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INDENTATION OF A TRANSVERSELY ISOTROPIC LAYER BY A TRUNCATED CONICAL PUNCH

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The indentation of a transversely isotropic layer on a two-parameter elastic foundation by a truncated conical punch is investigated. The author formulates the problem as the solution of dual intergal equations, which had to be solved approximatelly. The physical quantities which characterize the contact and the stress singularity are obtained. In the limiting case of the half-space problem exact solution is obtained. Some special cases such as cylindrical, conical, cylindrical with beveling punches also are considered. Numerical calculations are carried out for various cases of the material, such as cadmium, magnesium single crystals and E glass-epoxy and graphite-epoxy composites.

1. Introduction

In the elastic contact problems, there may be cases that the contact region depends on the magnitude of the external load (receding and advancing contacts). In the three dimensional contact problems with such contact regions, it is very difficult to determine analytically the solutions of the problem. The theoretical solution for the punch in the arbitrary shape in the three dimensional case, has not yet been made, to the author's knowledge. However, in the axisymmetric contact problems, the contact region becomes a circular or an annular and it is enough to determine only their radii.

In a truncated conical punch, the side surface is also contacted by the elastic layer and the contact area changes as the applied loading changes [1].

In the presented paper, the author analyzes the axisymmetric contact problem between the truncated conical punch and transversely isotropic layer on a two-parameter elastic foundation. The relationships among the contact stress, the resultant load, the displacement under the indenter and contact area are shown. The effect of transverse anisotropy is clarified.

2. Basic equations

Consider a transversely isotropic layer $0 \leqslant z \leqslant h$, with the planes of isotropy parallel to the boundaries. The stress-strain relationships of such a medium can be written in cylindrical coordinates (r, Θ, z) as follows:

$$
\begin{aligned}
\sigma_r &= c_{11}e_r + c_{12}c_\Theta + c_{13}e_z, \\
\sigma_\Theta &= c_{12}e_r + c_{11}e_\Theta + c_{13}e_z, \\
\sigma_z &= c_{13}e_r + c_{13}e_\Theta + c_{33}e_z, \\
\sigma_{rz} &= c_{44}e_{rz}, \\
\sigma_{\Theta z} &= c_{44}e_{\Theta z}, \\
\sigma_{r\Theta} &= \frac{1}{2}(c_{11}-c_{12})e_{r\Theta},
\end{aligned}
\tag{2.1}
$$

where c_{ij}'s are the elastic constants of a transversely isotropic solid body.

The foregoing strain e_{ij} can be first written in terms of the displacements and then substituted into the preceding equations to obtain the stress-displacement relationships. The relationships are finally used to the equilibrium equations to form a system of partial differential equations for the displacements.

In the problem with exial symmetry the displacements $(u, 0, w)$ are governed by the equations:

$$
\begin{aligned}
c_{11}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right] + (c_{13}+c_{44})\frac{\partial^2 w}{\partial r\,\partial z} + c_{44}\frac{\partial^2 u}{\partial z^2} &= 0, \\
c_{44}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + (c_{13}+c_{44})\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right] + c_{33}\frac{\partial^2 w}{\partial z^2} &= 0.
\end{aligned}
\tag{2.2}
$$

Introducing potential functions $\varphi_1(r, z)$ and $\varphi_2(r, z)$, [2] given by:

$$
u = \frac{\partial}{\partial r}(k\varphi_1 + \varphi_2), \qquad w = \frac{\partial}{\partial z}(\varphi_1 + k\varphi_2), \tag{2.3}
$$

the system of equations (2.2) is replaced by the following partial differential equations:

$$
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{s_i^2}\frac{\partial^2}{\partial z^2}\right)\varphi_i(r, z) = 0 \quad (i = 1, 2), \tag{2.4}
$$

provided that the dimensionless parameters s_1, s_2, k are given by the following equations:

$$
c_{33}c_{44}s^4 - [c_{11}c_{33} - c_{13}(c_{13}+2c_{44})]s^2 + c_{11}c_{44} = 0, \tag{2.5}
$$

$$
k = (c_{33}s_1^2 - c_{44})/(c_{13}+c_{44}). \tag{2.6}
$$

The components of the Cauchy stress tensor $\boldsymbol{\sigma}$ can be expressed in terms of derivatives of $\varphi_i(r, z)$:

$$
\begin{aligned}
\sigma_r &= -c_{44}(k+1)\frac{\partial^2}{\partial z^2}(\varphi_1+\varphi_2) - (c_{11}-c_{12})r^{-1}u, \\
\sigma_\Theta &= -c_{44}(k+1)\frac{\partial^2}{\partial z^2}(\varphi_1+\varphi_2) - (c_{11}-c_{12})\frac{\partial u}{\partial r}, \\
\sigma_z &= c_{44}(k+1)\frac{\partial^2}{\partial z^2}(s_1^{-2}\varphi_1 + s_2^{-2}\varphi_2), \\
\sigma_{rz} &= c_{44}(k+1)\frac{\partial^2}{\partial r\,\partial z}(\varphi_1+\varphi_2).
\end{aligned}
\tag{2.7}
$$

3. Boundary conditions

The problem to be solved is that of an elastic transversely isotropic layer of thickness h indented by a single rigid truncated conical punch on its upper surface. The conditions at the lower surface of the layer are those of a two-parameter elastic foundation. The axisymmetric face of a punch is assumed as $\varepsilon_0 - k_0(r-a)\,H(r-a)$, where k_0 is the slope of the side plane of the punch, $H(r-a)$ is the Heaviside's function and ε_0 is the measure of the depth penetration, while a is the radius of the plane contact region.

The boundary conditions for the elasticity problem can be written as follows (Fig. 1):

$$\begin{aligned}
&\text{(a)}\quad w(\varrho,0) = \varepsilon_0 - k_0 b(\varrho-\lambda)H(\varrho-\lambda); && 0 \leqslant \varrho \leqslant 1,\\
&\text{(b)}\quad \sigma_z(\varrho,0) = 0; && 1 < \varrho,\\
&\text{(c)}\quad \sigma_{zr}(\varrho,0) = 0; && \varrho \geqslant 0,\\
&\text{(d)}\quad \sigma_z(\varrho,0) \text{ is finite at } \varrho \to 1-0, &&\\
&\text{(e)}\quad \sigma_z(\varrho,\eta) = -k_n w(\varrho,\eta); && \varrho \geqslant 0,\\
&\text{(f)}\quad \sigma_{zr}(\varrho,\eta) = -k_t u(\varrho,\eta); && \varrho \geqslant 0.
\end{aligned} \tag{3.1}$$

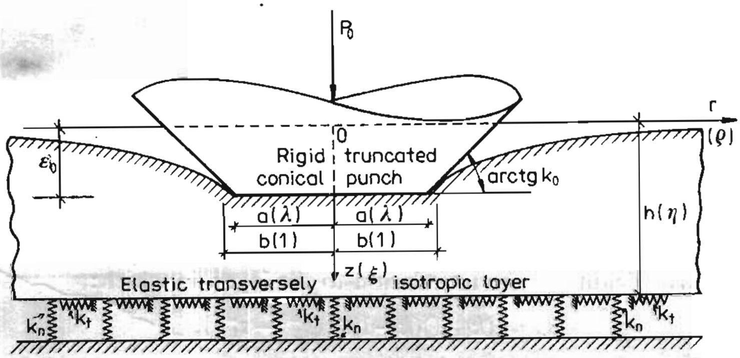

Fig. 1. Geometry of the problem

The resultant load due to the local indenter stresses is obtained as follows:

$$P_0 = -2\pi b^2 \int_0^1 \sigma_z(\varrho,0)\varrho\, d\varrho. \tag{3.2}$$

In above conditions the nondimensional variables and parameters are introduced:

$$\varrho = r/b, \quad \zeta = z/h, \quad \lambda = a/b, \quad \eta = h/b \tag{3.3}$$

In egns (e) and (f) in (3.1) k_n and k_t are the moduli of the linears reaction with the dimensions of stress per unit length. The regularity conditions at infinity also are assumed.

