SOLUTION OF SHALLOW SHELLS BY BOUNDARY ELEMENT METHOD. PROBLEM OF CORNERS

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1. Introduction

Boundary Element Method (BEM) is one of numerical methods which is finding now greater attention in mechanics [1, 2]. BEM is applied for the solution of the boundary problems described by the integral equations. The adventage of BEM which causes its computational applications is, that the BEM reduces geometrical size of the problem by one and consequently reduces the time of calculations. The unknown quantities are boundary displacements or forces depending on how the edge is supported. The solution of wider class of shells is restricted to the possibility of obtaining fundamental solutions (Green's function).

We will discuss the thin, shallow, spherical shell, which middle surface can be described approximatly by equation [5]:

$$f(x_1, x_2) = -\frac{1}{2}k(x_1^2 + x_2^2),\qquad (1.1)$$

where $k = 2d/a^2$ — constant curvature, d — height of the shell, a — radius of the bottom of the shell.

The fundamental solution is given in [4] and complete solution in [8].

The integral representation of the equilibrium equations of shells can be obtained by making use of Betti-Maxwell's reciprocal theorem [6]. If one of the state of forces is unit loading and corresponding fundamental solution is one of the state of the displacements and after making use of the properties of the Dirac's function we obtain the Somigliano's formulas for displacements:

$$u_j(G) = \int_C [T_i(K)\bar{u}_{ij}(K, G) + M_n(K)\bar{\varphi}_{nj}(K, G)]dC - \int_C [\bar{T}_{ij}(K, G)u_i(K) +$$

$$+ \bar{M}_{nj}(K, G)\varphi_n(K)]dC + \int_A P_i(F)\bar{u}_{ij}(F, G)dA + \sum_{r=1}^{k} R_r(K)\bar{u}_{3j}^r(K, G),\quad (1.2)$$

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$$\varphi_\mu(G) = \int\limits_C [T_i(K)\bar{u}_{i4}(K, G) + M_n(K)\bar{\varphi}_{n4}(K, G)]dC - \int\limits_C [\overline{T}_{i4}(K, G)u_i(K) +$$

$$+ \overline{M}_{n4}(K, G)\varphi_n(K)]dC + \int\limits_A P_i(F)\bar{u}_{i4}(F, G)dA + \sum_{r=1}^{k} R_r(K)\bar{u}_{34}^r(K, G), \qquad \begin{array}{l}(1.2)\\ \text{[cont]}\end{array}$$

where $\overline{T}_{i4} = \overline{T}_{i3,\mu}$, $\bar{u}_{i4} = \bar{u}_{i3,\mu}$, $\bar{\varphi}_{n4} = \bar{\varphi}_{n3,\mu}$, μ — arbitrary direction, C — boundary of the shell. A — the projection of the middle surface on the plane x_1, x_2, F — arbitrary point of the middle surface, G — point of application of the concetrated unit load, K — the searched point, the known fundamental solution is marked by bar.

2. Boundary integral equations

We obtain the boundary integral equations passing from inner point G of a shell to the point M which belongs to the shell edge. The formulas (1.2) become boundary integral equations:

$$\alpha u_j(M) = \int\limits_C [T_i(K)\bar{u}_{ij}(K, M) + M_n(K)\bar{\varphi}_{nj}(K, M)]dC - \int\limits_C [\overline{T}_{ij}(K, M)u_i(K) +$$

$$+ \overline{M}_{nj}(K, M)\varphi_n(K)]dC + \int\limits_A P_i(F)\bar{u}_{ij}(F, M)dA + \sum_{r=1}^{k} R_r(K)\bar{u}_{3j}^r(K, M),$$

$$\beta\varphi_n(M) = \int\limits_C [T_i(K)\bar{u}_{i4}(K, M) + M_n(K)\bar{\varphi}_{n4}(K, M)]dC - \int\limits_C [\overline{T}_{i4}(K, M)u_i(K) + \qquad (2.1)$$

$$+ \overline{M}_{n4}(K, M)\varphi_n(K)]dC + \int\limits_A P_i(F)\bar{u}_{i4}(F, M)dA + \sum_{r=1}^{k} R_r(K)\bar{u}_{34}^r(K, M),$$

where n — normal vector to the edge.

When we calculate the boundary forces or boundary displacements in the point $K = M$ arise the singularities, which can be avoided. The coefficients α and β in formulas (2.1) are equal 0.5 on smooth edge [7]. In the corner points Q the values of those coefficients are depending on the angle ω of the corner [7, 8].

