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NON-STANDARD ANALYSIS AND THE CONTINUOUS MEDIA

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1. Preliminary remarks

The A. Robinson's non-standard analysis seems to be a very convenient and efficient tool of formalization of the connections between the system of corpusculae of material body and the continuous medium, representing this body.

In the survey [5] of Polish studies in the above mentioned direction and in the paper [7] quoted in [5], there is also demonstrated the possibility of a similar formalization of the connections between porous medium and its skeleton (at given time instant).

In this paper we present certain notion of continuous medium and discuss its simple topological and measure theoretical properties.

We begin with a brief description of the non-standard formalism, used below. Starting from the set $X_0 = R$ (the real line) of individuals we obtain the universe of the standard model M of analysis as $X = \bigcup_{n=0}^{\infty} X_n$, where X_{n+1} equals to X_n plus the power set of X_n for any natural n. The set-theoretical epsilon-relation $\in$, restricted to X^2 is the only extralogical relation of the model M. The language of M contains also the constants, at least one for each element of X. For given infinite set W of indices and the $\mathscr{X}_0$-regular ultrafilter D on W, we define nonstandard model $^*M$ as the ultrapower of M modulo D. We do not suppose for $^*M$ the enlargement property and we make no hypotheses on saturation of $^*M$ in the powers, greater than $\mathscr{X}_1$.

The symbols $\hat{a}_w, \hat{A}_w, \dots$ stand for elements of the non-standard universe, that are equivalence classes of the functions $(a_w)_{w\in W}, (A_w)_{w\in W}, \dots$ respectively modulo D. The expressions of set theory, analysis etc., introduced in the paper are to be understood as the abbrevations of the expressions, defined in $^*M$. The standard elements of $^*X$ and the internal sets (relations, functions) in $^*M$ are defined as usual, the notation for standard elements is the usual one, too. We refer to [3] for details. If the argument concerns only the *elements of $^*X_n$ with given, finite n (as in the whole text below), then we can identify the relation $^*\in$ of $^*M$ with a suitable restriction of the set-theoretical epsilon.

2. Main definition and theorems

In all the paper U denotes a bounded open set in the space R^3 of the standard model M.

Definition 1. The set $P \subseteq U$ is said to be a continuous medium of the internal set $\hat{A}_w$ of points of $^*R^3$ in U if there holds inclusion $\hat{A}_w \cap {}^*U \subseteq {}^*P$.

We shall frequently use the following trivial

Lemma 1. The set $P \subseteq U$ is a continuous medium of $\hat{A}_w$ in U as above if there exists a set $H \in D$ such, that

$$U \cap \bigcup_{w \in H} A_w \subseteq P$$

(here and everywhere in the paper W and D are the set of indices and the ultrafilter, used in the description of $^*M$ in the section 1).

Proof. It is immediate. Indeed, $\hat{A}_w \cap {}^*U \subseteq {}^*P$ iff $A_w \cap U \subseteq P$ for each w from the set of indices, belonging to D, which is equivalent with the condition in the thesis, q.e.d.

The set $\hat{A}_w$ may be meant as a hyperfinite set of mass-points (see e.g. [5, 6]) or as the sume of (maybe, hyperfinite, too) internal family of balls of infinitesimal radius, intended as a model of a set of atoms. We can also interpret $\hat{A}_w$ as a skeleton of porous medium (see [7]). The notion of continuous medium defined in this section seems to be very naive and even too general. We prove, however, that under very natural hypotheses about the distribution of points of $\hat{A}_w$ in $^*U$ there is very few continuous media of $\hat{A}_w$ in U.

Let $\{K_n\}_{n \in {}^*N}$ be an internal sequence (indexed with extended naturals) of cubes K_n of the form $[a_n, b_n) \times [c_n, d_n) \times [e_n, f_n)$ ($[x, y)$ is the set of all $z \in {}^*R$ such that $x \leqslant z < y$). Suppose that:

1^0 there exists a positive infinitesimal $\hat{h}_w$ such that $b_n - a_n = d_n - c_n = f_n - e_n = \hat{h}_w$ for any $n \in {}^*N$,

2^0 if $m, n \in {}^*N$, $m < n$ then K_m, K_n are disjoint,

3^0 the extended space $^*R^3$ is a sum of all K_n's, $n \in {}^*N$.

