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INVARIANT FORMULATION OF A DISTORTIONAL MODEL OF PLASTIC HARDENING

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1. Introduction

The classical theory of plasticity relies on the description of evolution of the yield surface during plastic deformation. There are however several difficulties connected with the practical application of this concept. First, because experiments in metal plasticity reveal that this evolution is quite complex; the yield surface changes its size, its position in stress space and it also suffers strong distortional changes of its shape. This latter effect is reflected neither by classical kinematic-isotropic hardening models nor by kinematic-anisotropic hardening rules restricted to quadratic yield conditions, since in experiments it is observed that subsequent yield surfaces corresponding to complex load histories are, in general, nonsymmetric.

The second source of difficulties is more fundamental, since both from theoretical considerations as well as from experiments it follows that the classical formulation of the theory of plasticity, employing a single yield surface, may not be sufficient or at least iconvenient for the description of plastic hardening under variable loading conditions, particularly in the case of nonlinearly hardening materials. This is due to the fact that the distribution of generalized plastic hardening modulus in the stress space is, in general, discontinous. This may be accounted for by introducing, apart from the yield surface, certain additional surfaces (loading surfaces), separating the regions in the stress space with different laws of variation of hardening parameters. Such approach is used in various multi-surface hardening theories (e.g. Mróz (1967), Dafalias and Popov (1975)). It seems that it should also be applied in the theories of plasticity which make no use of the concept of yield surface.

In the multi-surface formulations simplified assumptions are usually made with regard to transformations of loading surfaces in the course of plastic deformation. Most frequently the kinematic-isotropic model is used here. However, from experiments it turns out again that loading surfaces undergo transformations similar to that displayed by yield surfaces and also in this case essential distortions of these surfaces may be observed.

The above remarks may be concluded by saying that the description of distortional changes of yield surfaces, reflecting more closely experimental findings, is necessary for a realistic description of hardening within any of the approaches mentioned. Some descriptons of distortional plastic hardening have already been proposed (e.g. Williams and Svensson (1971), Yoshida et al. (1978), Ortiz and Popov (1983), Watanabe (1987)).

The present paper is devoted to a further development of the concept of geometric description of distortional plastic hardening, first proposed by Życzkowski and Kurtyka (1984) and then discussed in subsequent papers (Kurtyka and Życzkowski (1985), Kurtyka (1988), Życzkowski and Kurtyka (1989)).

2. Aim of the work, notations and basic relations

The paper is concerned with the analysis of some invariant aspects of distortional models of plastic hardening. The main attention is paid to the study of the distortional model given by Kurtyka and Życzkowski (1985). The results of the paper apply, to some extent, also to other distortional models, as well as to kinematic- anisotropic hardening rules with quadratic yield conditions (e.g. Baltov and Sawczuk (1965).

The main purpose of the work is to formulate invariant forms of subsequent yield conditions for the hardening models considered. Derivation of the invariant forms is primarily based on the analysis of geometry of subsequent yield surfaces, which is performed with the use of representations of yield conditions in a five-dimensional deviatoric vectorial stress space (Ilyushin (1963).

Such a space is adequate for the analysis of the hardening models considered here, as they all describe pressure-insensitive materials, with initial and subsequent yield conditions expressible in terms of deviatoric parts of appropriate tensors. A further restriction is imposed by assuming that these models describe materials conforming to the Ilyushin postulate of isotropy (Ilyushin (1963)), which is equivalent with neglecting the influence of third deviatoric invariants of these tensors. The initial Huber-von Mises yield condition is thus assumed.

The present analysis of invariant formulations provides a simple way of deriving tensorial forms of hardening models formulated in the Ilyushin vectorial stress space (e.g. Yoshida et al. (1978)). Here such tensorial form will be derived for the distortional model considered (Kurtyka and Życzkowski (1985)) which has been given as yet only in vectorial formulation.