In our problem contact is maintained only by compressive stresses; in this unbonded frictionless contact problem the extent of the contact is the primary unknown quantity, it changes as the applied loading changes and the contact stress is finite at the end of the contact region, namely it is further imposed that the stresses be nonsingular at $\varrho \to 1-0$ (eq. (d) in (3.1)).

4. The displacement potentials

The method of Hankel transforms is used to satisfy the equations (2.4) and conditions (c), (e) and (f) in eqs (3.1). Then the displacement functions $\varphi_1(r, z)$ and $\varphi_2(r, z)$ are found to be:

$$\varphi_i(\varrho, \zeta) = (-1)^{i-1} \frac{s_1 s_2}{G_1(k+1)(s_1-s_2)s_i} \int_0^\infty x^{-1} w(x)[g_i(x\eta) \operatorname{ch} s_i x\eta\zeta - \\ - \operatorname{sh} s_i x\eta\zeta] J_0(x\varrho)\, dx; \quad (i = 1, 2), \tag{4.1}$$

where $G_1 = c_{44}$ is the shear modulus, along the z-axis, of the layer material and:

$$\begin{aligned} g_i(x\eta) &= \sum_{n=0}^{3} \varkappa_n b_n^{(i)}(x\eta) \Big/ \sum_{l=0}^{3} \varkappa_l m_l(x\eta), \\ b_0^{(i)}(x\eta) &= \operatorname{ch}(\alpha x\eta) - (-1)^i \beta^{-1}[\alpha \operatorname{ch}(\beta x\eta) - 2s_i], \\ b_{1,2}^{(i)}(x\eta) &= (x\eta)^{-1}[\operatorname{sh}(\alpha x\eta) \pm (-1)^i \operatorname{sh}(\beta x\eta)], \\ b_3^{(i)}(x\eta) &= (x\eta)^{-2}\beta^{-1}[(k^2 s_2 - s_1)\operatorname{ch}(\alpha x\eta) + (-1)^i (k^2 s_2 + s_1)\operatorname{ch}(\beta x\eta) - (-1)^i 2ks_i], \\ m_0(x\eta) &= \operatorname{sh}(\alpha x\eta) + \alpha\beta^{-1}\operatorname{sh}(\beta x\eta), \\ m_{1,2}(x\eta) &= (x\eta)^{-1}[\operatorname{ch}(\alpha x\eta) \mp \operatorname{ch}(\beta x\eta)], \\ m_3(x\eta) &= (x\eta)^{-2}\beta^{-1}[(k^2 s^2 - s_1)\operatorname{sh}(\alpha x\eta) - (k^2 s_2 + s_1)\operatorname{sh}(\beta x\eta)], \\ &\alpha = s_1 + s_2, \quad \beta = s_1 - s_2, \end{aligned} \tag{4.2}$$

are the functions of the elastic constants, and:

$$\varkappa_0 = 1, \ \varkappa_1 = k_n h(G_1 C)^{-1}, \ \varkappa_2 = k_t h(G_1 C s_1 s_2)^{-1}, \ \varkappa_3 = k_n k_t h^2 [G_1(k+1)]^{-2}, \tag{4.3}$$

describe the relative rigidity of the foundation to the layer, in which:

$$C = (k+1)(k-1)^{-1}(s_2^{-1} - s_1^{-1}), \tag{4.4}$$

is the function of the material parameters s_1, s_2 and k.

The normal stress and displacement needed for the solution of the mixed boundary conditions on the upper surface $z = 0$ of the layer can be obtained from the expressions of the potentials (4.1).

We obtain:

$$\begin{aligned} bG_1 Cw(\varrho, 0) &= \int_0^\infty w(x) J_0(x\varrho)\, dx, \\ b^2 \sigma_z(\varrho, 0) &= -\int_0^\infty x w(x)[1 - M(x\eta)] J_0(x\varrho)\, dx, \end{aligned} \tag{4.5}$$

where $M(x\eta)$ is the function involving material and geometrical parameters and is defined as:

$$M(x\eta) = 1 - \Big(\sum_{n=0}^{3} \varkappa_n f_n(x\eta)\Big) \Big/ \Big(\sum_{l=0}^{3} \varkappa_l m_l(x\eta)\Big),$$

$$f_0(x\eta) = \operatorname{ch}(\alpha x\eta) - 1 - \alpha^2\beta^{-2}[\operatorname{ch}(\beta x\eta) - 1], \tag{4.6}$$

$$f_{1,2}(x\eta) = (x\eta)^{-1}[\mathrm{sh}(\alpha x\eta) \pm \alpha\beta^{-2}\mathrm{sh}(\beta x\eta)], \tag{4.6 [cont.]}$$

$$f_3(x\eta) = (x\eta)^{-2}[(k^2 s_2 - s_1)\beta^{-1}\mathrm{ch}(\alpha x\eta) + \alpha\beta^{-2}(k^2 s_2 + s_1)\mathrm{ch}(\beta x\eta) - k\alpha^2\beta^{-2} + k].$$

We let:

$$p(x) = w(x)[1 - M(x\eta)], \tag{4.7}$$

where $p(x)$ is the new auxiliary function.
Then eqns (4.5) become:

$$bG_1 Cw(\varrho, 0) = \int_0^\infty p(x)[1 - g(x\eta)] J_0(x\varrho)\, dx,$$

$$b^2\sigma_z(\varrho, 0) = -\int_0^\infty xp(x) J_0(x\varrho)\, dx, \tag{4.8}$$

where:

$$g(x\eta) = 1 - \Big(\sum_{l=0}^{3} \varkappa_l m_l(x\eta)\Big) \Big/ \Big(\sum_{n=0} \varkappa_n f_n(x\eta)\Big), \tag{4.9}$$

with $m_n(x\eta)$ and $f_n(x\eta)$ being defined by eqns (4.2) and (4.6).

Hence, $p(x)$ is the only unknown which from eqns (a), (b) and (d) in (3.1) can be found.

5. The Fredholm integral equation

The problem of the type considered here has the character of the mixed boundary value problem with unknown radius b (1) of the contact region.

