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# PLASTIC STRAIN ENERGY UNDER CYCLIC MULTIAXIAL STATES OF STRESS

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In this paper the cyclic plastic strain energy density under multiaxial states of stress is analysed. A relationship is proposed which can be used to determine the plastic strain energy per cycle for non — Masing material for various stress range. In the investigation the generalized constitutive relations cyclic stress — strain for non — Masing material are used. The presented model based on general and basic principles and significant constants can be obtained from appropriate and common tests. Predictions of the proposed method with the experimental data of biaxial cyclic loading has shown good agreement.

## 1. Introduction

In total strain — controlled experiments it has been observed that after some number of cycles, usually less than half — life duration the steady state of the material is achived. For practical purposes, it may reasonably be assumed that the steady hysteresis loop does not vary with cycles [1 - 4]. Therefore, the idea of relating fatigue life to the plastic strain energy density (area of hysteresis loop) during a load cycle has been proposed. The work of Feltner and Morrow [5] need mentioning as the first significant contribution in which they have clearly pointed out the significance of plastic strain energy in the analysis of fatigue proporties of metals.

Halford [6] presented the relationship for plastic strain energy based on the cyclic stress — strain curve of the material. Esin and Jones [7] introduced the concept of statistical functions characterizing the micro — inhomogenity of stress and strains existing in a metal and used it to express the hysteresis energy. Jhansale and Topper [8] proposed an approach for describing the loop shapes of a non — Masing material. Abel and Muri [9] suggested that the hysteresis loop clousere failure may be an essential prerequite for the fatigue — crack initiation. The method of an analytical description of the steady hysteresis loops and cyclic strain curve from the test of one specimen only, was discussed in [10].

It is generally agreed, that plastic strain energy plays an important role in the damage process. The plastic strain energy density depends on the kind and range of the cyclic loading. The significance of the energy approach is in ability to unify microscopic and macroscopic testing data and to define multiaxial criteria. All, of therc abovc mentioned investigations consider only the uniaxial state of stress. The problem of plastic strain energy in multiaxial cyclic loading was considered in some works. Damali and Esin [11] discussed the extension of Esin's statistical formulation to the case of biaxial fatigue under proportional stressing. Garud [12] considered the plastic cyclic strain energy under complex loading based on a „new hardening rule". Lately, Lefebvre, Neal and Ellyin [13] according to $I_2$ (von Mises) theory presented the relation for the plastic energy dissipated during proportional loading.

It is objective of this paper to present the relationship, which can be used to determine plastic strain energy for both, Masing and non — Masing material under cyclic multiaxial states of stress. In the investigations, the general constitutive equations cyclic streess — strain [14] are used.

The significant constants in the proposed relationship can be obtained from appropriate and common tests.

## 2. Cyclic plastic strain energy

During strain fatigue, energy is dissipated because of plastic deformation. The cyclic plastic strain energy dissipated per unit volume during a given loading cycle for an element subjected to a cyclically varying stress and strain history $\Delta\sigma_{ij}$ and $\Delta\varepsilon_{ij}$ is:

$$\Delta W^p = \oint \Delta\sigma_{ij} d(\Delta\varepsilon^p_{ij}), \tag{1}$$

where $\Delta\varepsilon^p_{ij}$ is the plastic strain range. Because of incompressibility of plastic deformation (i.e. $d(\Delta\varepsilon_{kk}) = 0$, the foregoing relation can also be expressed as follows:

$$\Delta W^p = \oint \Delta S_{ij} d(\Delta\varepsilon^p_{ij}), \tag{2}$$

where $\Delta S_{ij} = \Delta\sigma_{ij} - (\delta_{ij}\Delta\sigma_{kk})/3$ is the deviatoric stress tensor.

The cyclic uniaxial stress — strain curve is generally obtained by plotting the stress corresponding to the positive (tensive) peaks of the hysteresis loop at half — life for various strain ranges. A widely used form is:

$$\frac{\Delta\varepsilon}{2} = \frac{\Delta\sigma}{2E} + \left(\frac{\Delta\sigma}{2K}\right)^{1/n'}, \tag{3}$$

where $\Delta\sigma/2$, $\Delta\varepsilon/2$ are the stress and strain amplitude, respectively, $E$ is the modulus of elasticity, $K$ is a strength coefficient, and $n'$ is the cyclic strain hardening exponent. A material is said to exhibit a Masing description when the branches of hysteresis loops can be described by the equation (3) magnified by a factor of two, i.e.:

$$\Delta\varepsilon = \frac{\Delta\sigma}{E} + 2\left(\frac{\Delta\sigma}{2K}\right)^{1/n'}. \tag{4}$$

The origin of the coordinate system in this case is transfered to the compressive tip of the hysteresis loop.