The invariant forms of yield conditions are derived here in a simple manner. First, they are expressed in terms of invariants of vectors in the Ilyushin space and then by invariants of corresponding tensors (deviators). Matrix notation is mainly used for the vectorial space and direct notation for the physical space. For the sake of clarity of indical notation which is also employed in both spaces, Latin letters are used for indices referring to physical coordinate system $(i, j, \ldots = 1,2,3)$, while Greek indices refer to coordinates of the five-dimensional Ilyushin stress vector space $(\alpha, \beta, \ldots = 1,2,\ldots 5)$. In this space the stress deviator $S = \{s_{ij}\}$ is represented as a stress vector $\sigma = \sigma_\alpha w_\alpha$, where w_α are base vectors of this space and where the components σ_α are defined here as :

$$\sigma_1 = \frac{3}{2} s_{11} , \quad \sigma_2 = \sqrt{3}\left(\frac{s_{11}}{2} + s_{22}\right) ,$$
$$\sigma_3 = \sqrt{3} s_{12} , \quad \sigma_4 = \sqrt{3} s_{23} , \quad \sigma_5 = \sqrt{3} s_{31} . \tag{2.1}$$

The inverse formulae, giving the components of the stress deviator as functions of the vectorial components, are :

$$s_{11}=\frac{2}{3}\sigma_1\,,\quad s_{12}=\frac{\sqrt{3}}{3}\sigma_3\,,\quad s_{22}=-\frac{1}{3}\sigma_1+\frac{\sqrt{3}}{3}\sigma_2\,,$$
$$s_{23}=\frac{\sqrt{3}}{3}\sigma_4\,,\quad s_{33}=-\frac{1}{3}\sigma_1-\frac{\sqrt{3}}{3}\sigma_2\,,\quad s_{31}=\frac{\sqrt{3}}{3}\sigma_5\,. \tag{2.2}$$

All stress-type deviatoric parameters of yield conditions will be represented in the stress space (2.1) as vectors in an analogous way. Derivation of invariant forms of yield conditions will be, however, mainly based on relations between vectorial and tensorial invariants. The basic relation here is that one defining the Ilyushin stress and strain spaces, i.e. a proportionality of the square of the norm of vector to the second basic deviatoric invariant. For the stresss vector defined by the formulae (2.1) this relation has the form

$$|\boldsymbol{\sigma}|^2=\boldsymbol{\sigma}^T\boldsymbol{\sigma}=\sigma_\alpha\sigma_\alpha=\frac{3}{2}s_{ij}s_{ij}=\frac{3}{2}\,\mathrm{tr}\,\mathbf{S}^2. \tag{2.3}$$

Due to this relation the Huber-von Mises initial yield condition is mapped in the stress vector space by the hypersphere:

$$|\boldsymbol{\sigma}|=\sigma_0\equiv R_0\,, \tag{2.4}$$

with the radius R_0 equal to the tensile yield stress σ_0.

A set of invariants of vectors includes norms of these vectors as well as their scalar products, related to mixed invariants of corresponding deviators. For any two stress-type vectors $\mathbf{x}=\{x_\alpha\}$ and $\mathbf{z}=\{z_\alpha\}$, with the corresponding deviators $\mathbf{X}=\{x_{ij}\}$ and $\mathbf{Z}=\{z_{ij}\}$:

$$\mathbf{x}^T\mathbf{z}=x_\alpha z_\alpha=\frac{3}{2}x_{ij}z_{ij}=\frac{3}{2}\,\mathrm{tr}\,\mathbf{XZ}\,. \tag{2.5}$$

The yield conditions of hardening models considered in the present paper can be expressed in terms of invariants of the type (2.3) and (2.5). There is no need for considering other stress invariants, as the materials studied here are assumed to be pressure-insensitive and the influence of third invariants is neglected. The parameters of subsequent yield surfaces will, in general, depend on the history of plastic strain deviator $\mathbf{E}^P=\{e^P_{ij}\}$. This problem is not treated here since only yield conditions at a given stage of plastic deformation are investigated. The space of plastic strain vector $e^p=\{e_\alpha\}$ may be introduced similarly as the stress vector space, but the coordinates of the plastic strain vector are defined in such a way that

$$\sigma^T e^p = \mathrm{tr}\, SE^p. \tag{2.6}$$

3. The distortional model

The distortional model proposed by Kurtyka and Życzkowski(1985) may be treated as a geometrical generalization of the kinematic-anisotropic hardening models with quadratic yield condition . In tensorial notation the latter yield condition is usually written in the form

