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MULTIMODAL OPTIMAL DESIGN OF STRUCTURAL ELEMENTS
(A SURVEY)

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1. Historical remarks

The history of multimodal optimization of structural elements began in 1976 when Olhoff & Rasmussen (1976) proved incorrectness of the Tadjbakhsh & Keller (1962) solution obtained for a clamped-clamped column and arrived at the correct one. The lowest critical force of the optimal column occured to be a double eigenvalue corresponding to two fundamental forms of in-plane buckling. This discovery -one of the most significant in structural optimization- opened new field for theoretical investigations and detailed numerical solutions.

Some corresponding observations have been made by Kiusalaas (1973) who optimized a beam on elastic foundation and by Szelag & Mróz (1979) who considered the optimal design of vibrating beams with unspecified *a priori* support action. Additional light has been brought to the problem by Masur & Mróz (1979,1980) who showed that a singular (non- differentiable) optimization problem arises here. At the same time Prager & Prager (1979) demontrated that the singular behaviour associated with repeated eigenvalues may appear for even a very simple finite-dimensional model of clamped - clamped column.

In 1978 M. Życzkowski suggested that the bimodality in structural optimization should be expected in more natural way for the optimal design of arches with respect to its stability. Further research activities in that direction have been undertaken by several of his collaborators. Results obtained by them are presented in the next chapters.

2. Nature of the problem

In general, the aim of the structural optimization is minimization of a functional which consists, as a rule, of three elements: material, manufacturing and exploitational costs. If the manufacturing costs are not particularly essential then the volume of the structural element constitutes a reasonable criterion for the cost as a design objective. Constraints imposed on the lowest buckling load of perfect structures corresponding to the first buckling mode or on the lowest natural frequency of free vibration belong to the case of eigenvalues as constraints. Variation of the shape of a structure in an optimization process affects also all other eigenvalues and may result in the lowering of a higher-order eigenvalue bellow the first one. Then the result of unimodal optimal design (i.e. with respect to single eigenvalue) is false and multimodal optimal design should be employed.

Sometimes a constraint (e.g. the lowest eigenvalue) may be interchanged with the design objective (e.g. volume - V) and the design problem may be stated as:

$$\max_{\phi(x)} [\min_{i} (P_i)], \qquad i=1..\infty, \qquad V=\text{const.}, \tag{2.1}$$

where $\phi(x)$ denotes a design variable (control function), P_i-the set of eigenvalues.

This max - min problem is non-differentiable, however. To avoid this difficulty Olhoff (1987) used a band formulation given by Bendsøe, Olhoff & Taylor (1983/1984) and Taylor & Bendsøe (1984). It consists in introducing an extra parameter P which ensures that one can formulate a standard differen- tiable problem even if repeated eigenvalue occurs. The problem (2.1) can be transformed into the problem of maximizing a bound P subject to the constraints $P_i \geq P$, $i=1..\infty$.

Such approach connected with the mathematical programming method is also presented by Olhoff (1989,1990).

Some general remarks dealing with the multimodal optimization of structural elements are presented in monographs by: Haug,Choi & Komkov

(1986), Banichuk (1986), Gajewski & Życzkowski (1988), Rozvany (1989) and Życzkowski (1990).

3. Methods of the solution of multimodal optimization problems

The basic method is connected with the modified calculus of variations. The appropriate non-stationary optimality conditions are derived by Olhoff & Rasmussen (1976) for conservative stability problems and by Masur & Mróz (1979) for conservative and non-conservative problems as well. The second method, widely used here, uses the Pontryagin's maximum principle. It leads to the same optimality conditions as obtained in previous papers (Błachut & Gajewski (1980)). The third and the most powerfull method, i.e. the sensitivity analysis, consists in computation of directional derivatives of a repeated eigenvalue. The theoretical aspects of this method were developed by Choi & Haug (1981 a,b,c), Haug & Rousselet (1981), Rousselet (1981 a,b) and Zolésio (1981).

