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NONHOMOGENOUS, CIRCULAR THICK-WALLED FULLY PLASTICIZED AT FAILURE CYLINDERS UNDER NONUNIFORMLY DISTRIBUTED PRESSURE

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1. Introduction

Nonhomogeneity of material for circular sections of-thick walled cylinders under combined loadings (uniformly distributed pressure, bending moment, axial force, influence of temperature) was sought in former papers Kordas and Wróblewski (1987), and Dollar and Kordas (1990).

The present elaboration is a further step in investigations on this problem; it means the problem is extended on cylinders loaded with nonuniformly distributed pressures: external $p_b(\vartheta)$ and internal $p_a(\vartheta)$ (lack of circular symmetry of pressure) - Fig.1 .

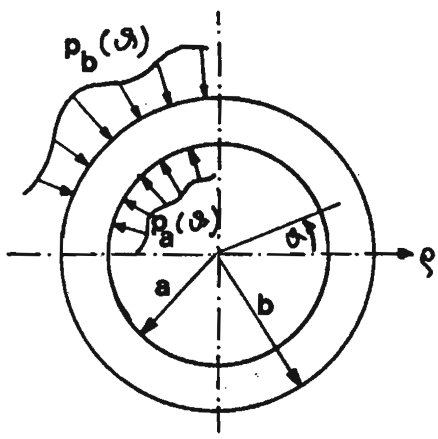

Fig.1. Circular cylinder and its load.

Distribution of plastic nonhomogeneity of the material that compensa tes pressure distribution ensuring full plastification of the circular section of the cylinder at failure is looked for.

Contrary to the method of small parameter applied in a number of papers on determination of shapes of bodies fully plasticized at ultimate carrying capacity the method of stress functions was here applied. As far as the method of small parameter gives solutions only slightly differing from solutions considered as trivial ones, solutions obtained by use of the method of stress function are more general.

A circular thick-walled cylinder of internal and external radii a and b respectively is considered. Material of cylinder is incompressible, perfectly elastic-plastic. Its yield point is sought by use of noncircular-symmetrical function.

With regard to the unlimited length and absence of loads along the axis of the cylinder (normal force, bending moment), a plane state of deformation is adopted. Stress function as well as pressures $p_a(\vartheta)$ and $p_b(\vartheta)$ were developed into trigonometric series (in order to ensure periodicity of solutions) and from the Huber-Mises-Hencky condition of plasticity the yield point was expressed by the coefficients of development of stress function. Making use of stress boundary conditions we obtained condition for coefficients of the stress function which satisfy the equations of internal equilibrium and the yield condition (a solution statically admissible).

For precise solution the condition of positiveness of power of plastic deformation is introduced; this condition gives a differential partial equations of the second order for the modulus of advancement plastic deformation $\phi(\rho,\vartheta)$.

With regard to lack of any boundary conditions for function $\phi(\rho,\vartheta)$ and a considerably complicated form of the equation another approach was applied.

Namely the form of the function $\phi(\rho,\vartheta)$ was assumed in such a way that in the whole area of variability its sign was the same. Subsequently from equation for ϕ the class of functions (coefficients of the yield point development) ensuring the assumed form φ was determined. From this class, functions satisfying the boundary conditions were chosen.

By application of this approach several examples were solved.

2. Basic equations and way of their solution

A thick-walled cylinder of internal radius a and external radius b was considered. The material applied was incompressible, perfectly elastic-plastic and its yield point was sought by use of noncircular-symmetrical function. The cylinder was loaded with internal and external load $p_a(\vartheta)$ and $p_b(\vartheta)$ respectively.

With regard to the unlimited length and absence of normal force and bending moment a plane-state of deformation was adopted. The state of stress is hence in every point of the cylinder determined by three components σ_r, σ_ϑ, $\tau_{r\vartheta}$ and $\sigma_z = (\sigma_r + \sigma_\vartheta)/2$. These are functions of the angle ϑ and dimensionless radius $\rho = r/b$. In the adopted polar coordinates ρ, ϑ, they satisfy two conditions of internal equilibrium:

$$\frac{\partial\sigma_r}{\partial\rho} + \frac{1}{\rho}\frac{\partial\tau_{r\vartheta}}{\partial\vartheta} + \frac{\sigma_r - \sigma_\vartheta}{\rho} = 0 ,$$

$$\frac{\partial\tau_{r\vartheta}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\sigma_\vartheta}{\partial\vartheta} + \frac{2\tau_{r\vartheta}}{\rho} = 0 . \qquad (2.1)$$

