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ON THE CONSTITUTIVE MODELLING OF ELASTIC-PLASTIC MICRO-PERIODIC MATERIALS

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1. Introduction

The aim of the paper is to outline an approach which enables one to describe the macro-properties of certain nonelastic (for example elastic-plastic) micro-periodic material composites. This problem has been studied, among the others, by Suquet (1985) and Marigo et al. (1987) via the general (asymptotic) homogenization method. The method outlined in this paper is based on the concept of the ideal constraints for stresses, Woźniak (1984) and takes into account the ideas of the nonstandard (microlocal) homogenization approach developed in a series of papers by Woźniak (1987, 1989), Kaczyński and Matysiak (1988), Matysiak and Nagorko (1989), Naniewicz (1987), Wągrowska (1988) and others. The main feature of the method is that it is relatively simple compared with the general (asymptotic) homogenization method and hence can be succesfully applied in the engineering practice. On the other hand it is an approximate method which makes it possible to obtain different, more or less approximate, macro-models of micro-periodic nonelastic material structures. The approach can be divided into the local (constitutive) and global modelling. In the local modelling we deal exclusively with one representative (heterogeneous) volume element and we describe the overall properties (macro-properties) of this element in term of certain macro-quantities. The global modelling constitutes an avenue leading from the laws of motion for the fields in the micro-periodic structure to the laws of motion for macro-fields introduced via the local modelling. In the present paper we restrict ourselves to the local (constitutive) modelling. All considerations are carried out within the range of infinitesimal strains.

Notations. Indices i,j,k run over 1,2,3 while indices a and A run over 1,...,n and 1,...,N, respectively; summation condition holds. For an arbitrary integrable function f(.) defined a.e. (almost everywhere) on the region V in R^3 we denote

$$\langle f\rangle = \frac{1}{\mathrm{vol}\ V}\int_V f(y)dy$$

where $dy \equiv dy_1dy_2dy_3$. For every symmetric tensor with components k_{ij} we define $k = (k_{ij}) \equiv (k_{11},k_{22},k_{33},k_{12},k_{23},k_{31})$ as an element of R^6. Similarly, the N-tuple of symmetric tensors with components $K^A_{ij} = K^A_{ji}$ is denoted by $K \equiv (K^1,\dots,K^N)$ where $K^A = (K^A_{ij}) \equiv (K^A_{11},K^A_{22},K^A_{33},K^A_{12},K^A_{23},K^A_{31})$.

2. Foundations

Let $V \equiv (-l_1/2,\ l_1/2) \times (-l_2/2,\ l_2/2) \times (-l_3/2,\ l_3/2)$ be a representative volume element (r.v.e) of the micro-periodic body, which is referred to the cartesian "micro"-coordinate system $Oy_1y_2y_3$. It is assumed that V is occupied by the heterogeneous, nonelastic material, governed by the constitutive relations the general form of which can be written down as:

$$\dot{e}(y) \in F(y;\ \sigma(y),\ \dot{\sigma}(y)),\ \text{for a.e. } y \in V, \tag{2.1}$$

where $\dot{e}(y) = (\dot{e}_{ij}(y))$, $\sigma(y) = (\sigma_{ij}(y))$, $\dot{\sigma}(y) = (\dot{\sigma}_{ij}(y))$ stand for the constitutive strain rates, stresses and stress rates, respectively, and where $F(y;\ \sigma,\ \dot{\sigma})$ for every σ, $\dot{\sigma}$ stands for a certain closed and convex (possibly empty) set in the strain rate space R^6. It can be observed that elastic-ideal plastic materials belong to the class of materials obtained via a specification of F(y;.).