By substitution of eqns (4.8) into boundary conditions (a) and (b) of eqns (3.1) we find that the function $p(x)$ must satisfy the dual integral equations:

$$w(\varrho, 0) = (G_1 Cb)^{-1} \int_0^\infty p(x)[1 - g(x\eta)] J_0(x\varrho)\, dx =$$

$$= \varepsilon_0 - k_0 b(\varrho - \lambda) H(\varrho - \lambda); \quad 0 \leqslant \varrho \leqslant 1, \tag{5.1}$$

$$\sigma_z(\varrho, 0) = -b^{-2} \int_0^\infty xp(x) J_0(x\varrho)\, dx = 0; \quad 1 < \varrho,$$

under the condition (d) of eqs. (3.1).
Multiplying both sides of the first and second of eqns (5.1) by $\varrho(t^2 - \varrho^2)^{-1/2} d\varrho$ and $\varrho(\varrho^2 - t^2)^{-1/2} d\varrho$, respectively, and integrating with respect to ϱ from 0 to t and from t to ∞, respectively, and differentiating the result of the first equation with respect to t and using the formulas [3]:

$$\frac{d}{dt} \int_0^t \frac{\varrho J_0(x\varrho)}{\sqrt{t^2 - \varrho^2}}\, d\varrho = \cos xt,$$

$$\int_t^\infty \frac{\varrho J_0(x\varrho)}{\sqrt{\varrho^2 - t^2}}\, d\varrho = \frac{\cos xt}{x}, \tag{5.2}$$

we obtain:

$$\int_0^\infty p(x)\cos xt\,dx = \int_0^\infty p(x)g(x\eta)\cos xt\,dx +$$

$$+CG_1 b\left[\varepsilon_0 - k_0 bt \arccos\frac{\lambda}{t}\cdot H(t-\lambda)\right]; \quad 0 \leqslant t \leqslant 1, \tag{5.3}$$

$$\int_0^\infty p(x)\cos xt\,dx = 0, \quad t > 1.$$

Making use of the Fourier inverse-transformation:

$$p(x) = \frac{2}{\pi}\int_0^1 p(t)\cos xt\,dt, \tag{5.4}$$

we get the following Fredholm integral equation of the second kind:

$$p(t) = \int_0^1 p(\tau)K(t,\tau)d\tau + f(t); \quad t,\tau \in \langle 0,1\rangle, \tag{5.5}$$

where the symmetrical Kernel is defined as follows:

$$K(t,\tau) = \frac{1}{\pi}\int_0^\infty g(x\eta)[\cos x(t+\tau) + \cos x(t-\tau)]\,dx, \tag{5.6}$$

and the function $f(t)$ is given by:

$$f(t) = G_1 Cb\left[\varepsilon_0 - k_0 bt \arccos\frac{\lambda}{t}\cdot H(t-\lambda)\right]; \quad 0 \leqslant t \leqslant 1. \tag{5.7}$$

The essential difficulty of the problem is that there is an unknown contact region, between the punch and the layer; the extent of this region has first to be found. It is seen that there is an elegant solution of the half-space problem which converges to the closed-form solution, according to $g(x\eta) \equiv 0$ for the half-space case.

6. The solution of the Fredholm equation

The solution of the integral equation of the Fredholm type (5.5) we obtain by an iterative procedure; this is valid when the ratio of the layer thickness to the radius of the contact region satisfies some condition. This condition is clarified below. The approximation of zero order corresponds to the solution of the half-space problem. This solution, at first, we obtain

6.1. The half-space problem. For the half-space case $g(x\eta) \equiv 0$ so that the governing equations (5.5) and (5.4) give exact form of the solution:

$$= f(t),$$

$$p_0(x) = \frac{2}{\pi} G_1 Cb[(\varepsilon_0 - k_0 b \arccos\lambda)] \frac{\sin x}{x} + k_0 b \int_\lambda^1 \frac{\sin xt}{x} \cdot \frac{d}{dt}\left(t \arccos\frac{\lambda}{t}\right) dt. \tag{6.1}$$

Substituting eq. (6.1) into $\sigma_z(\varrho, 0)$ of eq. $(5.1)_2$, we obtain:

$$\sigma_{z0}(\varrho, 0) = -\frac{2G_1 C}{\pi b}\left[(\varepsilon_0 - k_0 b \arccos\lambda)\frac{H(1-\varrho)}{\sqrt{1-\varrho^2}} + \right.$$

$$\left. + k_0 b \int_\lambda^1 \left(\arccos\frac{\lambda}{t} + \frac{\lambda}{\sqrt{t^2-\lambda^2}}\right)\frac{H(t-\varrho)}{\sqrt{t^2-\varrho^2}} dt\right]. \tag{6.2}$$

The stress $\sigma_z(\varrho, 0)$ is always zero in $\varrho > 1$. The first term in eqn (6.2) is singular at $\varrho \to \to 1-0$, while the second term tends to zero in this point. For the singularity to vanish at $\varrho \to 1-0$, it must be true that the coefficient of the first term of eqn (6.2) must be zero because $\sigma_{z0}(\varrho, 0)$ is finite at $\varrho \to 1-0$. Then the condition (d) of eqn (3.1) gives:

$$\varepsilon_0 = k_0 b \arccos\lambda. \tag{6.3}$$

Eqn (6.3) represents the nonlinear relation among ε_0, k_0, a and b because $\lambda = a/b$. For given k_0, ε_0 and a this equation yields the extent of the contact region b in terms of known quantities. Substituting eqn (6.3) into eqn (6.1) and (6.2), yields:

$$p_0(x) = \frac{2}{\pi} G_1 Cb^2 k_0 \int_\lambda^1 \frac{\sin xt}{x} \cdot \frac{d}{dt}\left(t \arccos\frac{\lambda}{t}\right) dt, \tag{6.4}$$

$$\sigma_{z0}(\varrho, 0) = -\frac{2}{\pi} G_1 Ck_0 \int_\lambda^1 \left(\arccos\frac{\lambda}{t} + \frac{\lambda}{\sqrt{t^2-\lambda^2}}\right)\frac{H(t-\varrho)}{\sqrt{t^2-\varrho^2}} dt; \quad 0 \leqslant \varrho \leqslant 1. \tag{6.5}$$

Using the elliptic integrals of the first kind $F(\varphi, s)$, [4], the stress $\sigma_{z0}(\varrho, 0)$ in the interval $0 \leqslant \varrho \leqslant 1$ is:

$$\sigma_{z0}(\varrho, 0) = -\frac{2}{\pi} G_1 Ck_0 \times$$

$$\times \begin{cases} F\left(\frac{\pi}{2}, \frac{\varrho}{\lambda}\right) - F\left(\arcsin\varrho, \frac{\varrho}{\lambda}\right) + \int_\lambda^1 \frac{\arccos\left(\frac{\lambda}{t}\right)}{\sqrt{t^2-\varrho^2}} dt; & 0 \leqslant \varrho < \lambda, \\ \frac{\lambda}{\varrho}\left[F\left(\frac{\pi}{2}, \frac{\lambda}{\varrho}\right) - F\left(\arcsin\varrho, \frac{\lambda}{\varrho}\right)\right] + \int_\varrho^1 \frac{\arccos\left(\frac{\lambda}{t}\right)}{\sqrt{t^2-\varrho^2}} dt; & \lambda < \varrho \leqslant 1. \end{cases} \tag{6.6}$$