Generally, if the origin of the coordinate axes is transformed to the tip of the negative (compressive) cycle, the positive branches of the half — life hysteresis loops do not fall on a monotonically increasing unique curve (non — Masing material). This observation is rather significant and it affects the maner in which the plastic strain energy per cycle is to be determined. Jhansale and Topper [8] proposed an approach for describing the loop shapes of non — Masing material. This approach was analysed in [2, 3] and is adopted herein. A „master" curve different from that of the cyclic curve is defined as one geometrically similar to the matched upper branches of the hysteresis loops with minimum proportional range.

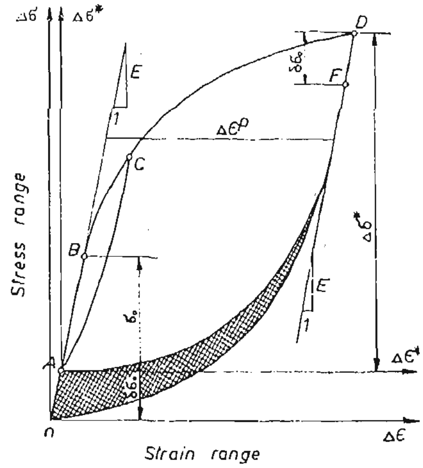


Fig. 1. Hysteresis loop for a non — Masing material

In the Figure 1 the uniaxial hysteresis loop (ABCA) for the „minimum" proportional range, $\sigma_0$ and that of (OABCDFO) with a bigger proportional range are presented. The equation of the „master" curve ABCD, where the origin A corresponds to the lower tip of the minimum proportional range hysteresis loop is given by [3]:

$$\Delta\varepsilon^* \doteq \frac{\Delta\sigma^*}{E} + 2\left(\frac{\Delta\sigma^*}{2K^*}\right)^{1/n^*}. \tag{5}$$

The increase in the proportional stress, $\delta\sigma_0$, can be obtained from:

$$\delta\sigma_0 = \Delta\sigma - \Delta\sigma^* = \Delta\sigma - 2K^*(\Delta\varepsilon^p/2)^{n^*}. \tag{6}$$

Then, the plastic strain energy density — the area of the hysteresis loop OABCDFO may be expressed as:

$$\Delta W^p = \frac{1-n^*}{1+n^*}(\Delta\sigma - \delta\sigma_0)\Delta\varepsilon^p + \delta\sigma_0\Delta\varepsilon^p. \tag{7}$$

For Masing material $\delta\sigma_0 = 0$, and equation (7) reduced to [6]

$$\Delta W^p = \frac{1-n'}{1+n'}\Delta\sigma\Delta\varepsilon^p. \tag{8}$$

For non — Masing material under cyclic multiaxial loading the relation between stress and strain components is given by [14]:

$$\frac{\Delta\varepsilon_{ij}^{*}}{2}=\frac{1+\nu}{2E}\Delta\sigma_{ij}^{*}-\frac{\nu}{2E}\delta_{ij}\Delta\sigma_{kk}^{*}+\frac{3}{2}\frac{(\overline{\Delta\sigma^{*}})^{\frac{1-n^{*}}{n^{*}}}}{(2K^{*})^{1/n^{*}}}\Delta S_{ij}^{*}, \tag{9}$$

where $\overline{\Delta\sigma^{*}}=(3/2\Delta S_{ij}^{*}\Delta S_{ij}^{*})^{1/2}$ is the effective range stress. The deviatoric stress tensor $\Delta S_{ij}^{*}$ is calculated as:

$$\Delta S_{ij}^{*}=\Delta\sigma_{ij}^{*}-\delta_{ij}(\Delta\sigma_{kk}^{*})/3. \tag{10}$$