Let on ∂V be given the stress boundary conditions

$$\sigma_{ij}(y)\ n_j(y) = T_{ij}n_j(y)\quad \text{for a.e. } y \in \partial V, \tag{2.2}$$

where n(y) is a unit normal to ∂V at y and T_{ij} are components of 3x3 symmetric matrix called macrostresses. The principle of virtual work for r.v.e. will be postulated in the form:

$$\int_V \sigma_{ij}(y) v_{(i,j)}(y) dy = \oint_{\partial V} T_{ij} n_j(y) v_i(y) da(y), \tag{2.3}$$

which is assumed to hold for every $v_i(\ )$ such that

$$v_i(y) = \delta_i + \epsilon_{ij} y_j + E_{ij} y_j + h_a(y) q_i^a, \quad y \in V, \tag{2.4}$$

where δ_i, $\epsilon_{ij} = -\epsilon_{ji}$, $E_{ij} = E_{ji}$, q^a_i are arbitrary constants and $h_a(\ )$ are given a priori sufficiently regular displacement shape functions defined such that

$$\langle h_{a,i} \rangle = 0, \quad h_a(y) = 0 \ \text{ for a.e.} \ y \in \partial V.$$

It means that the velocity field $\dot{u}(\ )$ defined on V is constrained by means of the formula

$$\dot{u}_i(y) = \dot{\delta}_i + \dot{\epsilon}_{ij} y_j + \dot{E}_{ij} y_j + h_a(y)\, \dot{q}^a_i ,$$

for some $\dot{\delta}_i$, $\dot{\epsilon}_{ij} = -\dot{\epsilon}_{ji}$, $\dot{E}_{ij} = \dot{E}_{ji}$, $\dot{q}^a_i$. Hence

$$\dot{u}_{(i,j)}(y) = \dot{E}_{ij} + h_{a,(i}\, \dot{q}^a_{j)} , \tag{2.5}$$

is the kinematical strain rate field in the r.v.e. For the particulars the reader is referred to Wozniak (1987).

Now assume that independently of the strain rate constrains (2.5) we also introduce the stress constrains given by

$$\sigma_{ij}(y) = \eta^A_{ijkl}(y)\, \Sigma^A_{kl} , \tag{2.6}$$

where $\Sigma^A_{kl} = \Sigma^A_{lk}$ are arbitrary constants and $\eta^A_{ijkl}(\ ) = \eta^A_{klij}(\ )$ are sufficiently regular stress shape functions defined a.e. on V, symmetric with respect to (i,j) and (k,l) which specify the constraints. Eq.(2.6) imply the stress rate constraints

$$\dot{\sigma}_{ij}(y) = \eta^{A}{}_{ijkl}(y)\ \dot{\Sigma}_{kl}{}^{A} . \tag{2.7}$$

The realization of the introduced stress constraints is assumed to be ideal, i.e., the condition

$$\int_V [\dot{u}_{(i,j)}(y) - \dot{e}_{ij}(y)]\ \tau_{ij}(y)\ dy = 0 , \tag{2.8}$$

holds for every $\tau_{ij}(\ )$ such that

$$\tau_{ij}(y) = \eta^{A}{}_{ijkl}(y)\ \bar{\Sigma}^{A}{}_{kl} , \tag{2.9}$$

for arbitrary $\bar{\Sigma}^{A}{}_{kl} = \bar{\Sigma}^{A}{}_{lk}$. It has to be emphasized that for the stress constraints the constitutive strain rates $\dot{e}_{ij}(y)$ do not coincide with the kinematical strain rates $\dot{u}_{(i,j)}(y)$, cf. Woźniak (1984).

Eqs. (2.1)-(2.9) where the multifunction F(y;.) and the shape functions ha(.), $\eta^{A}{}_{ijkl}(.\ )$ are assumed to be known, represent the foundations of the proposed macro-modelling method. Constants E_{ij}, $q^{a}{}_{i}$, $\Sigma^{A}{}_{kl}$, $\dot{\Sigma}^{A}{}_{kl}$ will be called the macro-constitutive quantities and Eqs.(2.5), (2.6) represent what can be called macro-micro localization conditions (cf. Suquet (1985)).

3. Results

In the sequel, for the sake of simplicity, we shall assume that the displacement and stress shape functions satisfy the extra conditions

$$\langle\eta^{A}{}_{ijkl}\ h_{a,j}\rangle = 0 ; \tag{3.1}$$

Moreover, we shall introduce the following macro-quantities

$$\dot{\Delta}^A_{kl} = \langle \eta^A_{ijkl}\, \dot{e}_{ij} \rangle \,, \tag{3.2}$$

and we denote $\Delta^A = (\Delta^A)$. For every $\Sigma = (\Sigma^1,\ldots,\Sigma^N)$, and $\dot{\Sigma} = (\dot{\Sigma}^1,\ldots,\dot{\Sigma}^N)$, we define a set (possibly empty)