3. Problem of corners

In the corners arise the singularities. The singularity order is the same as of the plate [3]. Therefore the corner coefficients α and β were determined by replacing the shell fundamental functions of forces, moments and displacements by corresponding plate fundamental solutions:

$$\bar{u}_{33} = \frac{1}{8\pi D} r^2 \ln r, \qquad (3.1)$$

$$\bar{u}_{34} = \bar{u}_{33,\xi_1}\mu_1 + \bar{u}_{33,\xi_2}\mu_2,$$

where $D = Eh^3/12(1 - \nu^2)$.

For $j = 3$ in the corner point Q the boundary integral equations have the form:

$$\alpha u_3(Q) = \int_C [T_i(K)\bar{u}_{l3}(K, Q) + M_n(K)\bar{\varphi}_{n3}(K, Q)]dC - \int_C [\bar{T}_{a3}(K, Q)u_a(K) +$$

$$+ \bar{T}_{33}(K, Q)u_3(K) + \bar{M}_{n3}(K, Q)\varphi_n(K)]dC + \int_A P_i(F)\bar{u}_{l3}(F, Q)dA + \sum_{r=1}^{k} R_r\bar{u}_{33}^r(K, Q),$$

(3.2)

$$\beta\varphi_n(Q) = \int_C [T_i(K)\bar{u}_{l4}(K, Q) + M_n(K)\bar{\varphi}_{n4}(K, Q)]dC - \int_C [\bar{T}_{a4}(K, Q)u_a(K) +$$

$$+ \bar{T}_{34}(K, Q)u_3(K) + \bar{M}_{n4}(K, Q)\varphi_n(K)]dC + \int_A P_i(F)\bar{u}_{l4}(F, Q)dA + \sum_{r=1}^{k} R_r\bar{u}_{34}^r(K, Q),$$

(3.3)

where $a = 1,2$.

We make use of the properties of the function $\bar{T}_{34}$ resulting from equilibrium equation:

$$\int_C T_{34}(K, Q)dC = 0,$$

(3.4)

then the equation (3.3) becomes.

$$\beta\varphi_n(Q) = \int_C [T_i(K)\bar{u}_{l4}(K, Q) + M_n(K)\bar{\varphi}_{n4}(K, Q)]dC - \int_C \{\bar{T}_{a4}(K, Q)u_a(K) +$$

$$+ \underline{\bar{T}_{34}(K, Q)[u_3(K) - u_3(Q)]} + \bar{M}_{n4}(K, Q)\varphi_n(K)\}dC + \int_A P_i(F)\bar{u}_{l4}(F, Q)dA +$$

$$+ \sum_{r=1}^{k} R_r(K)\bar{u}_{34}^r(K, Q).$$

(3.5)

In this way the order of the singularity in the underlined expression in (3.5) has been lowered.

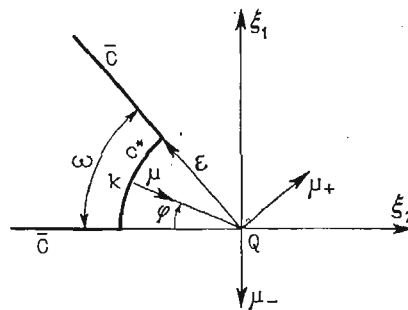The local coordinate system ξ_1, ξ_2 (Fig. 1.) is assumed.

Fig. 1. The local coordinate system

where:

$$\xi_1 = -\varepsilon\cos\varphi,$$
$$\xi_2 = \varepsilon\sin\varphi,$$
$$\mu(K) = [\cos\varphi, -\sin\varphi],$$
$$\mu_-(Q) = [0, -1],$$
$$\mu_+(Q) = [\sin\omega, \cos\omega].$$

(3.6)

In the surroundings of the point Q the boundary curve C of the shell is divided on boundary curve $\overline{C}$ and C^* (C^* is a circular sector around the point Q in diameter ε, Fig. 1.). The displacement function in the surroundings of point Q has been approximated by the linear function in the form:

$$u_3(K) = \frac{1}{\sin\omega}(u_{3\mu+}+\cos\omega\cdot u_{3\mu-})\xi_1-u_{3\mu-}\xi_2+u_3(Q),$$

$$\varphi_n(K) = \frac{\partial u_3(K)}{\partial n} = \frac{1}{\sin\omega}(u_{3\mu+}+\cos\omega\cdot u_{3\mu-})-u_{3\mu-}. \tag{3.7}$$