The stress $\sigma_{z0}(\varrho, 0)$ becomes an infinite compressive at $\varrho \to \lambda \pm 0$. The logarithmic singularity of the contact stress is:

$$\sigma_{z0}(\varrho, 0) \cong -\frac{2}{\pi} G_1 Ck_0 \ln\left[\frac{1}{1-\min\left(\frac{\varrho}{\lambda}, \frac{\lambda}{\varrho}\right)}\right]; \quad \varrho \to \lambda \pm 0. \tag{6.7}$$

Using eqns (6.4), (4.8)$_1$ and (6.3), the axial displacement $w(\varrho, 0)$ on the surface of the half-space $(g(x\eta) \equiv 0)$ is calculated to be:

$$w_0(\varrho, 0) = \frac{2}{\pi} k_0 b \int_\lambda^1 \frac{d}{dt}\left(t \arccos \frac{\lambda}{t}\right) dt \int_0^\infty \frac{\sin xt}{x} J_0(x\varrho) dx =$$

$$= \begin{cases} \varepsilon_0 - k_0 b(\varrho - \lambda) H(\varrho - \lambda); & 0 \leqslant \varrho \leqslant 1 \\ \dfrac{2}{\pi}\left[\varepsilon_0 \arcsin\left(\dfrac{1}{\varrho}\right) - k_0 b \displaystyle\int_\lambda^1 \arccos \frac{\lambda}{t} \cdot \frac{t}{\sqrt{\varrho^2 - t^2}} dt\right]; & \varrho \geqslant 1. \end{cases} \tag{6.8}$$

The gradient of $w_0(\varrho, 0)$ in $\varrho \geqslant 1$ with the aid of eqn (6.3) is:

$$\frac{dw_0(\varrho, 0)}{bd\varrho} = \frac{2}{\pi} \cdot \frac{\varepsilon_0}{b} \cdot \frac{\sqrt{\varrho^2 - 1}}{\varrho} - \frac{k_0}{\pi}\left[\frac{\pi}{2} + \arcsin\left(\frac{\lambda^2(1-\varrho^2) + \varrho^2(1-\lambda^2)}{\varrho^2 - \lambda^2}\right)\right]. \tag{6.9}$$

It is seen that as ϱ tends to $1+0$, the gradient of $w_0(\varrho, 0)$ tends to $-k_0$. Then the slope of $w_0(\varrho, 0)$ at $\varrho = 1$ coincides with that of the punch face. It means that the condition of $\sigma_z(\varrho, 0) =$ finite at the smoothly contact edge $\varrho = 1$ is equivalent to $[dw_0(\varrho, 0)/d\varrho]_{\varrho=1} = -k_0$.

By substitution of the stress (6.5) into the equilibrium condition of the punch (3.2), we obtain the resultant load:

$$P_0^{(0)} = 2G_1 C \varepsilon_0 b\left[1 + \frac{ak_0}{\varepsilon_0} \sin\left(\frac{\varepsilon_0}{k_0 b}\right)\right]. \tag{6.10}$$

The relationships among the resultant load P_0^0, the depth of the penetration ε_0 and the configuration of the punch k_0, a and the contact radius b are nonlinear.

6.2. The layer problem. The layer case is complicated by the presence of the boundary function $g(x\eta)$ in equations (5.5) and (5.6). An iterative solution to eqn (5.5) can be obtained if $\eta^{-1}(= b/h)$ is small. It is convenient to make the kernel (5.6) dimensionless. Introduce a new variable $\xi = x\eta$ and write the kernel (5.6) as follows:

$$K(t, \tau) = \frac{1}{\pi} \eta^{-1} \int_0^\infty g(\xi)[\cos \xi\eta^{-1}(t+\tau) + \cos \xi\eta^{-1}(t-\tau)] d\xi; \; K(t, \tau) \in L^2. \tag{6.11}$$

When expanded right hand side in eqn (6.11) in powers of η^{-1} this gives:

$$K(t, \tau) = \sum_{n=0}^\infty \frac{I_n}{\eta^{2n+1}} K^n(t, \tau); \quad t, \tau \in \langle 0,1 \rangle, \tag{6.12}$$

where:

$$K^n(t, \tau) = (-1)^n \frac{1}{2(2n)!} [(t+\tau)^{2n} + (t-\tau)^{2n}], \tag{6.13}$$

and:

$$I_n = \frac{2}{\pi} \int_0^\infty g(\xi) \xi^{2n} d\xi; \quad n = 0, 1, 2, \ldots \tag{6.14}$$

The power series in eqn (6.12) is absolutely and uniformly convergent for any values $t, \tau \in \langle 0, 1\rangle$, when the parameter $\eta(= h/b)$ satisfies the inequality:

$$\eta > C_1, \tag{6.15}$$

where:

$$C_1 = \sup_{0 \leq t, \tau \leq 1} |K(t, \tau)|. \tag{6.16}$$

The analysis of the layer problem by expansion in power series wit respect to the ratio $\eta^{-1}(= b/h)$ restricts the range of applicability of the obtained solution to these cases in which the ratio of the layer thickness to the contact radius of the punch satisfies the inequality (6.15). The governing integral equation (5.5) is exact, within the assumption of classical elasticity theory. Since this equation were derived using small-displacement approximations it must be ensured that $\varepsilon_0 < \bar{\varepsilon}_0$, where ε_0 is the measure of the depth penetration of the punch and $\bar{\varepsilon}_0$ is a predetermined value below which displacement are considered "small". Thus it is possible to place an upper bound on the ratio b/h. Using the results for the penetration of the punch, which are given below by eqns (6.21) or (6.24) we can shown the maximum allowable values of b/h to ensure small deflection as $b/h < < (\bar{\varepsilon}_0/h)/k_0 f(b/h, a/b, k_n, k_t$, material parameters), where $f(\cdot)$ are the real-valued functions involving material and geometrical parameters. Their full expressions are given below by eqns (6.28). If these conditions are violated the assumptions of infinitesimal linear elasticity are not applicable so that the present analysis is not meaningful. The inequality on b/h must be verified numerically; it was found to be satisfied.

The function $g(\xi)$, defined by eqns (4.2), (4.6) and (4.9) exponentially tends to zero as the value of ξ becomes large, is continuous for any $\xi \in (0, \infty)$ and its limit is bounded as ξ tends to zero. Then, we can easily evaluated the integrals (6.14) by a numerical method, integrating in finite interval.