$$\Phi\,(\Sigma,\dot{\Sigma}) = \{\ \Delta = (\Delta^1,\ldots,\Delta^N) \in R^{6N} : \Delta^A_{kl} = \langle \eta^A_{ijkl}\, e_{ij} \rangle,$$

$$\dot{e}(y) \in F(y;(\eta^A_{ijkl}\Sigma^A_{kl}),\ (\eta^A_{ijkl}\dot{\Sigma}^A_{kl}))\ , \tag{3.3}$$

$$\text{for a.e. } y \in V\}.$$

From Eqs.(2.1)-(2.9) under condition (3.1) and using (3.2) we derive the system of relations

$$T_{ij} = \langle \eta^A_{ijkl} \rangle \Sigma^A_{kl}, \quad \dot{T}_{ij} = \langle \eta^A_{ijkl} \rangle \dot{\Sigma}^A_{kl},$$

$$\dot{\Delta}^A_{kl} = \langle \eta^A_{ijkl} \rangle \dot{E}_{ij}\ , \tag{3.4}$$

between the introduced macro-quantities. Moreover, taking into account (2.1), (3.2) and (3.3) after some calculations, we arrive at

$$\dot{\Delta} \in \Phi(\Sigma,\dot{\Sigma}), \tag{3.5}$$

$\dot{\Delta} \equiv (\dot{\Delta}^1,\ldots,\dot{\Delta}^N)$. It can be also easily shown that

$$T_{ij}\dot{E}_{ij} = \Sigma^A_{kl}\dot{\Delta}^A_{kl} = \langle \sigma_{ij}\dot{u}_{(i,j)} \rangle. \tag{3.6}$$

Eqs.(3.4), (3.5) represent the main result of the proposed approach. Let us observe that Eqs.(3.4) are independent of material properties and interrelate the macro-stress T_{ij} macro-stress rates $\dot{T}_{ij}$ and macro-strain rates $\dot{E}_{ij}$ with the macro-quanities Σ^A_{kl}, $\dot{\Sigma}^A_{kl}$, $\dot{\Delta}^A_{kl}$ which can be called the partial macro-stresses, partial macro-stress rates and partial macro-strain rates, respectively. Eqs.(3.5) constitute the macro-description of material properties for micro-periodic composites; it interrelates partial macro-strain rates with partial macro-stresses and partial macro-stress rates. Hence Eqs.(3.4), (3.5) are macro-constitutive relations of this composite, treated as the interrelations between T_{ij}, $\dot{T}_{ij}$ and $\dot{E}_{ij}$. At last Eqs.(3.6) represents the Hill's macro-homogenity condition (cf. Suquet (1985)) which is a consequence of the proposed modelling.

It has to be emphasized that due to the condition (3.1) the macro-parameters $\dot{q}^{a}_{i}$ do not enter the macro-relations (3.4), (3.5). These parameters can be used in order to minimize, for example, the norm of the strain rate incompatibility $\| u_{(i,j)}(\ ) - e_{ij}(\ ) \|$ and/or the norm of the reaction forces due to the kinematical constraints (2.4); this problem will be treated separately.

4. Example

The general theory outlined in Secs.2,3 will be now illustrated by the example of the plane strain in the elastic-ideal plastic isotropic micro-periodic fibre reinformed composite. The cross section of the r.v.e. is represented by a rectangular $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2)$ on the plane 0 y_1y_2, with the rectangular fibre cross section $(-a_1/2, a_1/2) \times (-a_2/2, a_2/2)$. Setting $\xi_\alpha = a_\alpha/l_\alpha$, α=1,2 we shall assume that $\xi_1 \approx 1$ and $\xi_2 \ll 1$, i.e., that the fibres have a form of rod-like thin plates. Under these considerations the stress rate constraints (2.7)in the first approximation can be postulated in the form