The components of stress tensor $\Delta\sigma_{ij}^{*}$ may be estimated from the difference between $\Delta\sigma_{ij}$ and the increase in the proportional stress, $\overline{\delta\sigma_0}$. The increase in proportional stress, $\overline{\delta\sigma_0}$ can be obtained as:

$$\overline{\delta\sigma_0}\cong\frac{\overline{\Delta\sigma}}{2}-\sigma_0, \tag{11}$$

where $\overline{\Delta\sigma}=(3/2\Delta S_{ij}\Delta S_{ij})^{1/2}$, and $\sigma_0$ is the minimum proportional stress for uniaxial case. According to the $J_2$ theory the equivalent plastic strain can be defined as follows:

$$\overline{\Delta\varepsilon^{p}}=(2/3\Delta\varepsilon_{ij}^{p}\Delta\varepsilon_{ij}^{p})^{1/2}. \tag{12}$$

It is generally assumed that the effective cyclic stress — strain curve for multiaxial proportional loading takes the form analogous to equation (5), (see Fig. 2).

$$\overline{\Delta\varepsilon^{p}}=2\left(\frac{\overline{\Delta\sigma^{*}}}{2K^{*}}\right)^{1/n^{*}}. \tag{13}$$

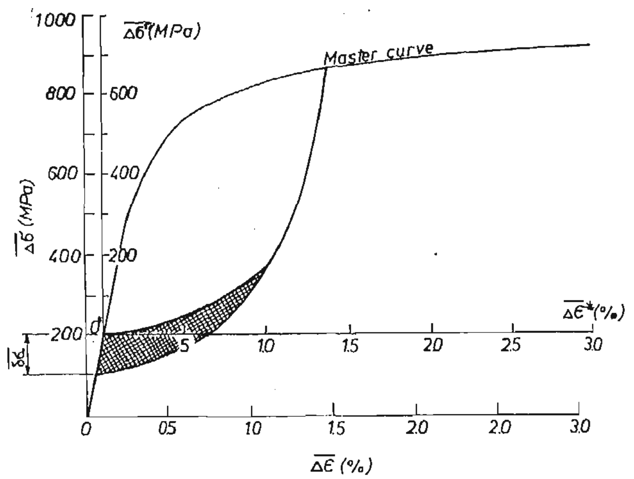


Fig. 2. A cyclic stress — strain relation for a A-516 Gr. 70 carbon steel

Therefore, the plastic strain energy for non — Masing material under multiaxial loading can be expressed by following equation:

$$\Delta W^{p}=\frac{1-n^{*}}{1+n^{*}}\overline{\Delta\sigma^{*}}\,\overline{\Delta\varepsilon^{p}}+\overline{\delta\sigma_0}\,\overline{\Delta\varepsilon^{p}}. \tag{14}$$

The above equation relates the plastic strain energy denisty under multiaxial loading with the components of the strain and the stress tensors. The softening or hardening of the material due to cyclic loading is included through the equation (9).

## 3. Comparison with experimental data

To examine the applicability of the proposed criterion we would require data on uniaxial cyclic — strain curves and amplitudes of strain and stress for multiaxial cyclic loading. The test results for A-516 Gr. 70 low alloy carbon steel presented in [2, 15] enable us to make relative comparison. This material as shown in [2, 3] does not follow the Masing hypothesis.

Fully reversed strain — controlled tests at six strain ratios ($\varrho = \Delta\varepsilon_t/\Delta\varepsilon_a$) have been carried on thin — walled tubular specimens using a hydraulic servo — controlled testing system. Specimens were cyclicaly loading in the axial direction while pressures were applied to the inside and outside alternatively during each half cycle. The strain ratio, $\varrho$, was kept constant during a given test. Figure 2 shown a master curve under uniaxial stress conditions. The strength coefficient $K^*$ and the hardening exponent $n^*$ of the Eq. (13) evaluated by a least — squares technique were:

$$K^* = 629.8 \text{ MPa} \quad \text{and} \quad n^* = 0.144. \tag{15}$$

The plastic components of strain tensor based on relationship (9) for the plane stress condition may be expressed as:

$$\Delta\varepsilon_x^{p*} = 2\frac{(\overline{\Delta\sigma^*})^{\frac{1-n^*}{n^*}}}{(2K^*)^{\frac{1}{n^*}}}\left(\Delta\sigma_x^* - \frac{1}{2}\Delta\sigma_y^*\right), \tag{16a}$$

$$\Delta\varepsilon_y^{p*} = 2\frac{(\overline{\Delta\sigma^*})^{\frac{1-n^*}{n^*}}}{(2K^*)^{\frac{1}{n^*}}}\left(\Delta\sigma_y^* - \frac{1}{2}\Delta\sigma_x^*\right), \tag{16b}$$

where

$$\overline{\Delta\sigma^*} = [(\Delta\sigma_x^*)^2 - \Delta\sigma_x^*\Delta\sigma_y^* + (\Delta\sigma_y^*)^2]^{1/2}. \tag{16c}$$