The boundary forces expressed in polar coordinates in this point are:

$$\overline{T}_{33}(K,\overline{Q}) = -\overline{Q}_r-\frac{1}{r}\,\overline{M}_{r\varphi,\varphi},$$

$$\overline{Q}_r = -D\left[\ddot{u}_{33,rr}+\frac{1}{r}\,\ddot{u}_{33,r}+\frac{1}{r^2}\,\ddot{u}_{33,\varphi\varphi}\right]_{,r},$$

$$\overline{M}_{r\varphi} = -(1-\nu)D\left(\frac{1}{r}\,\ddot{u}_{33,\varphi}\right)_{,r},$$

$$\overline{T}_{34}(K,Q) = -Q_r^*-\frac{1}{r}M_{r\varphi,\varphi}^* \tag{3.8}$$

$$Q_r^* = -D\left[\ddot{u}_{34,rr}+\frac{1}{r}\,\ddot{u}_{34,r}+\frac{1}{r^2}\,\ddot{u}_{34,\varphi\varphi}\right]_{,r},$$

$$M_{r\varphi}^* = -(1-\nu)D\left(\frac{1}{r}\,\ddot{u}_{34,\varphi}\right)_{,r},$$

$$\overline{M}_{n4} = -M_r^* = D\left[\ddot{u}_{34,rr}+\nu\left(\frac{1}{r^2}\,\ddot{u}_{34,\varphi\varphi}+\frac{1}{r}\,\ddot{u}_{34,r}\right)\right].$$

Taking into account formulas (3.6) we obtain:

$$\overline{T}_{33}(K,Q) = \frac{1}{2\pi\varepsilon},$$

$$\overline{T}_{34}(K,Q) = -\frac{3-\nu}{4\pi\varepsilon^2}(\cos\varphi\cdot\mu_1-\sin\varphi\cdot\mu_2), \tag{3.9}$$

$$\overline{M}_{n4}(K,Q) = -M_r^* = \frac{1+\nu}{4\pi\varepsilon}(\cos\varphi\cdot\mu_1-\sin\varphi\cdot\mu_2).$$

For $\varepsilon\to 0$ the integral along the boundary $\overline{C}$ in equations (3.3) and (3.5) becomes the principal value in Cauchy's sense. When we integrate the equation (3.2) along C^* ($\varepsilon\to 0$) we obtain the following quantities:

$$\lim_{\varepsilon\to 0}\int_{C^*}\overline{T}_{33}(K,Q)u_3(Q)\,dC^* = \frac{\omega}{2\pi}u_3(Q), \tag{3.10}$$

and from equation (3.5) for $\mu(Q) = \mu_-$:

$$\lim_{\varepsilon\to 0}\left\{\int_{C^*}\overline{T}_{34}(K,Q)[u_3(K)-u_3(Q)]\,dC^*+\int_{C^*}\overline{M}_{n4}(K,Q)\varphi_n(K)\,dC^*\right\} = \tag{3.11}$$

$$= \frac{1}{2\pi}(\omega\cdot u_{3\mu-}+\sin\omega\cdot u_{3\mu+}),$$

and for $\mu(Q) = \mu_+$:

$$\lim_{\varepsilon \to 0} \left\{ \int_{C_*} \overline{T}_{34}(K, Q)[u_3(K) - u_3(Q)] dC^* + \int_{C_*} \overline{M}_{n4}(K, Q)\varphi_n(K) dC^* \right\} = \tag{3.12}$$

$$= \frac{1}{2\pi} (\omega \cdot u_{3\mu+} + \sin\omega \cdot u_{3\mu-}),$$

In the case when $\omega = \pi$ expressions (3.10), (3.11), and (3.12) correspond to smooth edge and are equal 0.5.