$$
\begin{aligned}
&\dot{\sigma}_{11}(y) = \eta^{1}{}_{1111}(y)\dot{\Sigma}^{1}{}_{11} + \eta^{2}{}_{1111}(y)\dot{\Sigma}^{2}{}_{11},\\
&\dot{\sigma}_{12}(y) = \eta^{1}{}_{1212}(y)\,\dot{\Sigma}^{1}{}_{12} + \eta^{1}{}_{1221}(y)\dot{\Sigma}^{1}{}_{21},\\
&\dot{\sigma}_{22}(y) = \eta^{1}{}_{2222}(y)\dot{\Sigma}^{1}{}_{22},
\end{aligned}
\tag{4.1}
$$

with $\eta^{1}{}_{1212}(y) = \eta^{1}{}_{1221}(y) = 1/2$, $\eta^{1}{}_{2222}(y) = 1$, $\dot{\Sigma}^{1}{}_{12} = \dot{\Sigma}^{1}{}_{21}$ and where $\eta^{1}{}_{1111}(\ )$, $\eta^{2}{}_{2111}(\ )$ are characteristic function of the sets $V_1 = (-l_1/2, l_1/2) \times (-a_2/2, a_2/2)$ and $V_2 = V \setminus V_1$, respectively, By virtue of the symmetry conditions we also have $\eta^{1}{}_{2112}(y) = \eta^{1}{}_{2121}(y) = 1/2$;

all components η^{A}_{ijkl} not mentioned above are equal to zero.
Setting $\Sigma_{12} = \Sigma^{1}_{12}$, $\Sigma_{22} = \Sigma^{1}_{22}$ from Eq. (3.4) we get now

$$\begin{aligned} \dot{T}_{11} &= \xi_2 \dot{\Sigma}^{1}_{11} + (1-\xi_2)\dot{\Sigma}^{2}_{11}, \\ \dot{T}_{22} &= \dot{\Sigma}_{22}, \\ \dot{T}_{12} &= \dot{\Sigma}_{12}. \end{aligned} \tag{4.2}$$

Moreover from the known plane strain assumption for elastic-plastic materials we also obtain $T_{31} = T_{32} = 0$ and

$$\dot{T}_{33} = \frac{1}{2}\,[\xi_2\dot{\Sigma}^{1}_{11} + (1-\xi_2)\dot{\Sigma}^{2}_{11} + \dot{\Sigma}_{22}]. \tag{4.3}$$

For a.e. $y \in V$ we define the set k_y in R^3 setting:

$$k_y = \{\sigma \in R^3, (\sigma_{11}(y)-\sigma_{22}(y))^2 + 4(\sigma_{12}(y))^2 - 4k^2(y) \le 0\} \tag{4.4}$$

The constitutive relations (2.1) for the elastic-ideal plastic materials (with Huber-Mises-Hencky yield condition) take the well known form

$$\begin{aligned} \dot{e}_{ij}(y) &= \frac{1}{2\mu(y)}\dot{\sigma}_{ij}(y) - \delta_{ij}\frac{\lambda(y)}{2\mu(y)(2\mu(y)+3\lambda(y))}\dot{\sigma}_{kk}(y) + \lambda_{ij}(y), \\ \lambda_{ij}(y)\,&[\tau_{ij}-\sigma_{ij}(y)] \le 0; \quad \tau \in k_y \ \text{ and } \ \sigma \in k_y. \end{aligned} \tag{4.5}$$

In the sequel we define $\Sigma = (\Sigma^{1}_{11}, \Sigma^{2}_{11}, \Sigma_{22}, \Sigma_{12})$, $\bar{\Sigma} = (\bar{\Sigma}^{1}_{11}, \bar{\Sigma}^{2}_{11}, \bar{\Sigma}_{22}, \bar{\Sigma}_{12})$, $\dot{\Lambda} = (\dot{\Lambda}^{1}_{11}, \dot{\Lambda}^{2}_{11}, \dot{\Lambda}_{22}, \dot{\Lambda}_{12})$, $\Lambda = (\Lambda^{1}_{11}, \Lambda^{2}_{11}, \Lambda_{22}, \Lambda_{12})$ and we introduce the scalar product in R^4 of the form: $\Lambda\,\Sigma = \Lambda^{1}_{11}\Sigma^{1}_{11} + \Lambda^{2}_{11}\Sigma^{2}_{11} + \Lambda_{22}\Sigma_{22} + \Lambda_{12}\Sigma_{12}$.
Let us assume that $k_1 > k_2$ and introduce the set K in R^4 given by

$$\begin{aligned} K = \{\Sigma \in R^4, (\Sigma^{1}_{11} - \Sigma_{22})^2 + 4\Sigma^{2}_{12} - 4k^{2}_{2} \le 0 \ \wedge \\ \wedge\ (\Sigma^{2}_{11} - \Sigma_{22})^2 + 4\Sigma^{2}_{12} - 4k^{2}_{2} \le 0\}\,. \end{aligned} \tag{4.6}$$