$$F = N_{ijkl}(s_{ij} - a_{ij})(s_{kl} - a_{kl}) - 2k^2 = 0 , \tag{3.1}$$

where $\mathbf{N} = \{N_{ijkl}\}$ is a fourth order tensor of plastic anisotropy, $\mathbf{a} = \{a_{ij}\}$ is the deviatoric translation tensor (or back-stress) and k stands for the shear yield stress. In the stress space (2.1) this yield condition is mapped (under certain conditions imposed on the tensor $\mathbf{N}$) by a hyperellipsoidal surface , described by the quadratic form

$$F = (\sigma - a)^T C (\sigma - a) - 1 = 0 , \tag{3.2}$$

with a symmetric matrix $C = \{C_{\alpha\beta}\}$ corresponding to the anisotropy tensor $\mathbf{N}$ (with the components divided by the common factor $2k^2$) and a translation vector $a = \{a_\alpha\}$ representing the translation deviator .

In the general case of "deviatoric" anisotropy described by the yield condition (3.1) the tensor $\mathbf{N}$ has 15 independent components which in the vectorial representation (3.2) have simple geometrical interpretation. These are five, generally distinct eigenvalues γ_α of the matrix C, together with ten independent components of five mutually orthogonal eigenvectors $\hat{w}_\alpha$ of this matrix. This interpretation is clear when the matrix C is transformed into the canonical form

$$C = Q^T B Q ,\tag{3.3}$$

where $Q = \{Q_{\alpha\beta}\}$ is an orthogonal matrix ($Q^{-1} = Q^T, \det Q = \pm 1$) containing the eigenvectors ;

$$Q = \sum_{\alpha=1}^{5} w_{(\alpha)} \hat{w}^T_{(\alpha)} ,\tag{3.4}$$

and where $B = \{B_{(\alpha\alpha)}\}$ is a diagonal matrix with the eigenvalues γ_α being its non-zero components .

By using the transformation (3.3) the equation of the elliptic surface (3.2) can be given in the canonical form

$$\sum_{\alpha=1}^{5} \gamma_\alpha \hat{\tau}^2_\alpha - 1 = 0 ,\tag{3.5}$$

where

$$\hat{\tau}_\alpha = Q_{\alpha\beta}(\sigma_\beta - a_\beta) ,\tag{3.6}$$

are coordinates of the active stress vector

$$\tau = \sigma - a ,\tag{3.7}$$

in the moving reference frame $\hat{\sigma}_\alpha$, with the base vectors $\hat{w}_\alpha$ (Fig. 1).

The hyperellipsoid (3.2) may be treated as a surface resulting from a projective mapping of five concentric hyperspheres with five (generally different) radii R_α and with projective directions coinciding with directions of the eigenvectors $\hat{w}_\alpha$, being also directions of elliptic deformation of the yield surface . Such interpretation of the hyperellipsoidal surface, depicted in Fig.1.a for a two-dimensional subspace of the stress space (2.1), has been used for formulating the distortional model. A non-elliptic distortion of the yield surface has been obtained by allowing the hyperspheres, generating this surface in a similar manner, to be non-concentric (Fig.1.b) . The parameters responsible for the non-elliptic distortion of the yield surface are here five vectors of distortion d_α, describing displacements of centres of the hyperspheres with respect to the centre A of the yield surface - the origin of the moving reference frame $\hat{\sigma}_\alpha$.