Denote by mes the three-dimensional Lebesgue measure, by *mes its extension in $^*M$ and by st x the standard part of a finite hyperreal x. Denote, at the end, by A° the interior of a set $A \subseteq R^3$ in the natural topology of R^3.

We have the following theorems

Theorem 1. If for each $K_n \subseteq {}^*U$ the intersection $K_n \cap \hat{A}_w$ is non-empty, then any continuous medium P of $\hat{A}_w$ in U is dense in U.

Theorem 2. If for any K_n $^*U$ holds the inequality:

$$\mathrm{st}\left(\frac{{}^*\mathrm{mes}(K_n \cap \hat{A}_w)}{{}^*\mathrm{mes}(K_n)}\right) > 0,$$

then each Lebesgue measurable continuous medium P of $\hat{A}_w$ in U is of full Lebesgue measure in U.

Theorem 3. If for each $K_n \subseteq {}^*U$ holds the inequality:

$$\mathrm{st}\left(\frac{{}^*\mathrm{mes}(K_n \cap \hat{A}_w^0)}{{}^*\mathrm{mes}(K_n)}\right) > 0,$$

then each continuous medium P of $\hat{A}_w$ in U is of full Lebesgue measure in U and the set $U\setminus P$ is nowhere dense in U.

Remark. The occurence of *mes, a canonical extension of Lebesgue measure (not of the external measure) in $^*R^3$ in the inequalities of thms 2,3 implies measurability of the sets in question.

Proof of the thm 1. Let P be a continuous medium of $\hat{A}_w$ in U and let $B \subseteq U$ be an open ball with a center c and radius $r > 0$. Since $^*R^3$ is a sum of $K_n's$, there exists $p \in {}^*N$ such, that $^*c \in K_p$. Since $0 < \hat{h}_w < {}^*r$, we have $K_p \subseteq {}^*B$. Let $K_p = \hat{C}_w$, where any C_w, $w \in W$, is a cube in R^3 of the form $[s_w, t_w) \times [u_w, x_w) \times [y_w, z_w)$. Then $K_p \subseteq {}^*B$ and $K_p \cap \hat{A}_w$ non-empty imply, that $C_w \subseteq B$ and $C_w \cap A_w \neq \emptyset$ for each w from certain set $H_0 \in D$. Hence, if $U \cap \bigcup_{w\in H} A_w \subseteq P$ for certain $H \in D$, then $\bigcup_{w\in H} (C_w \cap A_w) \subseteq P$ and for each $w \in H_0 \cap H$, $C_w \cap A_w \subseteq P$ has non-void intersection with B. $H_0 \cap H \in D$ is non-empty set, which completes the proof.

Proof of the thm 2. Since U is bounded, the set I of all $n \in {}^*N$ such, that $K_n \subseteq {}^*U$, is hyperfinite. Hence, the set of all numbers:

$$c_n = \frac{{}^*\mathrm{mes}(K_n \cap \hat{A}_w)}{{}^*\mathrm{mes}(K_n)}, \quad n \in I,$$

is finite or hyperfinite and, according to the well-known non-standard result, has a least number $\hat{c}_w$. The standard part $2c$ of $\hat{c}_w$ is positive.