4. Multimodality in conservative stability problems

4.1. General remarks. Behaviour of loading in the course of buckling is an important factor both from the physical and mathematical point of view. In some cases the boundary value problems which determine the critical loading parameters are self-adjoint. In such cases the behaviour of loading is conservative and its critical value can always be determined by the static criterion of stability. The conservative stability constraints should protect the structure against any possible form of loss of stability i.e. the bifurcation and/or snap-through.

4.2. Columns. The classical bimodal solution of the optimization problem of clamped-clamped column has been found by Olhoff & Rasmussen (1976). It was generalized by Bochenek & Gajewski (1984) for the column elastically clamped with different flexibilities at each end. Prager's model have been used by Gajewski (1981) to demonstrate the bimodal optimization in a more general case, namely with respect to the frequency of transverse vibrations under axial compression. The analytical exact solution of the Olhoff- Rasmussen problem has been found by Seyranian (1983, 1984) and Masur (1984).

An optimally designed column for (x,y) in-plane buckling may loose its stiffness in the perpendicular (x,z) plane and subsequently may buckle in the second plane. In such cases optimization for buckling in two planes is necessary - it leads to multimodal optimization with a number of simultaneous modes equal to 2,3 or 4, depending on the mode of supports and additional geometrical constraints. Here we can talk about the bimodal optimization in each plane and the simultaneous mode design in two planes. Both, optimal design with respect to one control function and optimal design with respect to two control functions (width and depth) are possible. The numerical solutions to the problems under discussion were obtained by Bochenek (1987) and Bochenek & Nowak (1988).

Some other related problems were presented by Olhoff & Niordson (1979), Haug (1980, 1982), Choi & Haug (1981, 1982), Lam, Haug & Choi (1981), Olhoff (1981), Choi, Haug & Lam (1982), Banichuk & Barsuk (1983a, b, c), Teschner (1983a, b), Bratus & Seyranian (1983, 1984), Olhoff (1987, 1990), Parbery (1987), Gajewski (1990a).

4.3. Columns in elastic medium. The necessity of multimodal optimization is more visible for a column in an elastic medium; indeed, even for prismatic columns the number of half-wavelengths is *a priori* unknown and the occurence of two equally probable buckling modes may often be encountered. The bimodal optimization of a beam on Winkler's foundation was investigated for the first time by Kiusalaas (1973) (FEM) and next by Repin (1979) and Larichev (1982). A generalization given by Gajewski (1985) consists of an examination of two different mathematical approaches: Pontryagin's and sensitivity analysis. The ranges of validity of the unimodal and bimodal formulations in terms of the lower geometrical constraints and foundation coefficient were determined. The problem was studied by Plaut, Johnson & Olhoff (1986) for columns of an idealized sandwich cross-section and various boundary conditions. Quite recently, Shin, Plaut & Haftka (1987) used simultaneous analysis and design for the optimum design of a beam-column supported by an elastic foundation for the maximum buckling load. A similar problem was considered by Shin et al.(1988).

4.4. Arches. Problems of the optimal design of plane arches under stability constraints require the consideration of various forms of instability especially those due to bifurcation and snap-through. Moreover, the instability may occur in the plane of the arch and/or out of the plane. Therefore, multimodal formulation of the optimization problems should be considered here, as a rule.

Błachut and Gajewski (1981a) drew attention to the necessity of bimodal formulation for clamped-clamped funicular arches. Both symmetric and antisymmetric buckling modes were considered and the ranges of applicability of unimodal and bimodal optimization were found depending on geometrical constraints and on the steepness parameter. Many other examples were also presented by Olhoff (1982) and Olhoff & Plaut (1983).

Unimodal and bimodal optimization of extensible arches under snap-through, bifurcation and vibration constraints was carried out by Błachut and Gajewski (1981b) and Błachut (1982).