The minimum proportional range $\sigma_0$ for the A-516 Gr. 70 carbon steel is $\sigma_0 \cong 180$ MPa [3]. The predicted values of the plastic strain energy are compared with the experimental data for two strain ratios, $\varrho = 0$ and 1 in Fig. 3. It is seen that the correlation is rather good. Note that the required constants in Eq. (14), were obtained from the uniaxial test results.

## 4. Conclusions

A method for calculation of the cyclic plastic strain energy density under multiaxial loading has been presented. A relationship is derived which can be used to determine plastic strain energy density for a non — Masing material for various stress ranges. The Masing type of response also can be obtained as a particular case of the present model. The material constants used in the calculations can be obtained from the uniaxial test data.

The predictions of the proposed model are compared with the biaxial test data of A-516 Gr. 70 carbon steel, and the agreement is found to be good.

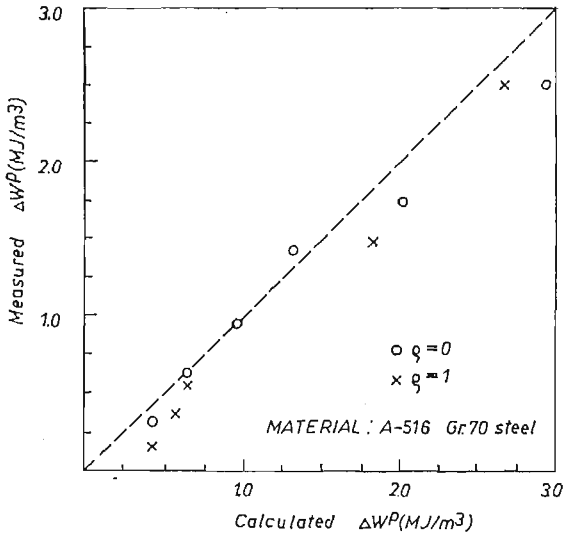


Fig. 3. Comparison between predicted and experimental data [15] for strain ratios 0 and 1

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Резюме

ЭНЕРГИЯ ПЛАСТИЧЕСКОЙ ДЕФОРМАЦИИ В УСЛОВИЯХ СЛОЖНОГО ЦИКЛИЧЕСКОГО НАГРУЖЕНИЯ

В работе представлено метод оценки энергии диссипации в цикле в условиях сложного циклического нагружения. Предлагаемый метод даёт возможность расчёта энергии пластической деформации для материалов описываемых моделем Масинга, а также таких, которые неподчиняются этой модели. Премуществом метода есть то, что коеффиценты использованные в расчётах можна получить на основе одноосных экспериментальных результатов.

Результаты расчёта циклической энергии пластической деформации полученные на осенов представленного метода были сравнены с даными экспериментальными в условиях плоского напряженного состояния.

Streszczenie

ENERGIA ODKSZTAŁCENIA PLASTYCZNEGO PRZY WIELOOSIOWYCH OBCIĄŻENIACH CYKLICZNIE ZMIENNYCH

W pracy przedstawiono metodę obliczania jednostkowej energii dysypacji przy obciążeniach cyklicznie zmiennych w warunkach złożonego stanu obciążenia. Przedstawiona metoda może być stosowana do obliczania energii odkształcenia plastycznego zarówno dla materiałów podlegających opisowi modelem Masinga i nie podlegających temu opisowi. Zaletą przedstawionej metody jest to, że stałe materiałowe zastosowane w obliczeniach mogą być otrzymane na podstawie wyników badań zagadnień jednoosiowych.

Wyniki obliczeń jednostkowej energii odkształcenia plastycznego uzyskane przy zastosowaniu przedstawionej metody zostały dla płaskiego stanu naprężeń porównane z odpowiednimi wynikami uzyskanymi z badań doświadczalnych.

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