For certain $H_1 \in D$ we have $c_w > c > 0$ whenever $w \in H_1$. Let P be a Lebesgue measurable continuous medium of $\hat{A}_w$ in U. Then for certain $H_2 \in D$ there is:

$$U \cap \bigcup_{w\in H_2} A_w \subseteq P.$$

Choose arbitrary point p in U. For certain $q \in {}^*N$ there is $^*p \in K_q$ and, since U is open, also $K_q \subseteq {}^*U$. Put $K_q = \hat{C}_w$, where each C_w is as in the proof of thm 1. Then for certain $H_3 \in D$ there is:

$$\frac{\mathrm{mes}(C_w \cap A_w)}{\mathrm{mes}(C_w)} \geqslant c_w \quad \text{and} \quad p \in C_w,$$

whenever $w \in H_3$. Since the set $H = H_1 \cap H_2 \cap H_3$ is infinite, and the length $\hat{h}_w$ of the edge of K_q is an infinitesimal, there is a sequence $\{w_n\}_{n=1}^{\infty}$ of $w_n's$ from H such, that $\lim_{n\to\infty} h_{w_n} = 0$. Let for each finite natural n E_n be an open cube with the edges of length $2h_{w_n}$ parallel to the axes of coordinates, such, that $p \in E_n$ and $C_{w_n} \subseteq E_n$. Then we have:

$$\frac{\mathrm{mes}(E_n \cap P)}{\mathrm{mes}(E_n)} \geqslant \frac{\mathrm{mes}(A_{w_n} \cap E_n)}{\mathrm{mes}(E_n)} \geqslant \frac{\mathrm{mes}(C_{w_n} \cap A_{w_n})}{\mathrm{mes}(E_n)} \geqslant \frac{1}{8} \cdot \frac{\mathrm{mes}(C_{w_n} \cap A_{w_n})}{\mathrm{mes}(C_w)}$$

$$\geqslant \frac{1}{8} \cdot c_{w_n} \geqslant c/8 > 0.$$

Hence,

$$\limsup_{n\to\infty} \left(\mathrm{mes}(E_n \cap P)/\mathrm{mes}(E_n)\right) \geqslant c/8 > 0.$$

Since P is Lebesgue measurable, $\mathrm{mes}(U\setminus P) = 0$ results from the last inequality above and from the Lebesgue density theorem.

Proof of the theorem 3. Let P be any continuous medium of $\hat{A}_w$ in U, where $\hat{A}_w$ satisfies the hypothese of the theorem. Denote the interior of A_w by B_w. There exists a set $H_1 \in D$ such, that P is a superset of the continuous medium $P_0 = U \cap \bigcup_{w \in H_1} B_w$ of the set $\hat{B}_w$ in U. $\hat{B}_w$ satisfies the condition of the theorem 2, P_0 is open and hence measurable, which completes the proof of the first thesis. Now, let p be a point in U and K and open cube in R^3 with center p, $K \subseteq U$. There exists $K_n = \hat{C}_w$ (any C_w is a cube in R^3 with the edges of length h_w) such, that $^*p \in K_n$. Thus, there exists a set $H_2 \in D$ such, that $\bigcup_{w \in H_2} (C_w \cap B_w) \subseteq P \cap K$ and $B_w \cap C_w$ is non-empty whenever $w \in H_2$. Left hand side of the last inclusion is an open set and we have proved, that each neighbourhood of each $p \in U$ includes an open ball that is disjoint with $U\setminus P$. Hence, $U\setminus P$ is nowhere dense, q.e.d.

3. Generalization

The ultrapower technique was essential above for obtaining proofs of thms 1, 2, 3. However, according to certain Frayne's theorem (corollary 4.3.13 in [1]), if M_1 is a proper elementary extension of the standard model M from the section 1, then M_1 can be elementarily embedded into an ultrapower $^*M$ of M modulo certain ultrafilter D. If M_1 contains non-standard naturals, D must be $\aleph_0$-regular. This makes possible generalization of the theorems 1, 2, 3 by weakening hypotheses. Let $^*M$ be arbitrary proper elementary extension of M, containing non-standard naturals, let U and $^*U$ be as in the section 2. Writing in the definition 1 A instead of $\hat{A}_w$ (A an internal subset of $^*R^3$) we can modify the definition of continuous medium. Let for internal $A \subseteq {}^*R^3$ A° be the internal set, $p \in A^\circ$ if there exists positive (maybe, infinitesimal) $r \in {}^*R$ such, that the internal ball B with center p and radius r is a subset of A. Let, at the end, the internal sequence of $K_n's$, $n \in {}^*N$, fulfils the conditions 1°, 2°, 3° of section 2 with a positive infinitesimal h instead of $\hat{h}_w$. Then, writing in the theorems 1, 2, 3 A instead of $\hat{A}_w$ and A^0 instead of $\hat{A}^0_w$, we obtain true theorems.