Additionally, if out-of-plane buckling is allowed for, then even bimodal formulation may be insufficient. This problem was discussed in detail by Bocheňek & Gajewski (1986) for an inextensible clamped-clamped circular arch; three- and four-modal solutions were obtained.

All the above-mentioned papers employed the magnitude of the cross-section as a design variable. Bochenek & Gajewski (1989) and Bochenek (1988) have tackled new and more complicated problems, namely the optimal design of arches for which both the depth and width of a rectangular cross-section were treated as two independent design functions. The arches were optimized with respect to in-plane and spatial buckling and their axes were assumed to be inextensible.

The thin-walled box-section circular arches loaded by a radial pressure were optimized by Bochenek (1989). Beside the overall both in-plane and out-of-plane buckling the local web and flange instability were taken into account. The bound formulation of the min-max problem and the mathematical programming method were used to the optimization of I section prismatic circular arch by Bochenek & Życzkowski(1989,1990).

Recently Wróblewski & Życzkowski (1989) has generalized previous problems introducing a nonlinear behaviour of the arch material. They considered the multimodal optimization against in-plane and spatial creep buckling.

Some related elastic optimization problems were also discussed by Błachut (1983), Błachut & Gajewski (1983), Bochenek & Gajewski (1983), Życzkowski & Gajewski (1983), Błachut (1984), Bochenek (1984), Gajewski & Życzkowski (1987, 1988).

4.5. Frames. Kiusalaas (1973) pointed out that the optimization of the portal clamped-clamped compressed frame with two elastic supports at both ends of the beam (of constant cross-sections of both columns and of the beam) leads to a bimodal solution. Bochenek & Gajewski (1982,1983) introduced functional design variables for columns and beam. They applied Pontryagin's principle and proved bimodality of the solution within a certain intervals of width to height ratio for simple portal frame without any additional elastic supports. This solution suggests that in cases of more complex frames the multimodal solutions should appear as a rule. Similar problem was considered by Szyszkowski, Watson & Fletkiewicz (1988).

4.6. Plates. The FEM technique was used by Szyszkowski, Watson & Fletkiewicz to derive the optimal condition for multimodal optimal design of rectangular plates. The number of modes affecting the optimal design was determined automatically during the iterative procedure.

4.7. Shells. Some primary observations on multimodal optimization of cylindrical shells were made by Medvedev (1980) and extended by Medvedev & Totsky (1984). Plaut & Johnson (1984) pointed out the possibility of a bimodal solution in the shape optimization of shallow shells with circular boundary. Quite recently Skrzypek & Bielski (1989) discussed both unimodal and bimodal optimization of elastic toroidal shells subject to buckling under external pressure. Geometrically nonlinear theory of finite displacements but small strains was applied. For some geometry parameters the optimal shape of the shell corresponds coalescense of bifurcation and snap-through pressures.