and in the presence of the adopted condition of full plastification at failure the Huber-Mises-Hencky condition in form of equation:

$$(\sigma_r - \sigma_\vartheta)^2 + 4\tau_{r\vartheta}^2 = \frac{4}{3}\sigma_0^2(\rho,\vartheta) , \qquad (2.2)$$

where $\sigma_0(\rho,\vartheta)$ denotes the sought yield point. The relations between the components of the state of stress, and the external loadings are determined by the stress boundary conditions

$$\sigma_r(1) = -p_b(\vartheta), \qquad \tau_{r\vartheta}(1) = 0 ,$$

$$\sigma_r(s) = -p_a(\vartheta), \qquad \tau_{r\vartheta}(s) = 0 , \qquad (2.3)$$

where $s = \frac{a}{b}$.

Adopting the stress function $F(\rho,\vartheta)$ so that

$$\sigma_r = \frac{1}{b^2}\left\{\frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \vartheta^2}\right\},$$

$$\sigma_\varphi = \frac{1}{b^2}\left\{\frac{\partial^2 F}{\partial \rho^2}\right\},$$

$$\tau_{r\varphi} = \frac{1}{b^2}\left\{\frac{1}{\rho^2}\frac{\partial F}{\partial \vartheta} - \frac{1}{\rho}\frac{\partial^2 F}{\partial\rho\partial\vartheta}\right\}, \qquad (2.4)$$

identical satisfaction of equations of internal equilibrium (2.1) is ensured.

The function $F(\rho,\vartheta)$ must be assumed so that the boundary conditions (2.3) are satisfied. The yield condition (2.2) permits determination of the looked for distribution of the yield points $\sigma_0(\rho,\vartheta)$.

The stress function $F(\rho,\vartheta)$ must be a periodic function and can be presented in form of trigonometrical series:

$$F(\rho,\vartheta) = \sum_{n=1}\left[\bar{R}_n(\rho)\sin n\vartheta + R_n(\rho)\cos n\vartheta\right] + R_0(\rho). \qquad (2.5)$$

Consideration of (2.5) in formulae (2.4) aims at expressing components of the state of stress:

$$\sigma_r = \frac{1}{b^2}\left\{\sum_{n=1}\left[\left(\rho^{-1}\bar{R}' - n^2\rho^{-2}\bar{R}_n\right)\sin n\vartheta + \left(\rho^{-1}R'_n + -n^2\rho^{-2}R_n\right)\cos n\vartheta\right] + \rho^{-1}R'_0\right\},$$

$$\sigma_\vartheta = \frac{1}{b^2}\left\{\sum_{n=1}\left[\bar{R}''_n\sin n\vartheta + R''_n\cos n\vartheta\right] + R''_0\right\},$$

$$\tau_{r\vartheta} = \frac{1}{b^2}\sum_{n=1}\left[-\left(\rho^{-1}\bar{R}'_n - \rho^{-2}\bar{R}_n\right)\cos n\vartheta + \left(\rho^{-1}R'_n - \rho^{-2}R_n\right) n\sin n\vartheta\right]. \qquad (2.6)$$

Expression of components of the state of stress in the yield condition (2.2) by means of formulae (2.6) permits to determine the sought yield point

$$\sigma_0^2(\rho,\vartheta) = \frac{3}{4b^2}\left\{\sum_{n=0}\left[\left(\bar{R}''_n - \rho^{-1}\bar{R}'_n + n^2\rho^{-2}\bar{R}_n\right)\sin n\vartheta + \right.\right.$$

$$\left.\left. + \left(R''_n - \rho^{-1}R'_n + n^2\rho^{-2}R_n\right)\cos n\vartheta\right]\right\}^2 + \frac{3}{b^2}\left\{\sum_{n=0}\left[\left(\rho^{-1}\bar{R}'_n - \right.\right.\right.$$

$$\left.\left.\left. - \rho^{-2}\bar{R}_n\right)n\cos n\vartheta - \left(\rho^{-1}R'_n - \rho^{-2}R_n\right) n\sin n\vartheta\right]\right\}^2. \qquad (2.7)$$

Pressures $p_a(\vartheta)$ and $p_b(\vartheta)$ are periodic functions and can be developed into Fourier's series:

$$p_a(\vartheta) = \sum_{n=1}\left(\bar{p}_{an}\sin n\vartheta + p_{an}\cos n\vartheta \right) + p_{a0} ,$$

$$p_b(\vartheta) = \sum_{n=1}\left(\bar{p}_{bn}\sin n\vartheta + p_{bn}\cos n\vartheta \right) + p_{b0} . \qquad (2.8)$$