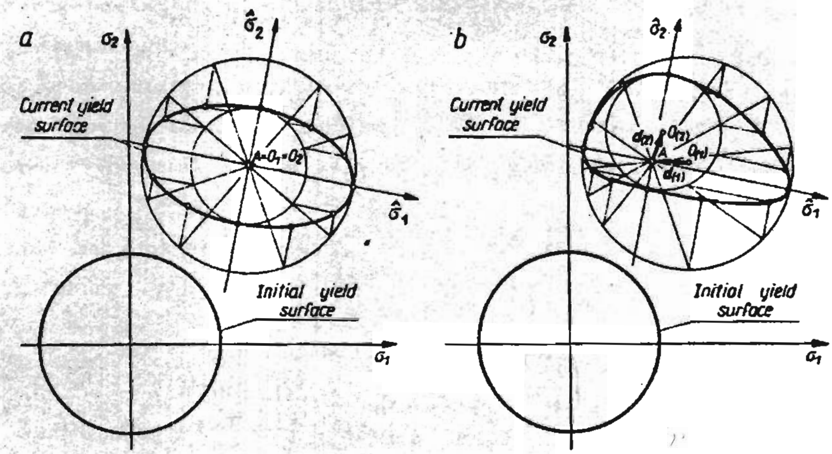

Fig. 1. Elliptic and distortional model of yield surfaces.

In general, the directions of distortional vectors $\mathbf{d}_\alpha$ may be arbitrary, while their lengths must satisfy the conditions

$$R^2_{(\alpha)} - |d_{(\alpha)}|^2 \geq 0 \ , \tag{3.8}$$

ensuring the location of the centre A within all the hyperspheres.

Such general distortional model is quite flexible, but it contains a large number of parameters defining the yield surface. It turns out, however, that satisfactory and still quite general description of experimental subsequent yield surfaces can be achieved by employing a much simpler distortional model (cf. Kurtyka and Życzkowski (1985), Kurtyka (1988)), namely such in which the distortional vectors $\mathbf{d}_\alpha$ are coaxial with the corresponding (i.e. having the same index α) eigenvectors $\hat{\mathbf{w}}_\alpha$ of elliptic distortions. This simpified model will be considered in the present paper.

The model is specified by the parameters of the elliptic surface (3.2) and, additionally, by five scalar parameters d_α , since in the simplified case

$$\mathbf{d}_\alpha = d_{(\alpha)} \hat{w}_{(\alpha)} \ , \quad \text{(no sum over } (\alpha)) \ . \tag{3.9}$$

As has been shown by Kurtyka and Życzkowski (1985) the yield condition for this model can be written in the form (3.2) and Eqs. (3.3-3.7) are valid also here, however the matrix **B** is now a functional matrix, with the diagonal components

$$B_{(\alpha\alpha)} = \gamma_{\alpha} = (R^2_{(\alpha)} + 2\, d_{(\alpha)}\, \hat{\tau}_{(\alpha)} - d^2_{(\alpha)})^{-1}, \tag{3.10}$$

where indices in parantheses are not subject to summation.

4. Invariant forms of yield conditions

4.1. Remarks on quadratic hardening rules. The distortional model considered here is a direct extension of kinematic-anisotropic hardening rules with elliptic yield conditions. In further developments of this model it would therefore be reasonable to use some results previously obtained for the elliptic model, particularly those concerning the tensor of plastic anisotropy **N** in the yield condition (3.1). Such approach may result, however, in restricted forms of the distortional model, since some forms of the tensor **N** which have been considered in literature give only quite restricted description of elliptic deformation of the yield surface. This is clearly seen in the case of hardening models based on Baltov-Sawczuk's rule (Baltov and Sawczuk (1965)), for which the tensor of plastic anisotropy can be written as a sum of isotropic (**I**) and anisotropic (**A**) tensors;

$$\begin{aligned} N_{ijkl} &= I_{ijkl} + A_{ijkl}, \\ I_{ijkl} &= \frac{1}{2}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\,\delta_{ij}\delta_{kl})\;, \\ A_{ijkl} &= A_o p_{ij} p_{kl}\;, \end{aligned} \tag{4.1}$$

this latter being an open product of a certain deviator $\mathbf{P}=\{p_{ij}\}$. In general, it may be specified by suitable rate equation (c.f. Axelsson (1979)) but in Baltov-Sawczuk's approach it is directly equal to plastic strain deviator $\mathbf{E}^p$. A_o is a scalar function of plastic strain, δ_{ij} is Kronecker's delta.