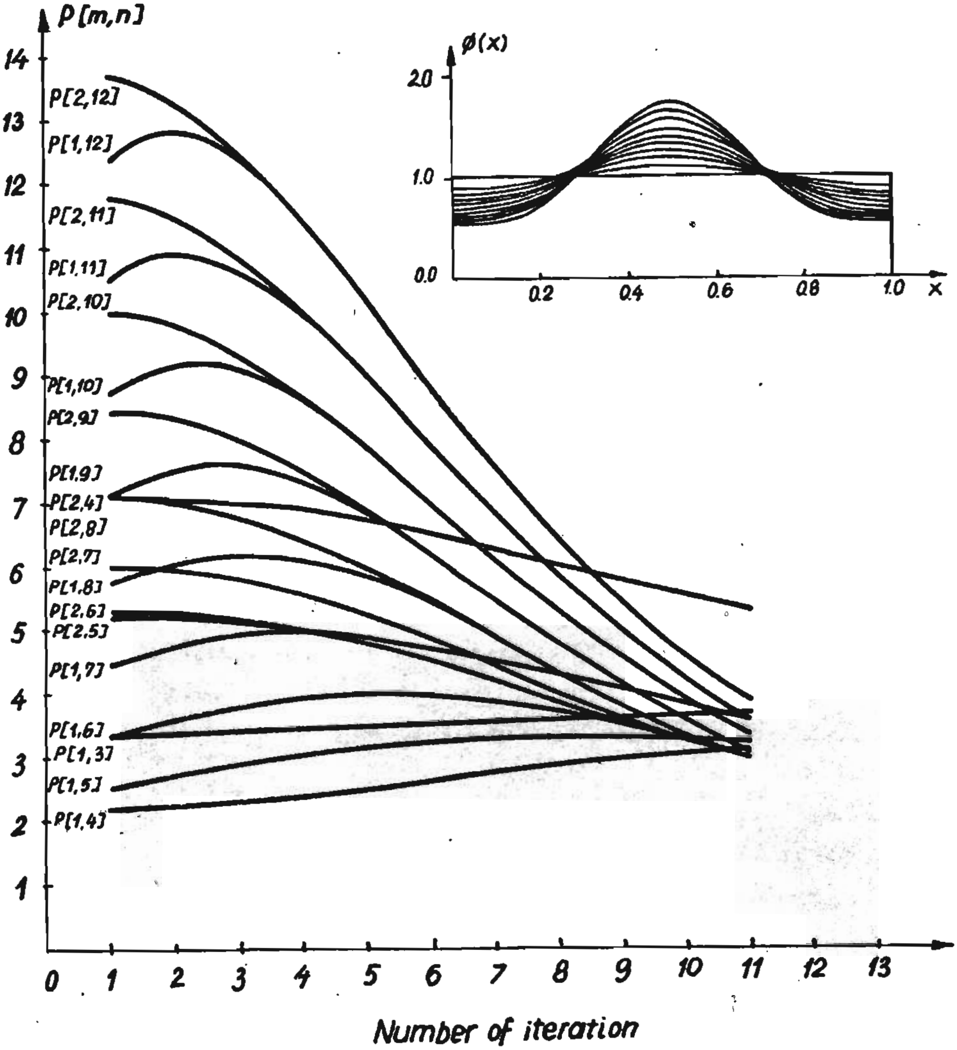

Fig. 1. Critical uniform constant pressures in subsequent steps of a cylindrical shell optimization.

The problem considered by Błachut (1987) is of slightly different nature. The main aim of that paper was to establish, parametrically, the maximal compressive axial buckling load of barrel-shaped shells. The wall thickness and the shell volume were kept constant. The buckling load increase was due to the change in the shell meridional curvature. It has been shown that some optimal solutions may correspond to equal axisymmetric collapse and bifurcation loads.

The possibility of appearence of a high eigenvalue multiplicity is illustrated in Fig. 1. Nineteen buckling loads P[m,n] (uniform constant pressure) calculated for an elastic cylindrical shell in subsequent optimization steps are shown (Gajewski (1990b)). The integers m and n denote the number of longitudinal and circumferential waves respectively.

5. Multimodality in vibration problems

5.1. Introductory remarks. Similarly to the buckling loads the frequencies of natural vibration of a structure are eigenvalues of a generalized eigenvalue problem, hence, they depend on design. Simple examples of finite-dimensional structural design problems with constraints on natural frequency were presented by Haug & Rousselet (1980), Haug & Choi (1981, 1982), Lam et al. (1981), Gajewski (1981) and Haug (1982). Appropriate optimality condition for finite multidegree of freedom vibrating systems was considered by Bartholomew & Pitcher (1984) and Khot (1985).

5.2. Columns. Bimodal formulation of the optimization problem of compressed vibrating columns was introduced by Gajewski (1981) and numerically solved by Bochenek & Gajewski (1984).

5.3. Beams on elastic foundation. The characteristic diagrams of the optimal frequency (under constant compressive load) versus the lower geometrical constraints for a column in an elastic medium were given by Gajewski (1985).

