{a_n'''}{a_0'''}\right) \tag{5.15}$$

where $u = -\varkappa a^2$ and δ are the measures of the concavity of the base of the punch and the penetration of the punch at $\varrho = \lambda$, respectively. The stress concentration factors at $\varrho = \lambda$ and $\varrho = 1$ are given by equations (5.5) and (5.15).

Only in the limiting case of the half-space problem with stress free surface outside the contact region we obtain from the above mentioned results the closed-form solution. It may be shown that these are

$$P = 4CG_1 a\left(\delta-\frac{1}{3}u\right), \quad \left(\geqslant P_0 \equiv \frac{32}{3}\,G_1 Cau\right)$$

$$\sigma_z(\varrho, 0) = -\frac{2}{\pi}\,CG_1 a^{-1}\,\frac{\delta-3u+4u\varrho^2}{\sqrt{1-\varrho^2}}, \quad \delta \geqslant 3u; \quad \varrho = \frac{r}{a}, \tag{5.16}$$

$$w_z(\varrho, 0) = [\delta-u(1-\varrho^2)]H(1-\varrho)+\frac{2}{\pi}\left\{[\delta-u(1-\varrho^2)]\arcsin\frac{1}{\varrho}-u\sqrt{\varrho^2-1}\right\}H(\varrho-1)$$

The condition that the entire punch surface makes contant with the half-space is $\sigma_z(\varrho, 0) \leqslant 0$ in $0 \leqslant \varrho \leqslant 1$. Then we obtain the condition $\delta \geqslant 3u$. The critical load P_0 means the minimum indented load for the entire face to contact. If $P < P_0$ or $\delta < 3u$, the contact region becomes annular. The above equations are valid for circular contact region. Additionally, if $u < 0$ and $\delta \geqslant -u$, the punch face is convex and the stress $\sigma_z(\varrho, 0)$ is always compressive on the contact region. If $u < 0$ and $\delta < -u$ the stress $\sigma_z(\varrho, 0)$ is compressive without the neighbourhood of $\varrho = 1$. The physical aspects of the corresponding isotropic problem are discussed by Barber [7], Shibuya [8] and for transversely isotropic material by author [9].

(iV) Parabolic punch

The general case of parabolic punch was presented above. Notice only, that these results for the half-space problem with stress free surface outside the contact region by purely limiting manipulations lead to exact solution

$$\delta = 2\varkappa a^2, \quad P = \frac{16}{3}\varkappa C G_1 a^3,$$

$$\sigma_z(\varrho, 0) = -\frac{4}{\pi} C G_1 \frac{\delta}{a}\sqrt{1-\varrho^2}; \quad 0 \leqslant \varrho \leqslant 1, \tag{5.17}$$

$$w_z(\varrho, 0) = \frac{\delta}{\pi}\left[(2-\varrho^2)\arcsin\left(\frac{1}{\varrho}\right) + \sqrt{\varrho^2-1}, \quad \varrho > 1; \quad \varrho = \frac{r}{a}\right.$$

6. Crack problem

The stress distribution produced by the indentation of a penny-shaped crack by an inclusion and tractions in a transversely isotropic layer can be investigated using the above mentioned results.

If we consider a layer of height $2h$ weakened by a penny-shaped crack of radius b located in the middle plane of the layer and opened by a thin symmetric rigid inclusion of profile $z = \pm(\delta - \varkappa b^2\varrho^2)$ and by tractions acting outside, then in such a crack problem formulae obtained in the previous sections can give its solution. By virtue of linear superposition, the stress field is equivalent to the field generated in the crack faces that are equal in magnitude but opposite in sign to the corresponding tractions in the uncracked layer. The last displacement and stress fields are obtained as the particular solution of the equilibrium equations. In particular: $u_{r0} = p_0 r c_{13}/c$, $w_{z0} = -p_0 z(c_{11}+c_{12})/c$, $\sigma_{z0} = -p_0$, if on the clamped-free faces of the layer the pressure p_0 is prescribed. Here c is a combination of the elastic constants, equal to $c = c_{33}(c_{11}+c_{12}) - 2c_{13}^2$.