Such a method of constructing the anisotropy tensor **A** results in a very simplified elliptic deformation. This is easily seen when the invariant form of Baltow-Sawczuk's yield condition

$$\mathrm{tr}(S - a)^2 + A_o \mathrm{tr}^2[(S - a)E^P] - 2k^2 = 0 , \tag{4.2}$$

is transformed into a vectorial invariant form. In view of Eqs. (2.3), (2.5) and (2.6) this latter is given by the equation

$$|\sigma - a|^2 + \frac{3}{2} A_o \left[(\sigma - a)^T e^P\right]^2 - \sigma_o^2 = 0 . \tag{4.3}$$

The scalar product in the square bracket may be rewritten as

$$(\sigma - a)^T e^P = |\sigma - a| \; |e^P| \cos \theta , \tag{4.4}$$

where θ is the angle between the active stress vector $(\sigma - a)$ and the plastic strain vector e^P. Eq. (4.3) can then be presented in the form

$$|\sigma - a|^2 \left(1 + \frac{3}{2} A_o |e^P|^2 \cos^2\theta \right) - \sigma_o^2 = 0 . \tag{4.5}$$

This equation describes a hyperellipsoid which is rotationally symmetric with respect to the direction of the plastic strain vector. Elliptic deformation of the yield surface occurs only in this direction, while in the subspace orthogonal to the vector e^P the yield surface remains a hypersphere. In this case four eigenvalues of the matrix C in Eq. (3.2) are equal to each other and only one eigenvector, coinciding with the direction of the vector e^P, is distinguished. This result is essential for our considerations as it shows that a general tensor of anisotropy **N** (for the class of materials considered here) cannot be defined in this way. Such a general tensor should be expressible in terms of five linearly independent tensors (deviators), defining five linearly independent directions of distortion - eigenvectors of the quadratic form, together with five different scalar functions for its five, generally different, eigenvalues.

Such representation of the tensor of plastic anisotropy for kinematic-anisotropic hardening can be revealed by transforming the general quadratic form (3.2) "back" to physical coordinates. However, we shall perform this transformation at once for the distortional model and the elliptic model will result as a special case.

4.2. Invariant form of the distortional yield condition. The simplest derivation of the invariant form of yield condition for the distortional model is obtained by starting from the canonical form (3.5), with $\gamma_{(\alpha)}$ given by Eq. (3.10). For that it should first be noted that the coordinates $\hat{\tau}_\alpha$ of the active stress vector are themselves invariants, as they can be written as the scalar products

$$\hat{\tau}_\alpha = (\sigma - a)^T \hat{w}_{(\alpha)} . \tag{4.5}$$

The scalar distortional parameters $d_{(\alpha)}$ are also invariants - scalar products of the distortional vectors $\mathbf{d}_{(\alpha)}$ and of the vectors of distortional directions $\hat{w}_\alpha$;

$$\mathbf{d}^T_{(\alpha)} \hat{w}_{(\alpha)} = d_{(\alpha)} \hat{w}^T_{(\alpha)} \hat{w}_{(\alpha)} = d_\alpha \quad \text{(no sum over } (\alpha)), \tag{4.6}$$

The remaining parameters in Eq. (3.10) - the radii $R_{(\alpha)}$ - are invariants as well. In view of that the canonical form of the distortional yield condition can be expressed in terms of vectorial invariants as follows

$$\sum_{\alpha=1}^{5} \gamma_{(\alpha)} \left[(\sigma - a)^T \hat{w}_{(\alpha)} \right]^2 - 1 = 0 , \tag{4.7}$$

where $\gamma_{(\alpha)} = (R^2_{(\alpha)} + 2d_{(\alpha)} (\sigma - a)^T \hat{w}_{(\alpha)} - d^2_{(\alpha)})^{-1}$, (4.8)

with no sum over (α) in (4.8).