5.4. Arches. Optimization of circular funicular arches, clamped at both ends, under vibration constraints was carried out by Błachut & Gajewski (1981a). Both symmetric and antisymmetric vibration modes were considered and bimodal optimization was allowed for. Shallow arches were also optimized under vibration constraints by Plaut & Olhoff (1983), where the form of the arch was varied for a given cross section, length and span. Both, the shape and material distribution were optimized by Olhoff (1983) and Olhoff & Plaut (1983). The parametrical optimization problem for the case of an in-plane vibration was solved by Błachut (1983,1984).

Introducing stretchability of arch axis Błachut & Gajewski (1981b) optimized circular arches under hydrostatic pressure for various loading parameters. Evolution of the optimal clamped-clamped circular arch shape is presented in Fig.2.
Similar problem for a simple four degree-of-freedom model was presented by Bochenek & Gajewski (1983) and Bochenek (1984). General statement of the problem was formulated in monograph by Gajewski & Życzkowski (1988).

5.5. Frames. A bimodal solution for a portal frame, treated as the eighteen-finite-element model subject to the natural frequency constraints and constraints on the cross-sectional area, was given by Haug, Choi & Komkov (1986).

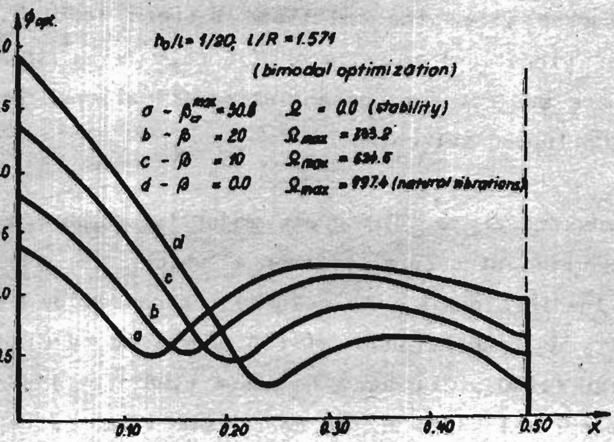

Fig.2. Optimal shapes of clamped-clamped vibrating circular arches.

5.6. Shells. Plaut, Johnson & Parbery (1984) considered thin shallow elastic shells with a given circular boundary. The material, surface area and the uniform thickness of the shell were specified. Such axisymmetric shape of the shell which provided maximum fundamental vibration frequency was the objective function. The optimality condition for bimodal formulation was derived by the calculus of variations. The maximization of natural frequency and multimodality of the forms of vibration for variable thickness orthotropic shells was also investigated by Medvedev (1985).

6. Multimodality in non-conservative stability problems

6.1. General remarks. The forms of stability loss of non-conservative systems (bifurcation or flutter) may depend on control variables, namely distribution of mass and stiffness, and hence they are unknown *a priori* before optimization. During an optimal design process equating two flutter critical forces or equating a flutter critical force with a bifurcation critical force may appear. However, it is difficult to predict which type of stability criterion should be employed.

A systematic introduction to optimization of non-conservative systems was given in a survey by Weisshaar & Plaut (1981). The so-called characteristic curves (i.e. the relations between loading parameters and frequencies of vibration) play important role here. In an optimization process they are subjected to essential modifications, leading to continuous or discontinuous switchings.

6.2. Columns and beams. The great majority of papers devoted to the optimal design of structural elements which are subjected to non-conservative loadings deals with columns compressed by a follower force. The possibility of discontinuous switching raised by Claudon (1975), Hanaoka & Washizu (1980), Claudon & Sunakawa (1981a,b) leads to the conclusion that the optimal shape should correspond to maximization of two critical flutter forces, equal to each other. The optimality condition in this case was derived by Masur & Mróz (1979) and by Błachut & Gajewski

(1980). Further details were given by Bogacz & Janiszewski (1985) and by Gajewski & Życzkowski (1988). Recently, Tada, Matsumoto & Oku (1989) formulated the shape determination problem, in which the objective was to maximize the critical load of the Beck's column under the condition of constant volume and the condition that the distance between characteristic curves for adjacent modes was kept wider than a certain value. The obtained shape corresponded to the equation of three subsequent critical flutter forces (the highest critical force so far). The same effect was observed by Seguchi et al. (1989) who an internal damping took into considerations.