Moreover define:

$$A_1 = \xi_2(1-\xi_1)\frac{4\mu_2+3\mu_2}{4\mu_2(2\mu_2+3\lambda_2)} + \xi_1\xi_2\frac{4\mu_1+3\lambda_1}{4\mu_1(2\mu_1+3\lambda_1)},$$

$$B_1 = -[\xi_2(1-\xi_1)\frac{3\lambda_2}{4\mu_2(2\mu_2+3\lambda_2)} + \xi_1\xi_2\frac{3\lambda_1}{4\mu_1(2\mu+3\lambda)}],$$

$$A_2 = (1-\xi_2)\frac{4\mu_2+3\lambda_2}{4\mu_2(2\mu_2+3\lambda_2)}, \tag{4.7}$$

$$B_2 = -(1-\xi_2)\frac{3\lambda_2}{4\mu_2(2\mu_2+3\lambda_2)},$$

$$C = \xi_1\xi_2\frac{1}{2\mu_1} + \frac{1-\xi_1\xi_2}{2\mu_2},$$

and

$$A = \begin{bmatrix} A_1, & 0, & B_1, & 0 \\ 0, & A_2, & B_2, & 0 \\ B_1, & B_2, & A_1+A_2, & 0 \\ 0, & 0, & 0, & C \end{bmatrix}. \tag{4.8}$$

Under foregoing denotations it can be shown that the constitutive relations (3.5) lead to

$$\dot{\Delta} = A\dot{\Sigma} + \Lambda, \tag{4.9}$$

where $\Lambda = (\Lambda^1{}_{11}, \Lambda^2{}_{11}, \Lambda_{22}, \Lambda_{12})$ is an arbitrary vector in R^4 that satisfies the condition

$$\Lambda\ (\bar{\Sigma} - \Sigma) \le 0, \quad \bar{\Sigma} \in K \ \text{and} \ \Sigma \in K. \tag{4.10}$$

From (4.9), (4.10), we conclude that the general macro-equations (3.5) are now specified to the form:

$$\dot{\Delta} \in \Phi(\Sigma,\dot{\Sigma}), \quad \Phi(\Sigma,\dot{\Sigma}) = A\dot{\Sigma} + \partial\ \mathrm{ind}_K\ (\Sigma), \tag{4.11}$$

which has to hold together with Eqs.(4.2) and with $\Delta = (\Delta^1_{11}, \Delta^2_{12}, \Delta_{22}, \Delta_{12}) = (\xi_2 E_{11}, (1-\xi_2)E_{11}, E_{22}, E_{12})$.
The foregoing example has only an illustrative meaning but can be a basis for more detailed analysis which will be given separately.

5. Final remarks

In this paper we have sketched only certain aspects of the local (constitutive) modelling for non-elastic composite periodic materials with the constitutive law of the form given by multyequation (2.1). For the sake of simplicity we have also introduced the extra condition (3.1) by means of which only stress constraints (2.6) were involved in the resulting macro-constitutive relations (3.4), (3.5). More general line of the approach to the macro-modelling of composites involving constraints both for strains and stresses will be given in forthcoming papers.

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Summary

O MODELOWANIU KONSTYTUTYWNYM SPRĘŻYSTO-PLASTYCZNYCH MIKRO-PERIODYCZNYCH MATERIAŁÓW

W pracy zaproponowano metodę modelowania lokalnego dla sprężysto-plastycznych mikroperiodycznych kompozytów. Wprowadzając odpowiednie więzy kinematyczne oraz naprężeniowe wyprowadzono relacje konstytutywne pomiędzy wielkościami prędkości makro-odkształceń cząstkowych oraz makro-naprężeniami cząstkowymi i ich prędkościami.

Rozważania zostały zilustrowane przykładem sprężysto-idealnie plastycznego mikro-periodycznego kompozytu.