Now, this yield condition may be rewritten with the use of tensorial invariants. However, first it is necessary to give a tensorial interpretation of the directional vectors $\hat{w}_{(\alpha)}$. In physical space these vectors are represented by a set of five non-dimensional deviators which will be denoted by $\hat{W}_{(\alpha)} = \{\hat{w}_{(\alpha)ij}\}$. Since the vectors $\hat{w}_{(\alpha)}$ are orthonormal,

$$\hat{w}^T_{(\alpha)} \hat{w}_{(\beta)} = \delta_{\alpha\beta} , \tag{4.9}$$

where $\delta_{\alpha\beta}$ is Kronecker's delta in Ilyushin's coordinates, thus, in view of Eq. (2.5), the directional deviators must also satisfy the following "orthonormality conditions"

$$\mathrm{tr}\, \hat{W}_{(\alpha)} \hat{W}_{(\beta)} = \frac{2}{3} \delta_{\alpha\beta} \text{ , for } \alpha,\beta = 1,2,\ldots 5 \text{ .} \tag{4.10}$$

By employing again the relation (2.5), the invariants (4.5) can now be expressed as tensorial invariants of the active stress deviator (S - a) and of the directional deviators $\hat{W}_{(\alpha)}$:

$$\hat{\tau}_{\alpha} = (\sigma - a)^T \hat{w}_{(\alpha)} = \frac{3}{2} \mathrm{tr}\left[(S - a)\hat{W}_{(\alpha)}\right] \text{ .} \tag{4.11}$$

Substitution of these invariants into Eq. (23) yields the final invariant form of yield condition for the distortional model

$$\frac{9}{4} \sum_{\alpha=1}^{5} \gamma_{(\alpha)} \mathrm{tr}^2\left[(S - a)\hat{W}_{(\alpha)}\right] - 1 = 0 \text{ ,} \tag{4.12}$$

where $\gamma_{(\alpha)}$ are invariant functions of the active stress deviator. They are given by

$$\gamma_{(\alpha)} = \left\{ R^2_{(\alpha)} + 3d_{(\alpha)} \mathrm{tr} \left[(S - a)\hat{W}_{(\alpha)}\right] - d^2_{(\alpha)}\right\}^{-1} \text{,} \tag{4.13}$$

with no sum over (α) in (4.13).

The scalar parameters $R_{(\alpha)}$ and $d_{(\alpha)}$ could also be expressed in terms of quantities defined in physical space, as they are related to yield stresses under certain specified loading conditions, however, in Eq. (4.13) they are retained in the form of parameters defined in the vectorial stress space, where they have clear geometrical interpretation.

The above invariant form of the distortional yield condition (4.12, 4.13) may be easily written in indicial notation, formally identical to that of the quadratic yield condition (3.1), namely

$$F = L_{ijkl} (s_{ij} - a_{ij}) (s_{kl} - a_{kl}) - 1 = 0 \text{ .} \tag{4.14}$$

However here the tensor $L=\{L_{ijkl}\}$, corresponding to the tensor **N** of plastic anisotropy in Eq. (3.1), will not be a constant tensor, but a certain invariant tensorial function of the active stress deviator. The form of these function can be obtained directly from Eq. (4.12) by rewriting the squares of the invariants in indicial notation, where from

$$L = \frac{9}{4} \sum_{\alpha=1}^{5} \gamma_{(\alpha)} (\hat{W}_{(\alpha)} \otimes \hat{W}_{(\alpha)}) ,$$
$$L_{ijkl} = \frac{9}{4} \sum_{\alpha=1}^{5} \gamma_{(\alpha)} (\hat{w}_{(\alpha)ij} \hat{w}_{(\alpha)kl}), \tag{4.15}$$

while $\gamma_{(\alpha)}$ are expressed now as

$$\gamma_{(\alpha)} = \left\{ R_{(\alpha)}^{2} + 3d_{(\alpha)} (S_{ij} - a_{ij}) \hat{w}_{(\alpha)ij} - d_{(\alpha)}^{2} \right\}^{-1} , \tag{4.16}$$

again without summation over (α) in (4.16).

4.3 Discussion of special cases. The formulae (4.15) and (4.16) are the most general forms of the tensor of plastic anisotropy for the distortional model considered, corresponding to general type of distortion, when all parameters d_α are different from zero. In this case the distortional yield surfaces do not exhibit any symmetry in the vectorial stress space. Should some d_α vanish then the yield surface deformations in certain directions will be restricted to symmetrical elliptic deformations.