Seyranian (1980, 1982a, b, 1990) analysed flexural-torsional flutter of a thin-walled wing in fluid flow. He employed the sensitivity analysis and found the optimum realizing equal critical forces of flutter and of divergence.

6.3. **Annular plates.** The optimization problem of annular plates compressed by uniformly distributed non-conservative forces was formulated by Gajewski (1990a). Both the precritical membrane state and the small

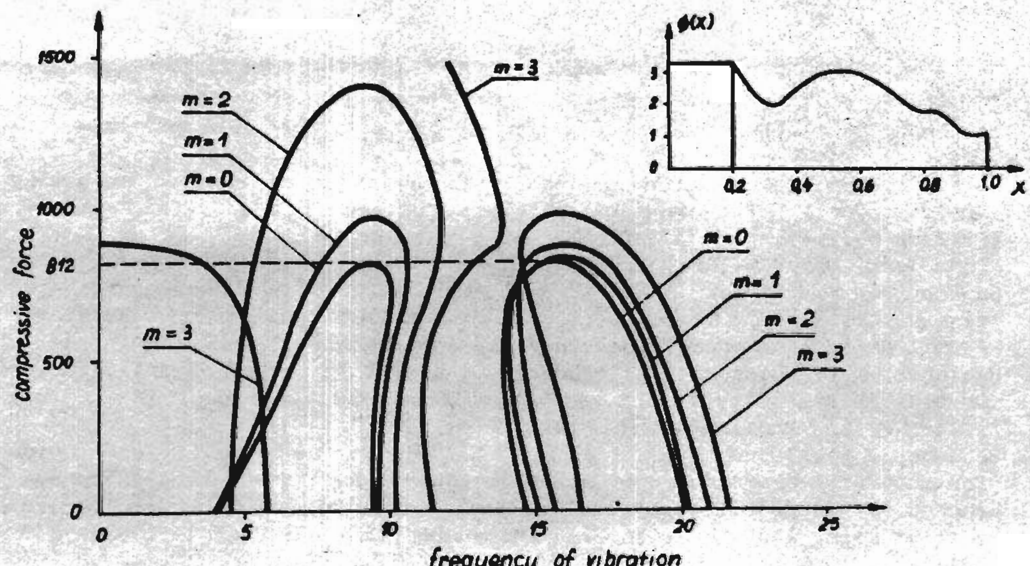

Fig. 3. A shape of the annular plate and related characteristic curves.

transverse vibration were taken into account. In general, the kinetic criterion of stability had to be applied. Using the sensitivity analysis and consecutive iterations Gajewski & Cupiał (1990) observed the typical effects already known from the optimal design of columns. Some new effects connected with evolution of characteristic curves depending on the number of circumferential waves can be seen in Fig.3. Some relating results have already been presented by Gajewski (1990a).

7. Final remarks

The aim of this survey paper was to present, as far as possible, a full list of references concerning the multimodal optimal design of structural elements, despite of a "repeated character" of some papers. The contribution to multimodal optimization topics by authors from the "Cracow optimization school" - mainly coordinated by Professor M. Życzkowski - is evidently seen.

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Summary

WIELOMODALNA OPTYMALIZACJA ELEMENTÓW KONSTRUKCYJNYCH*

W pracy przedstawiono rozwój problematyki optymalnego kształtowania elementów konstrukcyjnych z uwagi na wielokrotne wartości własne. W sposób bardzo krótki omówiono zagadnienia optymalizacji w przypadkach obciążeń konserwatywnych i niekonserwatywnych oraz dla elementów drgających. Zamieszczono wykaz znanych autorowi publikacji, dotyczących omawianej problematyki.

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