4. Numerical solutions

On the boundary of the thin shell two boundary conditions are known. Introducing the boundary conditions into equations (2.1) we obtain four integrall equations which we can solve numerically. The shell contour is divided into N elements and the middle surface into L elements, then we obtain $4\overline{N} + k$ linear algebraic equations ($\overline{N}$ — number of knots and k — number of corners) as follows:

$$\alpha u_j = \sum_{j=1}^{N} \int_{C_j} (\overline{u}_{ij} dC_j) T_i + \sum_{j=1}^{N} \int_{C_j} (\overline{\varphi}_{nj} dC_j) M_n - \sum_{j=1}^{N} \int_{C_j} (\overline{T}_{ij} dC_j) u_i +$$

$$- \sum_{j=1}^{N} \int_{C_j} (\overline{M}_{nj} dC_j) \varphi_n + \sum_{l=1}^{L} \int_{A_l} (\overline{u}_{ij} dA_l) P_i + \sum_{r=1}^{k} R_r \overline{u}_{3j}^r,$$

$$\beta\varphi_n = \sum_{j=1}^{N} \int_{C_j} (\overline{u}_{i4} dC_j) T_i + \sum_{j=1}^{N} \int_{C_j} (\overline{\varphi}_{n4} dC_j) M_n - \sum_{j=1}^{N} \int_{C_j} (\overline{T}_{i4} dC_j) u_i + \tag{4.1}$$

$$- \sum_{j=1}^{N} \int_{C_j} (\overline{M}_{n4} dC_j) \varphi_n + \sum_{l=1}^{L} \int_{A_l} (\overline{u}_{i4} dA_l) P_i + \sum_{r=1}^{k} R_r \overline{u}_{34}^r.$$

The number of boundary elements and the choice of the basic functions depend on the shape of the contour and on the demands for the accuracy of the solution. The solution of the set of linear algebraic equations (4.1) allow us to determine the boundary values of displacements and forces at assumed knots. Somigliano's formulas (1.2) permit to determine the displacements and forces inside the shell.

5. Examples

The shown above algorithm of solution of shallow shell was applied, among others, to the solution of a spherical shell on rectangular plan. The contour of shell was divided into 4, 6 and 10 elements and the basic function was a Lagrange's polinomial of third order. The dimensions of the shell are following: rectangular plan 9.0×6.0 m, thickness of the shell $h = 0.10$ m, height of the shell $d = 0.10$ m. The shell is uniformly loaded $q = 1.0$ kN/m². Material constants are $E = 2.0 \times 10^7$ kN/m², $\nu = 0.16$.
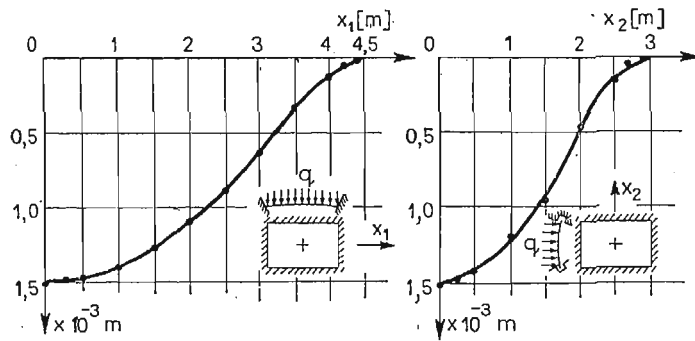
Fig. 2. Vertical displacements

Table 1.

	u_3 [m]	4 elements	6 elements	10 elements
	0,0	1,502 E−3	1,499 E−3	1,494 E−3
	0,25	1,497 E−3	1,494 E−3	1,489 E−3
	0,5	1,482 E−3	1,479 E−3	1,474 E−3
	1,0	1,409 E−3	1,407 E−3	1,402 E−3
	1,5	1,287 E−3	1,285 E−3	1,280 E−3
x_1	2,0	1,118 E−3	1,116 E−3	1,110 E−3
	2,5	8,936 E−4	8,925 E−4	8,862 E−4
	3,0	6,334 E−4	6,329 E−4	6,255 E−4
	3,5	3,579 E−4	3,579 E−4	3,488 E−4
	4,0	1,208 E−4	1,210 E−4	1,096 E−4
	4,2	5,562 E−5	5,579 E−5	4,309 E−5
	4,4	1,812 E−5	1,814 E−5	5,174 E−6
	0,0	1,502 E−3	1,499 E−3	1,494 E−3
	0,25	1,482 E−3	1,479 E−3	1,474 E−3
	0,5	1,424 E−3	1,421 E−3	1,416 E−3
	1,0	1,200 E−3	1,197 E−3	1,193 E−3
x_2	1,5	8,656 E−4	8,633 E−4	8,598 E−4
	2,0	4,838 E−4	4,817 E−4	4,791 E−4
	2,5	1,533 E−4	1,482 E−4	1,464 E−4
	2,7	6,565 E−5	5,743 E−5	5,573 E−5
	2,9	1,440 E−5	6,625 E−5	5,356 E−6

The calculated vertical displacements are shown in Fig. 2. and Fig. 3. The numerical results are given in Tables 1 and 2 where the contour of the shell is divided into 4, 6 and 10 eliments. The graphs are completed for 4 elements division of the edge contour. The two discussed shells are supported in a different manner on the edge as is seen in graphs.