The solution of the layer case was obtained in iterative form, all quantities being expressed in power series in the parameter $\eta^{-1}(= b/h)$. We have the following recursive relation to calculate of the auxiliary function $p(t)$:

$$p_{r+1}(t) = p_0(t) + \int_0^1 p_r(\tau) K(t, \tau) d\tau; \quad r = 0, 1, 2, \ldots, \tag{6.17}$$

where $p_0(t)$ is the solution of the half-space problem, which is given by eqn (6.1) and (5.7). Applying the expansion (6.12) and (6.13) and the iterative formula (6.17), we obtain the approximate solutions correct to $0(\eta^{-9})$ or $0(\eta^{-10})$:

$$\begin{aligned} p_1(t) &= p_0(t) + G_1 C b \eta^{-1} \hat{p}_1(t), \\ p_2(t) &= p_0(t) + G_1 C b \eta^{-1} [\hat{p}_1(t) + \eta^{-1} \hat{p}_2(t)], \end{aligned} \tag{6.18}$$

where $p_1(t)$ and $p_2(t)$ are the first and second approximations, respectively, and:

$$\hat{p}_1(t) = \varepsilon_0(a_0 + a_1 t^2 + a_2 t^4 + a_3 t^6) - k_0 b(b_0 + b_1 t^2 + b_2 t^4 + b_3 t^6) + 0(\eta^{-8}), \tag{6.19}$$

$$\hat{p}_2(t) = \varepsilon_0(c_0 + c_1 t^2) - k_0 b(d_0 + d_1 t^2) + 0(\eta^{-8}). \tag{6.20}$$

describe the differences of the approximate solutions of the layer problem and exact solution of the half-space problem, respectively. The coefficients a_i, b_i, c_i and d_i are the functions involving material and geometrical parameters; their full expressions are given in the Appendix. Substituting eqn (5.4) into eqn (4.8), using the solutions (6.18) and (6.19), after simplification (i.e. through use of the Weber-Schafheitlin integral [3]), we arrive at the following solutions:

1st approximation:

$$\varepsilon_0[1+\eta^{-1}(a_0+a_1+a_2+a_3)] = k_0 b[\arccos\lambda+\eta^{-1}(b_0+b_1+b_2+b_3)], \tag{6.21}$$

$$\sigma_{z1}(\varrho,0) = \sigma_{z0}(\varrho,0)+\frac{4}{\pi}G_1Cb^{-1}\eta^{-1}\sqrt{1-\varrho^2}\left\{\varepsilon_0\left[a_1+\frac{2}{3}a_2+\frac{3}{5}a_3+4\left(\frac{1}{3}a_2+\right.\right.\right.$$
$$\left.\left.+\frac{1}{5}a_3\right)\varrho^2+\frac{8}{5}a_3\varrho^4\right]-k_0b\left[b_1+\frac{2}{3}b_2+\frac{3}{5}b_3+4\left(\frac{1}{3}b_z+\frac{1}{5}b_3\right)\varrho^2+\frac{8}{5}b_3\varrho^4\right]\right\};$$
$$\varrho\in\langle 0,\lambda)\cup(\lambda,1\rangle, \tag{6.22}$$

$$w_1(\varrho,0)= w_0(\varrho,0)+\frac{2}{\pi}\eta^{-1}\left\{\varepsilon_0\left[\arcsin\left(\frac{1}{\varrho}\right)\cdot\left(a_0+\frac{1}{2}a_1\varrho^2+\frac{3}{8}a_2\varrho^4+\right.\right.\right.$$
$$\left.\left.+\frac{5}{16}a_3\varrho^6\right)-\frac{1}{2}\sqrt{\varrho^2-1}\left(a_1+\frac{1}{2}a_2+\frac{1}{3}a_3+\frac{1}{4}\varrho^2\left(3a_2+\frac{5}{4}a^3\right)+\frac{5}{8}a_3\varrho^4\right)\right]$$
$$-k_0b\left[\arcsin\left(\frac{1}{\varrho}\right)\cdot\left(b_0+\frac{1}{2}b_1\varrho^2+\frac{3}{8}b_2\varrho^4+\frac{5}{16}b_3\varrho^6\right)-\frac{1}{2}\sqrt{\varrho^2-1}\left(b_1+\frac{1}{2}b_2+\right.\right.$$
$$\left.\left.\frac{1}{3}b_3+\frac{1}{4}\varrho^2\left(3b_2+\frac{5}{4}b_3\right)+\frac{5}{8}b_3\varrho^4\right)\right]$$
$$\left.-\int_0^\infty\left[g(x\eta)J_0(x\varrho)\int_0^1\hat{p}_1(t)\cos xt\,dt\right]dx\right\};\quad \varrho\geqslant 1. \tag{6.23}$$

2nd approximation:

$$\varepsilon_0[1+\eta^{-1}(a_0+a_1+a_2+a_3)+\eta^{-2}(c_0+c_1)] =$$
$$= k_0b[\arccos\lambda+\eta^{-1}(b_0+b_1+b_2+b_3)+\eta^{-2}(d_0+d_1)], \tag{6.24}$$

$$\sigma_{z2}(\varrho,0) = \sigma_{z1}(\varrho,0)+\frac{4}{\pi}G_1Cb^{-1}\eta^{-2}\sqrt{1-\varrho^2}\,(\varepsilon_0c_1-k_0bd_1); \tag{6.25}$$
$$\varrho\in\langle 0,\lambda)\cup(\lambda,1\rangle,$$

$$w_2(\varrho,0) = w_1(\varrho,0)+\frac{2}{\pi}\eta^{-2}\left\{\varepsilon_0\left[\arcsin\left(\frac{1}{\varrho}\right)\cdot\left(c_0+\frac{1}{2}c_1\varrho^2\right)\right.\right.$$
$$\left.-\frac{1}{2}c_1\sqrt{\varrho^2-1}\right]-k_0b\left[\arcsin\left(\frac{1}{\varrho}\right)\cdot\left(d_0+\frac{1}{2}d_1\varrho^2\right)-\frac{1}{2}d_1\sqrt{\varrho^2-1}\right] \tag{6.26}$$
$$\left.-\int_0^\infty\left[g(x\eta)J_0(x\varrho)\int_0^1\hat{p}_2(t)\cos xt\,dt\right]dx\right\};\quad \varrho\geqslant 1.$$

The improper integrals in eqns (6.23) and (6.26) can be evaluated in finite interval by a numerical method, because the integrands involving the function $g(x\eta)$ converge exponentially to zero as the value of x becomes large.
The above mentioned approximations are valid for layer case provided that the inequality:

$$b/h < (\bar{\varepsilon}_0/h)/k_0 f_{1,2}(b/h,\, a/b,\, k_n,\, k_t,\, \textit{material parameters}), \tag{6.27}$$

is not violated.
In an upper bound (6.27) which can be placed on the ratio of contact radius to layer thickness, the ratio $\bar{\varepsilon}_0/h$ is the relative allowable displacement under the indenter below which displacement are considered "small" and the functions $f_{1,2}(\cdot)$ are defined as follows:

$$\begin{aligned} f_1(\cdot) &= [\arccos\lambda+\eta^{-1}(b_0+b_1+b_2+b_3)][1+\eta^{-1}(a_0+a_1+a_2+a_3)]^{-1}, \\ f_2(\cdot) &= [\arccos\lambda+\eta^{-1}(b_0+b_1+b_2+b_3)+\eta^{-2}(d_0+d_1)][1+\eta^{-1}(a_0+ \\ &\quad +a_1+a_2+a_3)+\eta^{-2}(c_0+c_1)]^{-1}; \qquad \lambda = a/b,\ \eta^{-1} = b/h, \end{aligned} \tag{6.28}$$

in the first and second approximations, respectively, where a_i, b_i, c_i and d_i are defined in the Appendix. The expression (6.27) shows the maximum allowable values of b/h to ensure small deflections and determines the range of applicability of the presented approximate results. In the problems solved through the course of this study, the contact radius b changes as the applied load changes. If the imposed condition (6.27) is not violated, the problems studied are indeed linear contact problems and the results obtained approximates the physical quantities which characterize the contact quite well for engineering applications. The conditions on b/h, must be verified numerically; it was found to be satisfied.