The formulae (4.15) and (4.16) also give the general form of the tensor of plastic anisotropy for the kinematic-anisotropic hardening models with elliptic yield conditions. This case is obtained by setting in (4.16) all $d_\alpha = 0$, which results in L being now a constant tensor. Comparing this tensor with the tensor of anisotropy given by Baltov-Sawczuk (Eq. (4.1)) it is seen that, apart from non-essential differences connected with defining the tensor L as a tensor having the dimension of one over stress squared (with N being non-dimensional), the main difference is that L is constructed from five open products of five directional deviators (instead of one as in (4.1)) fulfilling the ortogonality conditions (4.10), multiplied by five different scalar functions (instead of one function A in (4.1)). The second difference is that the tensor L, unlike N, does not contain any isotropic fourth-order tensor. In fact this tensor is not necessary if the elliptic deformation is general, i.e. if all R_α are different, and all $\hat{W}_\alpha$ are specified. This difference is also reflected by the absence of the isotropic invariant $tr(S - a)^2$ of the active stress deviator in the general invariant form (Eqs. (4.12,4.13)), whereas it apears in the invariant form of Baltov-Sawczuk's model (Eq. (4.2)). This invariant is redundant in the general representation but it may appear in the invariant forms of yield conditions corresponding to restricted cases of both elliptic and non-elliptic deformation. Such forms may be usefull for studying simplified models of plastic hardening and they can be derived from the general form (4.12).

For this aim let us assume that from the set of five scalar parameters R_α (α=1,2,... 5) only some are different, say R_α for α=1,...β < 4, and the remaining ones are equal to each other; $R_\alpha \equiv R_o$ for $\alpha = \beta$+1,..5, then the yield condition (4.12) may be written in the form

$$\sum_{\alpha=1}^{\beta} \gamma_{(\alpha)} \, tr^2\left[(S - a)\hat{W}_{(\alpha)}\right] + \frac{1}{R_o^2} \sum_{\alpha=\beta+1}^{5} tr^2\left[(S - a)\hat{W}_{(\alpha)}\right] - \frac{4}{9} = 0 , \qquad (4.17)$$

where $\hat{W}_\alpha$ for $\alpha = \beta$+1,...5 are now unspecified (but still fulfilling the conditions (4.10)). The above form is valid also in the case of non-elliptic distortion, provided that the parameters d_α in Eq. (4.13) are vanishing for $\alpha = \beta$+1,...5. Now, making use of the following identity

$$\sum_{\alpha=1}^{5} tr^2\left[(S - a)\hat{W}_{(\alpha)}\right] = \frac{2}{3} tr(S - a)^2 , \qquad (4.18)$$

we may express the second sum in Eq. (4.17) by the invariants appearing in the first sum. After some rearrangements the yield condition may be written in the form

$$tr(S - a)^2 + \frac{3}{2} \sum_{\alpha=1}^{\beta} (\gamma_{(\alpha)} R_o^2 - 1) \, tr^2\left[(S - a)\hat{W}_{(\alpha)}\right] - \frac{2}{3} R_o^2 = 0 , \qquad (4.19)$$

which is analogous to the Baltov-Sawczuk hardening rule, but not restricted to elliptic deformation and not only to one direction of that deformation.

By setting β=1 in Eq. (4.19), i.e. by specyfing only one direction of distortion, one obtains the most simple distortional model of the kind considered here, being a distortional counterpart of Baltov-Sawczuk's hardening model. The model of this type is represented in the Ilyushin vectorial stress space by a surface which is rotationally symmetric with respect to the direction of distortion.

Even in this simple form the model accounts for most effects observed in experiments (such as Bauschinger's effect, cross effect, translation, rotation and distortion of the yield surface) and it may be used as a starting point for constructing effective forms of evolution laws for parameters involved in the description. The set of parameters contains in this case two deviatoric tensors ($\hat{W}_1$ and a) and three scalar parameters: R_1, R_o, d_1. The restricted invariant form (4.19) may also be presented in tensorial notation which can be obtained either from Eq. (4.19) or by a direct transformation of the tensor L given by Eq. (4.15). Decomposing this tensor in a similar manner as in Eq. (4.17) into two parts and using the identity