4. On the existence of porosity

Define, analogously as in [5] the porosity $\pi(p)$ of $\hat{A}_w$ at the point $p \in U$ as a standard part of the F-limit:

$$F-\lim_{n\to\infty} {}^*\mathrm{mes}(\hat{A}_w \cap {}^*I_n)/{}^*\mathrm{mes}({}^*I_n),$$

where for $p = (x, y, z)$ and a finite natural n, $I_n = \left(x-\frac{1}{n}, x+\frac{1}{n}\right) \times \left(y-\frac{1}{n}, y+\frac{1}{n}\right) \times \left(z-\frac{1}{n}, z+\frac{1}{n}\right)$ is an open interval in R^3 (see [4] for the definition of F-limit). By

$\mathrm{st}_3\hat{p}_w$ we denote in this section the point $p \in R^3$ such, that the euclidean distance between $\hat{p}_w$ and $*p$ is infinitesimal.

Denote by S the σ-algebra of all Borel subsets of U. We prove the following

Theorem 4. If $\hat{A}_w \in *S$, then the porosity function π is defined at almost all (with respect to the Lebesgue measure) points of the set U.

Proof. Denote by S_1 the σ-algebra of subsets of $*U$, generated by $*S$ and let the internal sequence $\{K_n\}_{n\in *N}$ be as in section 2. For any $E \subseteq U$ let $\tilde{E}$ be a counterimage of E with respect to the mapping st_3.

We sketch, for convenience, the proof, that $\tilde{E} \in S_1$ whenever $E \in S$. Let $G \subseteq U$ be open, let $d(a, A)$ be the distance of. point a from the set A in the euclidean metric of R^3 and let for $n \in N$, G_n be the sum of those exactly K_q's that $*d(\hat{p}_w, *U\setminus G)) \geqslant \frac{1}{*n}$ for any $\hat{p}_w \in K_q$. Then $\tilde{G} = \bigcup_{n=1}^{\infty} G_n$, $\tilde{G} \in S_1$. The family of all sets $\tilde{E}$, where $E \in S$, is a σ-algebra of subsets of $\tilde{U}$, generated by the sets $\tilde{G}$, where $G \subseteq U$ are open. Of course, this σ-algebra is a subfamily of S_1. Put $m_0(B) = *\mathrm{mes}(B \cap \hat{A}_w)$ for any $B \in *S$. $(*U, *S, m_0)$ is then an internal measure space in a sense of [2] and we can extend $\mathrm{st}\, m_0$ to the Loeb measure m_1 defined on S_1.

Let, at the end, $m_2(E) = m_1(\tilde{E})$ for each $E \in S$. Then for a sequence $\{E_n\}$ of pairwise disjoint sets from S, $\tilde{E}_n$'s are pairwise disjoint and $m_2(\bigcup_{n=1}^{\infty} E_n) = m_1(\widetilde{\bigcup_{n=1}^{\infty} E_n}) = m_1(\bigcup_{n=1}^{\infty} \tilde{E}_n) = \sum_{n=1}^{\infty} m_1(\tilde{E}_n)$, m_2 is a measure on S. If $E \in S$ is of Lebesgue measure zero, then for any $*\varepsilon > 0$, and open $G \subseteq U$ such, that $\mathrm{mes}(G) > \varepsilon$ and $E \subseteq G$ we have $\tilde{G} \subseteq *G$, for G open, and $m_2(E) \leqslant m_2(G) = m_1(\tilde{G}) \leqslant m_1(*G) = \mathrm{st}*\mathrm{mes}(*G \cap \hat{A}_w) \leqslant \mathrm{st}*\mathrm{mes}(*G) < \varepsilon$. Hence, $m_2(E) = 0$ too and m_2 is absolutely continuous with respect to Lebesgue measure restricted to S. It follows from the Radon-Nikodym theorem, that m_2 has at Lebesgue almost all points $p = (x, y, z) \in U$ the Radon-Nikodym derivative with respect to mes, denote it by $f(p)$, equal to $\lim_{n\to\infty} m_2(I_n)/\mathrm{mes}(I_n)$, where $I_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \times \left(y - \frac{1}{n}, y + \frac{1}{n}\right) \times \left(z - \frac{1}{n}, z + \frac{1}{n}\right)$ is an open interval and $I_n \subseteq U$ for n large enough. We have for any standard natural n and for:

$$J_n = \left(x - \frac{1-2^{-n}}{n}, x + \frac{1-2^{-n}}{n}\right) \times \left(y - \frac{1-2^{-n}}{n}, y + \frac{1-2^{-n}}{n}\right) \times$$

$$\left(z - \frac{1-2^{-n}}{n}, z + \frac{1-2^{-n}}{n}\right), *J_n \subseteq \tilde{I}_n \subseteq *I_n \text{ and } m_1(*I_n \setminus *J_n) \leqslant$$

$$\mathrm{mes}(I_n \setminus J_n) < 1/(8^n \mathrm{mes}(I_n)).$$

Hence, for n large enough:

$$\left| \frac{m_2(I_n)}{\mathrm{mes}(I_n)} - \mathrm{st}\, \frac{{}^*\mathrm{mes}({}^*I_n \cap \hat{A}_w)}{{}^*\mathrm{mes}({}^*I_n)} \right| < 8^{-n}$$

and $f(p)$ equals to $\pi(p)$ the standard part of the F-limit of ${}^*\mathrm{mes}({}^*I_n \cap \hat{A}_w)/{}^*\mathrm{mes}({}^*I_n)$, q.e.d.

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Р е з ю м е

НЕСТАНДАРТНЫЙ АНАЛИЗ И СПЛОШНЫЕ СРЕДЫ

Для внутреннего множества $\hat{A}_w$ точек нестандартного расширения трехмерного пространства и открытого множества U стандартного пространства определяем непрерывную среду $\hat{A}_w$ в U. как стандартное множество P, в расширении* P которого заключены все общие точки $\hat{A}_w$ и $^*U$, Для этого абстрактного определения даются возможные физические истолкования, в том числе связанные с понятием пористой среды.

Даются условия гарантирующие, что непрерывная среда

1° плотная в U,

2° измеримая непрерывная среда полной меры в U,

3° всякая непрерывная среда полной меры в U и её внутренность плотная в U,

Доказаны соответствующие теоремы. Доказана даже теорема существования почти всюду функции пористости в смысле работы [7].

S t r e s z c z e n i e

ANALIZA NIESTANDARDOWA I OŚRODKI CIĄGŁE

Dla zbioru wewnętrznego $\hat{A}_w$ punktów rozszerzenia niestandardowego przestrzeni trójwymiarowej definiujemy ośrodek ciągły zbioru $\hat{A}_w$ w otwartym podzbiorze U standardowej przestrzeni trójwymiarowej jako zbiór P standardowych punktów przestrzeni, w którego rozszerzeniu $^*P$ zawarte są wszystkie punkty wspólne $\hat{A}_w$ i $^*U$.

Dla tej definicji podajemy możliwe interpretacje fizykalne, między innymi w terminach ośrodka porowatego.

Podano w pracy warunki dostateczne na to, by

1° ośrodek ciągły P był gęsty w U,

2° mierzalny ośrodek ciągły P był pełnej miary w U,

3° każdy ośrodek ciągły był pełnej miary w U i miał wnętrze gęste w U.

Podano dowody odpowiednich twierdzeń. Udowodniono także przy dodatkowych założeniach o $\hat{A}_w$ twierdzenie o istnieniu prawie wszędzie funkcji porowatości w sensie pracy [7].

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