In both approximations the total load P_0 is:

$$\begin{aligned} P_0^{(1)} = P_0^{(0)} - 8G_1 Cb\eta^{-1}\Big\{\varepsilon_0\Big(\frac{1}{3}a_1+\frac{2}{5}a_2+\frac{13}{35}a_3\Big) \\ -k_0 b\Big(\frac{1}{3}b_1+\frac{2}{5}b_2+\frac{13}{35}b_3\Big)\Big\}, \end{aligned} \tag{6.29}$$

$$P_0^{(2)} = P_0^{(1)} - \frac{8}{3}G_1 Cb\eta^{-2}(\varepsilon_0 c_1 - k_0 b d_1). \tag{6.30}$$

Equations (6.21), (6.24) and (6.29), (6.30) represent the relationships among P_0, ε_0, k_0, a, h and b, which are nonlinear. The contact stress in the layer case has the singularity such as in the half-space problem; it is the logarithmic singularity given by eqn (6.7).

7. The special cases

The solution of the special cases are summarized:
(a) Indentation of an half-space by a cylindrical punch. Particulary, taking $b \to a+\delta_1(1 \to \lambda+\delta,\ \delta \to 0)$ and $\varepsilon_0 = \text{const}$, we have:

$$\arccos\lambda = \sqrt{2\delta}, \qquad \varepsilon_0 = k_0\sqrt{2\delta}\,a, \qquad p_0(x) = \frac{2}{\pi}G_1 Ca\varepsilon_0\frac{\sin x}{x},$$

$$P_0^{(0)} = 4G_1 C\varepsilon_0 a \tag{7.1}$$

$$\sigma_{z0}(\varrho, 0) = -\frac{2}{\pi} G_1 C \frac{\varepsilon_0}{a} \frac{1}{\sqrt{1-\varrho^2}} H(1-\varrho), \quad (7.1)\ [\text{cont.}]$$

$$w_0(\varrho, 0) = \varepsilon_0 \left[H(1-\varrho) + \frac{2}{\pi} \arcsin\left(\frac{1}{\varrho}\right) H(\varrho - 1) \right].$$

(b) Indentation of a half-space by a conical punch
Taking $a \to 0 (\lambda \to 0)$, we have:

$$p_0(x) = G_1 C b^2 k_0 \frac{\cos x - 1}{x^2}, \qquad \varepsilon_0 = \frac{\pi}{2} k_0 b,$$

$$P_0^{(0)} = 2 G_1 C \varepsilon_0 b = \pi G_1 C b^2 k_0, \quad (7.2)$$

$$\sigma_{z0}(\varrho, 0) = -G_1 C k_0 \operatorname{arcch}\left(\frac{1}{\varrho}\right) \cdot H(1-\varrho),$$

$$w_0(\varrho, 0) = (\varepsilon_0 - k_0 b \varrho) H(1-\varrho) + \frac{2}{\pi} \varepsilon_0 \left[\arcsin\left(\frac{1}{\varrho}\right) - \varrho + \sqrt{\varrho^2 - 1} \right] H(\varrho - 1).$$

The results (7.1) and (7.2) are the solutions for a half-space loaded by an indenter cylindrical or conical. These results show that the depth ε_0 of penetration and the displacement $w_0(\varrho, 0)$ are dependent on the material parameter $G_1 C$, where as the contact stress $\sigma_{z0}(\varrho, 0)$ is independent on the elastic constants of the material in a case of cylindrical punch. In contrast with those, it is found in a case of conical punch the contact stress depends on the elastic properties of the material, because the radius of the contact region depends on the parameter $G_1 C$. Taking $G_1 C = G/(1-\nu)$ we obtain the results of the isotropic case, when the half space is an isotropic medium with shear modulus G and Poisson's ratio ν, which agree with the known results [5, 6]. From Eqns (7.1) and (7.2) we can understand that a difference between two solutions of the isotropic and anisotropic half-space contact problem is in the function of elastic constant only.
(c) Indentation of an layer by a conical punch
Taking $g(x\eta) \neq 0$ and $\lambda \to 0$, we have:

$$i_n(0) = \pi/2(n+1), \qquad (\text{see Appendix}),$$

$$\varepsilon_0 [1 + \eta^{-1}(a_0 + a_1 + a_2 + a_3)] = \frac{\pi}{2} k_0 b [1 + \eta^{-1}(\bar{b}_0 + \bar{b}_1 + \bar{b}_2 + \bar{b}_3)],$$

or: (7.3)

$$\varepsilon_0 [1 + \eta^{-1}(a_0 + a_1 + a_2 + a_3) + \eta^{-2}(c_0 + c_1)] =$$

$$= \frac{\pi}{2} k_0 b [1 + \eta^{-1}(\bar{b}_0 + \bar{b}_1 + \bar{b}_2 + \bar{b}_3) + \eta^{-2}(\bar{d}_0 + \bar{d}_1)],$$

in the first and second approximations, respectively.
Full expressions of the coefficients $\bar{b}_i$ and $\bar{d}_i$ are given in the Appendix.
Substituting the relations (7.3) into eqns (6.22), (6.23) and (6.29) or eqns (6.25), (6.26) and (6.30) and replacying, at first b_i, d_i and $k_0 b$ by $\bar{b}_i$, $\bar{d}_i$ and $\pi k_0 b/2$, respectively, we obtain the solutions for a layer indented by a conical punch. The relations among ε_0, k_0, h and b are nonlinear.

(d) Indentation of a layer by a cylindrical punch
For the layer indented by a cylindrical punch we have:

$$i_n(1-\delta) \to \frac{2n-1}{2^{n+3}}\sqrt{2\delta} \qquad (n = 3, 5, 7), \qquad i_1(1-\delta) \to \frac{1}{4}\sqrt{2\delta}\,,\ \delta \to 0,$$

(see Appendix), (7.4)

$$\varepsilon_0[1+\eta^{-1}(a_0+a_1+a_2+a_3)] = k_0 a\sqrt{2\delta}\,[1+\eta^{-1}(\bar{\bar{b}}_0+\bar{\bar{b}}_1+\bar{\bar{b}}_2+\bar{\bar{b}}_3)],$$

or:

$$\varepsilon_0[1+\eta^{-1}(a_0+a_1+a_2+a_3)+\eta^{-2}(c_0+c_1)] =$$
$$= k_0 a\sqrt{2\delta}\,(1+\eta^{-1}(\bar{\bar{b}}_0+\bar{\bar{b}}_1+\bar{\bar{b}}_2+\bar{\bar{b}}_3)+\eta^{-2}(\bar{\bar{d}}_0+\bar{\bar{d}}_1)],$$

in the first and second approximations, respectively. Substituting eqn (7.4) into eqns (6.22), (6.23) and (6.29) or (6.25), (6.26) and (6.30) and replacying, at first, b_i, d_i and $k_0 b$ by $\bar{\bar{b}}_i$, $\bar{\bar{d}}_i$ (see Appendix) and $k_0 b\sqrt{2\delta}$, respectively, we obtain the solutions for a layer indented by the cylindrical punch.