Apart from the displacement and stress there are the stress intensity factors which characterize the crack problem. These are given by Eq. (4.12) for smooth inclusion and by Eq. (5.5) for cylindrical inclusion. In the special cases the stress intensity factors at the inner and outer boundaries of the annulus $a \leqslant r \leqslant b$ are given by

$$L_a = -\frac{2}{\pi}\frac{\delta G_1 C}{b}\sqrt{\frac{a}{1-\lambda^2}}\left[1+\frac{1}{a_0''}\sum_{n=1}^{\infty} a_n''\right],$$

$$L_b = \frac{2}{\pi}\frac{\delta G_1 C}{b}\sqrt{\frac{b}{1-\lambda^2}}\left[1+\frac{1}{a_0''}\sum_{n=1}^{\infty}(-1)^n a_n''\right], \tag{6.1}$$

when the crack is opened only by cylindrical inclusion ($p_0 = 0$) and

$$L_a = -\frac{2p_0}{\pi}\sqrt{\frac{a}{1-\lambda^2}}\left(\frac{\sum_{n=0}^{\infty}(-1)^n a_n'}{\sum_{n=0}^{\infty}(-1)^n a_n''}\sum_{n=0}^{\infty} a_n'' - \sum_{n=0}^{\infty} a_n'\right), \tag{6.2}$$

$$L_b = 0,$$

for cylindrical inclusion and pressure

$$p_0 \geqslant p_{0cr} = CG_1 \frac{\delta}{ba_0'} \cdot \frac{1+\frac{1}{a_0''}\sum_{n=1}^{\infty}(-1)^n a_n''}{\sum_{n=1}^{\infty}(-1)^n\left(\frac{a_n'}{a_0'}-\frac{a_n''}{a_0''}\right)}, \tag{6.3}$$

where the critical load p_{0cr} means the minimum pressure for the tip $\varrho = \lambda$ of the crack to contact. When the pressure at the layer surface is above the critical load we have an annular contact region between the crack faces, the outer circumference of which coincides with the crack tip and the inner radius of which will shrink with increasing load.

The crack problem for high loads can therefore be treated by contact problem of the layer and rigid base with protrusion, which is discussed in the other author's paper [6].

The stress in ensity factor L_a is allways negative, decreases with the increase of the external pressure to value (6.2) and for the pressure above the value (6.3) depends on the load as in the contact problem [6]. In contrast L_b is positive and decreases to zero.

On the other hand, in the case of the tension of the layer with cylindrical inclusion in the crack, there may be cases that the inclusion surface makes partially contact or does not make contact with the elastic medium. Let p_{0cr}' be the tension giving the state in which the layer contacts the surfaces of the inclusion without neighbourhood of the point $\varrho = 0$. Then for thick layer Eq. (5.6) and (5.8) give

$$p_{0cr}' = \frac{I_0^0(0)+\frac{1}{a_0''}\sum_{n=1}^{\infty} a_n'' I_0^n(0)}{1+a_0'\sum_{n=1}^{\infty} I_0^n(0)\left(\frac{a_n'}{a_0'}-\frac{a_n''}{a_0''}\right)}. \tag{6.4}$$

When the tensile load p_0 is above the critical value (6.4), the contact area will be an annulus, the inner circumference of which increases with the increase of the load, and when the load is above some value p_{0cr}'' the elastic body does not make contact with the surface of the inclusion. These critical loads can be found from the condition $w_c(\lambda, 0) = \delta$, where $w_c(\lambda, 0)$ denotes the displacement of the penny-shaped crack in the point $\varrho = \lambda$. For a thick layer these critical loads are given by formula

$$p_{0cr}'' = \frac{\pi}{2} G_1 C \frac{\delta}{b} \cdot \frac{1}{\sqrt{1-\lambda^2}}. \tag{6.5}$$

Very interesting case in which the tension is in the interval $p_{0cr}' < p_0 < p_{0cr}''$ and the contact region on the cylindrical inclusion is annular is not included in our solutions.

By substitution of Eq. (6.5) for $\lambda = 0$ into Eq. (5.12) we obtain $L_a \to 0$ and $L_b = 2p_0\sqrt{b}/\pi$.

The physical aspects of the corresponding isotropic problem are discussed by Tweed [10] and Gladwell [11].

7. A rigid inclusion pressed between two layers

The solution presented here may be applied to the following problem: two identical transversely isotropic layers are pressed together by a pressure; a rigid obstacle lies between the layers. The solution for each layer may be obtained as the superposition of two fields. The first corresponds to a stress field in the z-direction namely $\sigma_z(r, z) = -p_0 f(r)$. The second may be expressed by means of the above mentioned results provided that the compatibility equation $\{dw_z(r, 0)/dr\}_{r=b} = 0$ is satisfied. This is the equation giving b, the extent of the contact region. Thus, on using Eqs. (3.5) and (3.7) we find $F(\pi) = 0$ or

$$\sum_{n=0}^{\infty} (-1)^n a_n = 0, \tag{7.1}$$

which with the aid of the notation (3.19) may be written

$$2CG_1 \varkappa b \sum_{n=0}^{\infty} (-1)^n a_n''' - c \sum_{n=0}^{\infty} (-1)^n a_n'' + p_0 \sum_{n=0}^{\infty} (-1)^n a_n' = 0. \tag{7.2}$$

Assuming the ratio of the inner to the outer radius of the uncontact region $\lambda = a/b$ and the ratio $\eta = h/b$ an solving for given external pressure distribution the equations (3.20) we can obtain the parameters a_n', a_n'' and a_n'''. Then, Eqs. (7.2) and (3.24) yield the value of the pressure p_0 and Eqs (3.8) and (3.19) the value of δ in terms of known quantities; the values which give this contact state.