Consideration of formulae (2.6) and (2.8) in boundary conditions (2.3) leads to boundary conditions for coefficients of stress functions : $\bar{R}_n(\rho)$, $R_n(\rho)$, $R_0(\rho)$:

$$R'_0(1) = - b^2p_{b0} , \qquad R'_0(s) = - b^2sp_{a0} ,$$

$$R_n(1) = R'_n(1) , \qquad R_n(s) = R'_n(s) ,$$

$$\bar{R}_n(1) = \bar{R}'_n(1) , \qquad \bar{R}_n(s) = \bar{R}'_n(s)\, s ,$$

$$p_{an} = b^{-2}s^{-2}R_n(s)\,(n^2-1) , \qquad p_{bn} = b^{-2}R_n(1)\,(n^2-1) ,$$

$$\bar{p}_{an} = b^{-2}s^{-2}\bar{R}_n(s)\,(n^2-1) , \qquad \bar{p}_{bn} = b^{-2}\,\bar{R}_n(1)\,(n^2-1) . \qquad (2.9)$$

Each function $F(\rho,\vartheta)$ whose coefficients of development satisfy conditions (2.9) ensure satisfaction of equations of inner equilibrium (2.1) as well as of stress boundary condition (2.3). Such a solution is, however, only a statically admissible one and to be considered a precise one, the condition of positiveness of power of plastic deformations $W=\frac{2}{3}\sigma_o^2\dot{\phi}$ must be checked. This resolves itself into determination of the sign $\dot{\phi}$ (modulus of plastic deformations advancement). From the law of shape change in the plane state of deformation, and from the compatibility condition the following equation for the function determining the modulus of advancement of plastic deformations $\dot{\phi}(\rho,\vartheta)$ is developed:

$$\frac{\partial^2}{\partial\vartheta^2}\left[\phi\,(\sigma_r-\sigma_\vartheta)\right] - \rho^2\frac{\partial^2}{\partial r^2}\left[\phi\,(\sigma_r-\sigma_\vartheta)\right] - 4\,\frac{\partial^2}{\partial r\partial\vartheta}\left[\rho\phi\tau_{r\vartheta}\right] +$$

$$- 3\rho\,\frac{\partial}{\partial r}\left[\phi\,(\sigma_r-\sigma_\vartheta)\right] = 0. \qquad (2.10)$$

Substituting into this equation formulae (2.6) for the components of stress state the following equation is obtained:

$$(\dot\phi - \rho^2\phi'')\sum_{n=0}\left[(\rho^{-1}\bar R_n' - \bar R_n'' - n^2\rho^{-2}\bar R_n)\sin n\vartheta + (\rho^{-1}R_n - R_n'' - n^2\rho^{-2}R_n)\cos n\vartheta\right] +$$

$$- 4\dot\phi'\sum_{n=0}\left[(\rho^{-1}\bar R_n - \bar R_n')n\cos n\vartheta - (\rho^{-1}R_n - R_n)n\sin n\vartheta\right] +$$

$$+ 2\phi\sum_{n=0}\left\{\left[\bar R_n''-\rho^{-1}\bar R_n'+(2-n^2)\rho^{-2}\bar R_n\right]n\cos n\vartheta - \left[R_n''-\rho^{-1}R_n'+(2-n^2)\rho^{-2}R_n\right]n\sin n\vartheta\right. +$$

$$+ \phi'\sum_{n=0}\left\{\left[2\rho^2\bar R_n''' + \rho\bar R_n'' - (2n^2+1)\bar R_n' + 3n^2\rho^{-1}\bar R_n\right]\sin n\vartheta\right. +$$

$$\left. + \left[2\rho^2 R_n''' + \rho R_n'' - (2n^2-1)R_n' + 3n^2\rho^{-1}R_n\right]\cos n\vartheta\right\} +$$

$$+ \phi\sum_{n=0}\left\{\left[\rho^2\bar R_n'''' + 2\rho\bar R_n''' - (2n^2+1)\bar R_n'' + \rho^{-1}(2n^2-1)\dot R_n' + \rho^{-2}(n^4-4n^2)\bar R_n\right]\sin n\vartheta\right. +$$

$$+ \left\{\left[\rho^2 R_n'''' + 2\rho R_n''' - (2n^2+1)R_n'' + \rho^{-1}(2n^2-1)\dot R_n' + \rho^{-2}(n^4-4n^2)R_n\right]\cos n\vartheta = 0\,,\right.$$

$$(2.11)$$

where the dot denotes the derivative with respect to the angle, and prime the derivative with respect to the dimensionless radius ρ.