(e) Indentation of a half-space or of a layer by a cylindrical punch with beveling.
Assuming, that:

$$\varepsilon_0 - k_0 b \arccos\lambda = F(\varepsilon_0, k_0, a, b) > 0, \tag{7.5}$$

we obtain the results for a cylindrical punch with beveling. In this case, the contact stress $\sigma_{z0}(\varrho, 0)$ is given by eqn (6.2) for the half-space problem (zero order approximation) and by eqns (6.22) or (6.25) for the layer problem.
The resultant load, in this case, is:

$$P_0^{(0)} = 2G_1 Cb\,[2\varepsilon_0 + k_0 b\,(\lambda\sqrt{1-\lambda^2} - \arccos\lambda)]. \tag{7.6}$$

Substituting eqn (7.6) into eqn (6.29) or (6.30) we obtain the total load P_0 for the layer problem.
The displacement $w_0(\varrho, 0)$ in $\varrho \geqslant 1 (r \geqslant b)$ is given by eqn (6.8) and the radial gradient of the displacement in the interval $\varrho > 1$ is:

$$\frac{dw_0(\varrho,0)}{bd\varrho} = -\frac{2}{\pi\varrho b}\cdot\frac{\varepsilon_0 - k_0 b\varrho^2\arccos\lambda}{\sqrt{\varrho^2-1}} -$$
$$-\frac{k_0}{\pi}\left[\frac{\pi}{2}+\arcsin\frac{\lambda^2(1-\varrho^2)+\varrho^2(1-\lambda^2)}{\varrho^2-\lambda^2}\right]; \quad \varrho > 1. \tag{7.7}$$

In this case, the gradient of $w_0(\varrho, 0)$ tends to infinity at $\varrho \to 1+0$. For the condition (6.3) the radial slope (7.7) corresponds to eqn (6.9).

(f) Contact problems for a thick layer indented by a pair of presented punches.
Considering the limiting case $\varkappa_1 \to \infty$ and $\varkappa_2 \to 0$ we obtain the boundary function in the form:

$$g(x\eta) \to g_0(x\eta) = 1 - \frac{\operatorname{ch}\alpha x\eta - \operatorname{ch}\beta x\eta}{\operatorname{sh}\alpha x\eta + \alpha\beta^{-1}\operatorname{sh}\beta x\eta}. \tag{7.8}$$

Evaluating the integral (6.14) for the function (7.8) and substituting these integrals into the presented results we obtain the solutions of the contact problems for a thick plate of thickness $2h$ indented by a pair of truncated conical or cylindrical or conical or cylindrical with beveling punches on its upper and lower surfaces.

8. Isotropic medium

All the result obtained in this paper can also be applied to completely isotropic bodies, provided that $s_1 \to 1$, $s_2 = 1$, i.e. $\alpha = 2$, $\beta \to 0$ and $k \to 1$. By evaluation of the limits by means of de L'Hospital's rule in the above mentioned expressions we can obtain without any difficulty all the boundary functions the relative rigidities of foundation to layer and the material parameter C, which assume the form:

$$g(x) = 1 - \sum_{n=0}^{3} \frac{\varkappa_n m_n(x)}{\varkappa_n f_n(x)}, \tag{8.1}$$

where:

$$\varkappa_0 = 1, \quad \varkappa_1 = k_n h(1-\nu)/G, \quad \varkappa_2 = k_t h(1-\nu)/G, \varkappa_3 = k_n k_t h^2/4G^2 \tag{8.2}$$

$$f_0(x) = \text{ch}\,2x - 2x^2 - 1, \quad f_{1,2}(x) = x^{-1}(\text{sh}\,2x \pm 2x), \tag{8.3}$$

$$f_3(x) = x^{-2}[(3-4\nu)\text{ch}\,2x + 2x^2 + 1 + 4(1-\nu)(1-2\nu)],$$

$$m_0(x) = \text{sh}\,2x + 2x, \quad m_{1,2}(x) = x^{-1}(\text{ch}\,2x \mp 1),$$
$$m_3(x) = x^{-2}[(3-4\nu)\text{ch}\,2x - 2x], \tag{8.4}$$

$$C = (1-\nu)^{-1}, \tag{8.5}$$

when the layer is an isotropic medium with shear modulus G and Poisson's ratio ν. The solution of a transversely isotropic case leads to the solution of the isotropic one.

9. Numerical results

Numerical results are presented for the relation among the resultant load P_0, the depth of penetration ε_0, the contact radius b and the punch configuration a and k_0 and for different materials such as cadmium and magnesium crystals [7], fiber-reinforced composite materials with the fiber direction normal to the plane punch face, E glass-epoxy and graphite-epoxy [8] and comparative isotropic material, which occupied the half-space region.

To expedite numerical evaluation, we determine $P_0^{(0)}/2G_1Ca\varepsilon_0$ and $\lambda = a/b$ due to ak_0/ε_0 as shown in Fig. 2. The values $P_0^{(0)}/2G_1Ca\varepsilon_0$ and $\lambda = a/b$ become infinity and zero, respectively, as ak_0/ε_0 tends to zero. We can consider two states in such a case. The first state, in which k_0 and ε_0 are constant and a tends to zero corresponds to the indentation by a conical punch. In this case, $P_0^{(0)}$ and b are finite. The second state, in which a and ε_0 are constant and k_0 tends to zero corresponds to the indentation by a semi-infinite punch with a little inclination.

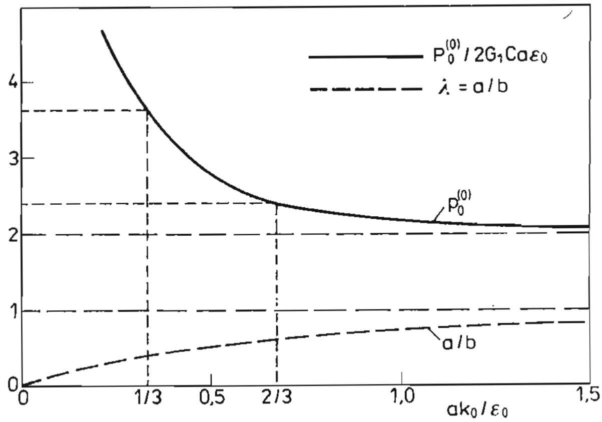

Fig. 2. The relation among a, b, k_0, $P_0^{(0)}$ and ε_0

The values $P_0^{(0)}/2G_1 Ca\varepsilon_0$ decrease monotonously with an increasing ak_0/ε_0, and tend to 2 as ak_0/ε_0 becomes large (cylindrical punch). In this case, we take $\lambda \to 1$.

The effect of the material dissimilarity is used by the constants $G_1 C$. Their values are: $2.03G_i$; $2.51G_i$; $1.28G_i$; $1.57G_i$; $G_i/(1-\nu)$, where $G_i = 10^4$ MPa, for cadmium, magnesium, E glass-epoxy, graphite-epoxy and isotropic materials, respectively. The effect of material anisotropy is apparent.

Fig. 3 shows the contact stress distributions in the two cases of $a/b = 1/3$ and $a/b = 2/3$, respectively. The pressure in the region $\varrho < \lambda$ is smaller than that for the cylindrical punch.