$$\sum_{\alpha=1}^{5} (\hat{W}_{(\alpha)} \otimes \hat{W}_{(\alpha)}) = \frac{2}{3} I \quad , \tag{4.20}$$

which holds for any set of five deviators $\hat{W}_\alpha$ fulfilling the conditions (4.10), one obtains in the restricted case the tensor L in the form

$$L = \frac{3}{2R_o^2} \left[I + \frac{3}{2} \sum_{\alpha=1}^{\beta} (\gamma_{(\alpha)} R_o^2 - 1) \, (\hat{W}_{(\alpha)} \otimes \hat{W}_{(\alpha)}) \right] , \tag{4.21}$$

now containing the isotropic fourth order tensor I given by Eq. (4.1).

5. Concluding remarks

In the above derivation of invariant forms of the distortional model, which are also valid as general forms for elliptic yield criteria, geometrical invariants of the yield surface have been employed. From the analysis presented here it follows that the general case of deviatoric anisotropy considered here may be described by a set of five invariants;

$tr(S-a)\hat{W}_{(\alpha)}$, $\alpha = 1,2..5$. Such invariants may be used for deriving distortional hardening rules different from the proposed distortional model. In this respect it is worth-pointing out that this set of invariants may be replaced by the equivalent set of five invariants

$$\left[tr(S - a)^2 \right]^{\frac{1}{2}}, \qquad \cos \theta_\alpha , \qquad \alpha = 1,2,3,4 , \tag{5.1}$$

which can be obtained from the set of basic invariants by representing these invariants as scalar products of corresponding vectors. The invariants $\cos \theta_\alpha$ are defined by the relations

$$tr(S - a)\, \hat{W}_{(\alpha)} = \frac{2}{3} (\sigma - a)^T \hat{w}_{(\alpha)} = \tag{5.2}$$
$$= \frac{2}{3} |\sigma - a|\, |\hat{w}_{(\alpha)}| \cos \theta_\alpha = \left[\frac{2}{3} tr(S - a)^2 \right]^{\frac{1}{2}} \cos \theta_\alpha ,$$

where in view of the condition (4.18)

$$\sum_{\alpha=1}^{5} \cos^2\theta_\alpha = 1 , \tag{5.3}$$

and only four of these invariants are independent.

Invariants of this type have been employed by Ortiz and Popov (1983) who proposed a rather generally formulated idea of describing distortional effects with the use of trigonometric series of such invariants. The effective form has been given by the authors only for the case of simple distortion, with only one distortional deviator $\hat{W}_\alpha$ being specified. Another proposal of this type, also restricted to simple distortion, is due to Yoshida et al. (1978). Their model can be interpreted (and also generalized) in terms of the invariants considered here.

Concluding the present paper it is worth to add two remarks :

1. In the further development of the concept of distortional plastic hardening the most important problem is to specify the tensorial parameters $\dot{\hat{W}}_\alpha$, responsible for anisotropic changes of the yield surface. Apart from experimental efforts in this direction, one possible way of investigations is connected with Ilyushin's idea, based on the use of geometrical invariants of strain trajectory in vectorial representation

of plastic processes.

2. The geometrical invariants of the yield surface, employed in the present approach, provide a certain measure of aquired plastic anisotropy of a material. Indeed, as pointed out by Boehler (1985), the type of anisotropy can be precisely defined, whereas there is no precise definition of the degree of anisotropy and this may be mainly judged through anisotropic changes of yield surfaces. In this respect it may therefore be interesting to investigate the type of anisotropy described by particular distortional models, and to relate the present results with the methods of description of plastic anisotropy employed within the theory of tensor function representations.

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Summary

NIEZMIENNICZE SFORMUŁOWANIE DYSTORSYJNEGO MODELU WZMOCNIENIA PLASTYCZNEGO

W pracy omówiono niezmiennicze postacie zapisu warunku plastyczności geometrycznego modelu wzmocnienia plastycznego (Kurtyka, Życzkowski 1985). Podano zapis tego warunku przy użyciu niezmienników odpowiadających wektorowej reprezentacji procesów plastycznych Iliuszina i na tej podstawie sformułowano równoważny zapis poprzez niezmienniki tensorowe.