Two approaches aiming at satisfying the condition of positiveness of the modulus ϕ are possible. First the functions $\bar R_n$ and R_n can be adopted in such a way as to satisfy the boundary conditions (2.9) and subsequently equation (2.11) is solved examining the sign of ϕ.

This, however, encounters considerable difficulties, science in a general case the equation is nonlinear and conditions on function $\phi(\rho,\vartheta)$ are missing.

The form of function $\phi(\rho,\vartheta)$ can be also assumed so that its sign would be identical in the whole area of variability (e.q. by analogy with a homogenous circular cylinder under constant pressure, for which $\phi=c\rho^2$ or ϕ=const). Subsequently from equation (2.11) the class of function $\bar{R}_n$ and R_n which would ensure the assumed form of ϕ can be determined and from it can be chosen functions satisfying the boundary conditions (2.9). In the examples the second approach was obtained. Namely distributions of plastic nonhomogeneity of material corresponding to full plastification of circular sections of cylinders at various loading were found.

3. Examples

In the analyzed examples the form of the function $\phi(\rho,\vartheta)$ was assumed like for a homogenous circular cylinder under constant pressure ($\phi=c\rho^2$), and then from equation (2.11) the class of functions (coefficients of yield point development) which ensure the assumed form of ϕ were determined and from this class, functions satisfying the boundary conditions (2.9) were selected.

Analysis of formulae (2.7-2.11) permits drawing interesting conclusions. We will not, however, discuss it here, presenting only its results in graphical form. For example a circular cylinder loaded with constant pressure p_a and p_b at edges can be fully plastified for a constant yield stress, or dependent only on one variable ρ (circular - symmetrical nonhomogeneity of the yield point), or for noncircular - symmetrical distributions of the yield stress. In Figs 2 and 3 distributions of radial stress $\sigma_r(\rho,\vartheta)$ and distributions of yield point $\sigma_0(\rho.\vartheta)$ were shown in axometric projections for a cylinder loaded with constant pressure $p_a(\vartheta)=3p$ and $p_b(\vartheta)=0$ at given n=4, s=0.5. The diagram $\sigma_r(\rho,\vartheta)$ is simultaneously an illustration of loading the cylinder. In diagrams the adopted scale was p=2.5[mm]. Distribution of radial stress $\sigma_r(\rho,\vartheta)$ and of the yield point $\sigma_0(\rho,\vartheta)$ for a circular cylinder loaded on the edges with variable pressures $p_a(\vartheta)=p(6 + 2\cos5\vartheta)$, $p_b(\vartheta)=p(2 + \cos5\vartheta)$, s=0.5, n=5 are shown in Figs 4 and 5.

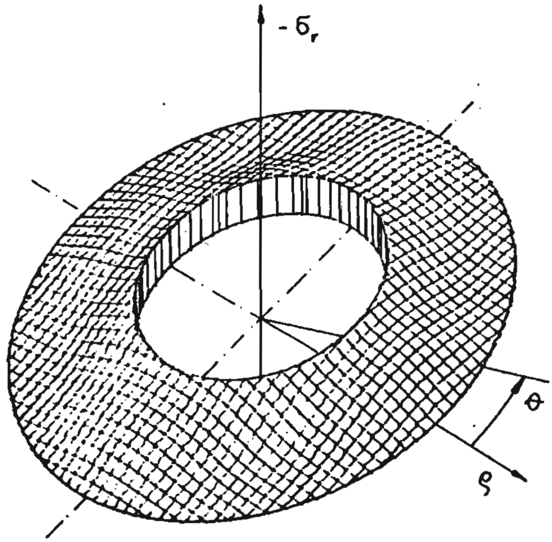

Fig. 2. Distribution of radial stress;
$p_a(\theta)=3p$, $p_b(\theta)=0$, $n=4$, $s=0.5$.