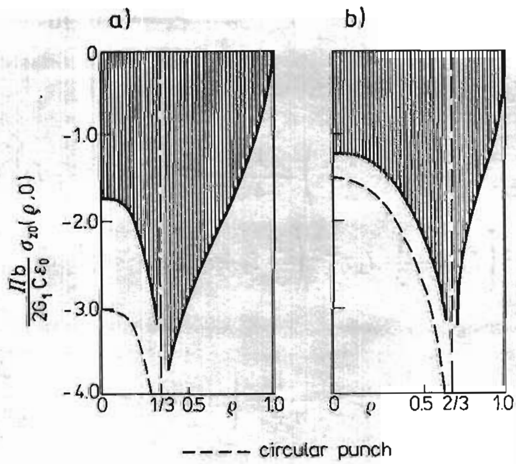

Fig. 3. The distribution of contact stress for $\lambda = 1/3$ (a) and $\lambda = 2/3$ (b)

The results, which are shown, correspond to the half-space case. The presented solutions may be used also to show the qualitative and quantitative effects of the plate thickness, boundary conditions and anisotropy on the contact behaviour and the load-contact length relation in the contact problem between the transversely isotropic layer and the rigid indenters such as cylindrical, conical, truncated conical and cylindrical with beveling. The solution may be also applied to many other problems, including contact problems for the transversely isotropic layer indented by a pair of the presented punches.

Appendix

The parameters that appear in equations (6.19) and (6.21) - (6.30) are defined as:

$$a_0 = \sum_{n=0}^{3} (-1)^n \frac{1}{(2n+1)!} I_n \eta^{-2n}, \qquad a_1 = \frac{1}{2} \sum_{n=1}^{3} (-1)^n \frac{1}{(2n-1)!} I_n \eta^{-2n},$$

$$a_2 = \frac{1}{24} \eta^{-4}\left(I_2 - \frac{1}{6} I_3 \eta^{-2}\right), \qquad a_3 = -\frac{1}{7!} I_3 \eta^{-6};$$

$$b_0 = \sum_{n=0}^{3} (-1)^n \frac{1}{(2n)!} I_n \eta^{-2n} i_{2n+1},$$

$$b_1 = -\frac{1}{2} \eta^{-2}\left(i_1 I_1 - \frac{1}{2} i_3 I_2 \eta^{-2} + \frac{1}{24} i_5 I_3 \eta^{-4}\right),$$

$$b_2 = \frac{1}{24} \eta^{-4}\left(i_1 I_2 - \frac{1}{2} i_3 I_3 \eta^{-2}\right), \qquad b_3 = -\frac{1}{6!} I_3 i_1 \eta^{-6}$$

$$c_0 = I_0\left(I_0 - \frac{1}{2} I_1 \eta^{-2} + \frac{1}{19} I_2 \eta^{-4}\right) + \frac{1}{36} I_1 \eta^{-4}\left(I_1 - \frac{1}{10} I_2 \eta^{-2}\right),$$

$$c_1 = -\frac{1}{2} I_1\left(I_0 - \frac{1}{2} I_1 \eta^{-2}\right)\eta^{-2},$$

$$d_0 = i_1 I_0^2 - I_0 I_1 \eta^{-2}\left(\frac{1}{3} i_1 + \frac{1}{2} i_3\right) + \frac{1}{4} I_1^2 \eta^{-4}\left(\frac{1}{5} i_1 + \frac{1}{3} i_3\right) + \frac{1}{24} I_0 I_2 i_5 \eta^{-4}$$

$$d_1 = -\frac{1}{2} I_1 \eta^{-2}\left[I_0 i_1 - \frac{1}{2} I_1 \eta^{-2}\left(\frac{1}{3} i_1 + i_3\right)\right],$$

where:

$$i_n(\lambda) = \int_{\lambda}^{1} \tau^n \arccos\left(\frac{\lambda}{\tau}\right) d\tau = \frac{1}{n+1}\left[\arccos\lambda - \lambda \int_{\lambda}^{1} \frac{\tau^n}{\sqrt{\tau^2-\lambda^2}}\, d\tau\right] =$$

$$= \frac{1}{n+1}\left[\arccos\lambda - \frac{\lambda\sqrt{1-\lambda^2}}{n}\left\{\begin{array}{ll} 1; & n=1 \\ 1+2\lambda^2; & n=3 \\ 1+\frac{4}{3}\lambda^2+\frac{8}{3}\lambda^4; & n=5 \\ 1+\frac{6}{5}\lambda^2+\frac{8}{5}\lambda^4+\frac{16}{5}\lambda^6; & n=7 \end{array}\right\}\right],$$

$$i_n(\lambda) \in (i_n(1-\delta),\ i_n(0)> \qquad \delta \to 0,$$

$$i_1(1-\delta) \to \frac{1}{4}\sqrt{2\delta}\,, \qquad i_n(1-\delta) \to \frac{2n-1}{2^{n+3}}\sqrt{2\delta}\,; \qquad n = 3, 5, 7$$

$$i_n(0) = \pi/[2(n+1)]; \qquad n = 1, 3, 5, 7$$

The parameters that appear in the equations (7.3) and (7.4) are defined by the coefficients b_i and d_i by replacying $i_n(\lambda)$ by $1/(n+1)$ for $\bar{b}_i$ and $\bar{d}_i$, and $i_n(\lambda)$ by 1/4, for $n = 1$, and by $(2n-1)/2^{n+3}$, for $n = 3, 5, 7$, for $\bar{\bar{b}}_i$ and $\bar{\bar{d}}_i$.

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Резюме

ВДАВЛИВАНИЕ ШТАМПА В ВИДЕ СРЕЗАННОГО КОНУСА В ТРАНСВЕРСАЛЬНО — ИЗОТРОПНОЙ СЛОЙ

Рассмотрено контактную задачу для трансверсально-изотропного слоя находящегося на двухпараметрическом упругом основании и штампа в виде срезанного конуса. Автор сформулировал задачу как решение дуальных интегральных уравнений, которые решил приближенно. Физические величины характеризующие рассмотренную контактную задачу и сингулярность контактных напряжений представлено формулами, которые для слоя апроксимационные. Некоторые особенные случайи как штамп цилиндрический, в виде конуса, либо цилиндра ограниченного срезанным конусом тоже рассмотрено. Численные расчёты выполнены для кристалов кадмия и магния и композитов эпоксидовых армированных волокнами стекла и графита.

Streszczenie

KONTAKT WARSTWY POPRZECZNIE IZOTROPOWEJ I STEMPLA O KSZTAŁCIE STOŻKA ŚCIĘTEGO

Rozpatrzono zagadnienie kontaktowe dla poprzecznie izotropowej warstwy na dwuparametrowym sprężystym podłożu i stempla w kształcie stożka ściętego. Autor sformułował zagadnienie jako rozwiązanie dualnych równań całkowych, które rozwiązał w sposób przybliżony.

Fizyczne wielkości charakteryzujące rozpatrzony kontakt i osobliwość naprężeń przedstawiono w postaci wzorów, które dla zagadnienia półprzestrzeni są ścisłe natomiast dla warstwy mają charakter aproksymacyjny. Pewne szczególne przypadki takie jak: stempel walcowy, stożkowy, walcowy ze ścięciem zostały także rozpatrzone. Przedstawiono wyniki liczbowe dla różnych materiałów takich jak: kadm, magnez i kompozytów epoksydowych zbrojonych włóknami ze szkła lub grafitu.

Praca wpłynęła do Redakcji dnia 3 grudnia 1982 roku.