The presented results can not be compared with the numerical results taken from the literature because of lack of such examples. It is to state that the number of boundary elements when the basic functions are polinomials of third order have the slight influence on the calculated displacements inside the shell.
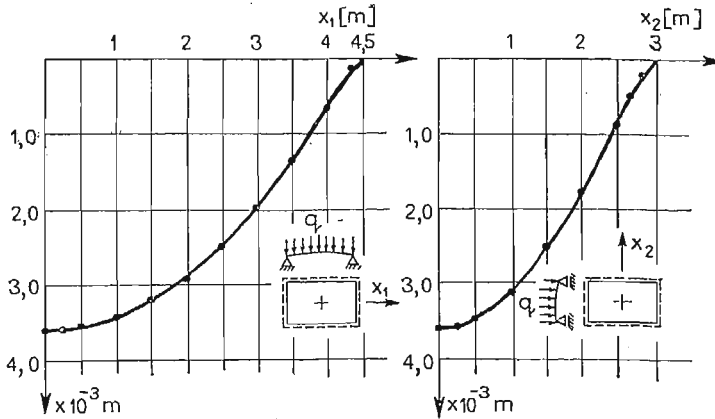
Fig. 3. Vertical displacements

Table 2.

u_3 [m]		4 elements	6 elements	10 elements
	0,0	3,615 E−3	3,681 E−3	3,689 E−3
	0,25	3,606 E−3	3,670 E−3	3,678 E−3
	0,5	3,578 E−3	3,636 E−3	3,644 E−3
	1,0	3,452 E−3	3,491 E−3	3,500 E−3
	1,5	3,237 E−3	3,249 E−3	3,258 E−3
x_1	2,0	2,927 E−3	2,911 E−3	2,922 E−3
	2,5	2,506 E−3	2,470 E−3	2,483 E−3
	3,0	1,980 E−3	1,935 E−3	1,952 E−3
	3,5	1,354 E−3	1,317 E−3	1,338 E−3
	4,0	6,687 E−4	6,502 E−4	6,704 E−4
	4,2	3,937 E−4	3,837 E−4	3,977 E−4
	4,4	1,449 E−4	1,449 E−4	1,284 E−4
	0,0	3,615 E−3	3,681 E−3	3,689 E−3
	0,25	3,584 E−3	3,650 E−3	3,658 E−3
	0,5	3,492 E−3	3,560 E−3	3,567 E−3
	1,0	3,128 E−3	3,201 E−3	3,207 E−3
x_2	1,5	2,545 E−3	2,624 E−3	2,628 E−3
	2,0	1,777 E−3	1,859 E−3	1,862 E−3
	2,5	8,850 E−4	9,560 E−4	9,629 E−4
	2,7	5,242 E−4	5,726 E−4	5,810 E−4
	2,9	2,101 E−4	1,864 E−4	1,967 E−4

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Р е з ю м е

РАЗРЕШЕНИЕ ЗАДАЧИ ПОЛОГИХ ОБОЛОЧЕК МЕТОДОМ ГРАНИЧНЫХ УРАВНЕНИЙ.
УГЛОВАЯ ПРОБЛЕМА

В статье представлены основы метода граничных уравнений применены к цифровому анализу пологих оболочек. Подробно обсуждена проблема углов и связанные с ней особенности. Для иллюстрации метода приведены численные решения двух сферических оболочек, распределеных на прямоугольном плане.

Streszczenie

ROZWIĄZANIE ZAGADNIENIA POWŁOK MAŁOWYNIOSŁYCH ZA POMOCĄ METODY
ELEMENTÓW BRZEGOWYCH. ZAGADNIENIE NAROŻY

Przedstawiono podstawy metody elementów brzegowych w zastosowaniu do analizy numerycznej powłok małowyniosłych. Omówiono szczegółowo problem naroży i związanych z narożami osobliwości. Dla zilustrowania metody zamieszczono rozwiązania numeryczne dwóch powłok sferycznych na rzucie prostokątnym.

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