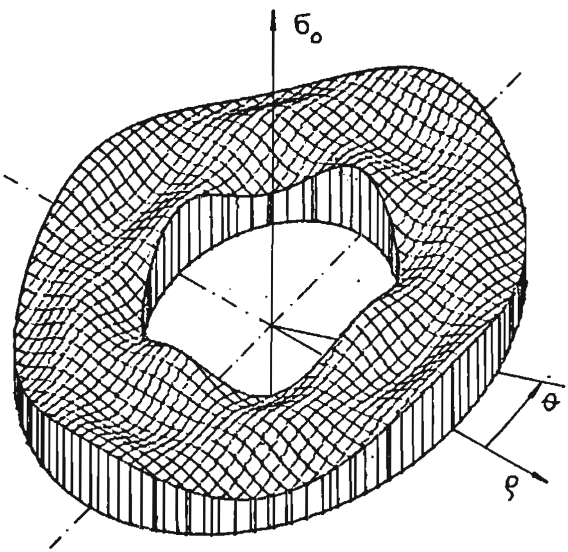

Fig. 3. Sought distribution of the yield point;
$p(\theta)=3p$, $p(\theta)=0$, $n=4$, $s=0.5$.

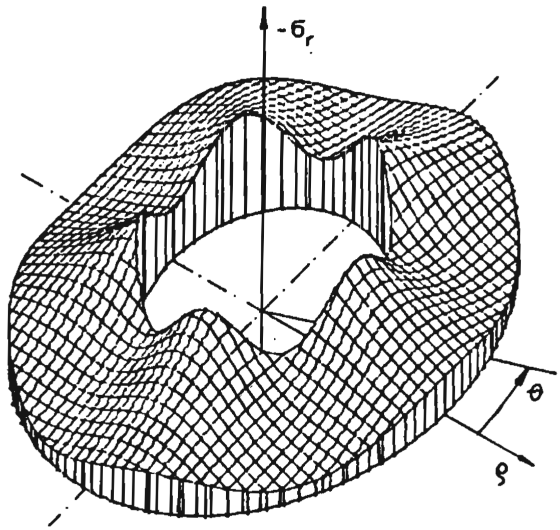

Fig. 4. Distribution of radial stress;
$p_a(\theta)=p(6+2\cos5\theta)$, $p_b(\theta)=p(2+\cos5\theta)$, $n=5$, $s=0.5$

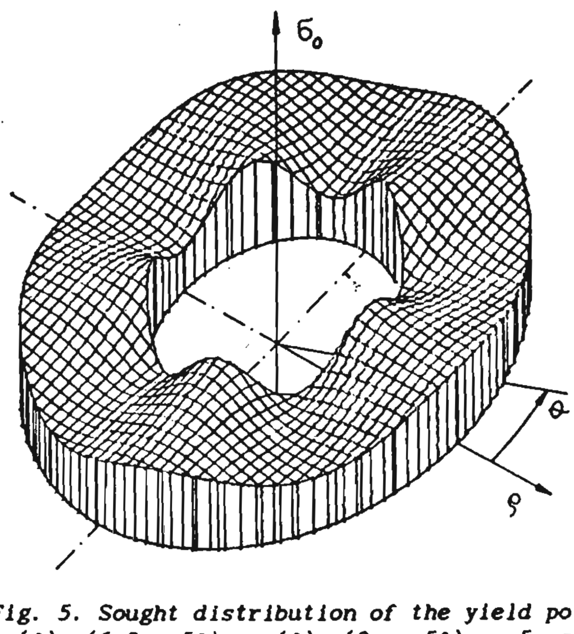

Fig. 5. Sought distribution of the yield point;
$p\ (\theta)=p(6+2\cos5\theta)$, $p\ (\theta)=p(2+\cos5\theta)$, $n=5$, $s=0.5$

References

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Summary

NIEJEDNORODNE KOŁOWE CYLINDRY GRUBOŚCIENNE, CAŁKOWICIE UPLASTYCZNIONE W STADIUM ZNISZCZENIA

W pracy poszukuje się niejednorodności plastycznej materiału, która zapewnia całkowite uplastycznienie kołowych przekrojów cylindrów grubościennych, obciążonych nierównomiernie rozłożonymi ciśnieniami: zewnętrznym $p_b(\vartheta)$ i wewnętrznym $p_a(\vartheta)$. Zastosowano metodę funkcji naprężeń. Funkcję tę, podobnie jak ciśnienia, rozwinięto w szeregi trygonometryczne (celem zapewnienia okresowości rozwiązań). Poszukiwaną granicę plastyczności wyrażono z warunku plastyczności Hubera-Misesa-Hencky'ego poprzez współczynniki rozwinięcia.