Full details of the other corresponding problems may be found in the articles by Alblas [12], Gladwell [13, 14] and author [6].

8. Isotropic case

All the results obtained in this paper can also be applied for completely isotropic bodies.

Setting $\alpha = s_1 + s_2 = 2$ and evaluating the limit under $\beta = s_1 - s_2 \to 0$ in Eq. (2.23), we get

$$\begin{Bmatrix} h(x)-1 \\ h_1(x) \end{Bmatrix} = \frac{1}{\sinh 2x + 2x + c_1 x^{-1}(\cosh 2x - 1)} \times \\ \times \begin{Bmatrix} -[\cosh 2x - 2x^2 - 1 + c_1 x^{-1}[\sinh 2x + 2x)] \\ 2(\sinh x + x\cosh x) \end{Bmatrix}. \tag{8.1}$$

For an isotropic material the parameter C reduces to $(1-\nu)^{-1}$ and the relative rigidity of the foundation to the layer is $c_1 = c_0 h(1-\nu)/G$. Here G is the shear modulus and ν is Poisson's ratio.

9. Conclusions

It has been demonstrated that a large class of unbonded contact problems may be reduced to the solution of the infinite systems of simultaneous algebraic equations.

On the basis of the presented results the effect of arbitrary loading outside of an indenter, of the boundary conditions and transverse anisotropy on the contact behaviour and the load-contact length relation can be clarified.

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Резюме

КОНТАКТНАЯ ЗАДАЧА ЖЁСТКОГО ТЕЛА И ТРАНСВЕРСАЛЬНО-ИЗОТРОПНОГО СЛОЯ

Рассматривается задача трансверсально-изотропного слоя контактируемого с жёстким телом. Нижний край слоя упруго подпёртый. На верхней площадке дано нормальное перемещение внутри круговой области с неизвестным радиусом, вокруг которого выступают нормальные напряжения в области кольца и исчезают нормальные перемещения на остальной части верхнего края слоя.

Задача сформулирована как решение тройных интегральных уравнений. При решении этих уравнений используются дифференциальное, интегральное и рядовое представления неизвестной функции, которая удовлетворяет два из трёх уравнений точно, в то время как третье ведёт к трём бесконечным системам алгебраических уравнений относительно коэффициентов введенных в представлении.

Физические величины, которые характеризуют контакт и коэффициенты интенсивности напряжения представлены при помощи коэффициентов — решении алгебраических уравнений.

Рассмотрены некоторые задачи о штампе, включении и трещине в трансверсально-изотропном слое.

Streszczenie

KONTAKT MIĘDZY SZTYWNYM CIAŁEM I POPRZECZNIE IZOTROPOWĄ WARSTWĄ

Rozpatrzono zagadnienie warstwy poprzecznie izotropowej kontaktującej się z ciałem sztywnym. Dolna płaszczyzna warstwy jest sprężyście podparta. Na górnej powierzchni warstwy dane jest normalne przemieszczenie wewnątrz kołowego obszaru o nieznanym promieniu; na zewnątrz tego obszaru występują normalne naprężenia, a na pozostałej części tej powierzchni przemieszczenia normalne są równe zeru. Na obu brzegach warstwy naprężenia styczne nie występują.

Autor sformułował zagadnienie jako rozwiązanie potrójnych równań całkowych. W celu rozwiązania ich wprowadzono taką różniczkową, całkową i szeregową reprezentację poszukiwanej funkcji, która spełnia dwa z trzech równań ściśle, podczas gdy trzecie równanie prowadzi do trzech nieskończonych układów równań algebraicznych względem współczynników wprowadzonych w reprezentacji. Fizyczne wielkości, które charakteryzują kontakt oraz współczynniki intensywności naprężenia wyznaczono za pomocą rozwiązań układów równań algebraicznych.

Rozpatrzono pewne zagadnienia stempla, inkluzji i szczeliny dla poprzecznie izotropowej warstwy.

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