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CONTROLLED MOTION OF FINITE ELEMENTS

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We are considering the finite elements controlled motion, in view of further applications of it into problems of the control of deformations. Using differential–geometric techniques based on the Lie brackets and the Lie algebras we have formulated certain statements concerning controllability and minimization of the number of controlling inputs.

1. General considerations

To make the treatment self-contained let us begin with a brief reminder of elementary concepts in control. As asual, the state and control variables will be denoted respectively by $x = (x^1, ..., x^n)$ and $u = (u^1, ..., u^m)$.

The state and control spaces will be denoted by $\mathcal{P}$ and $\mathcal{C}$, respectively.

We say that the control system

$$\frac{dx}{dt} = f(x, u), \tag{1.1}$$

i.e., analytically

$$\frac{dx^i}{dt} = f^i(x^1, ..., x^n; u^1, ..., u^m),$$

is controllable if for any pair of states x_{in} and x_{fin}, there exists a control signal $u(t)$ and a motion $x(t)$ such that the equations given above are satisfied, and

$$x(t_{in}) = x_{in},$$
$$x(t_{fin}) = x_{fin},$$

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t_{in} and t_{fin} being, respectively, an initial and a final (in general non–specified) instants of time.

We say that the system is locally controllable, if for any state x there exists a neighbourhood U such that the system is controllable in U, i.e., any pair of states in U may be joined with an appropriate control process $(x(t), u(t))$, satisfying equations of motion.

We say that the system is dimensionally controllable if, starting from an arbitrary initial state and using appropriate control signals, we can attain n–dimensional regions of the state space. This means, roughly speaking, that the m variables u^a, $a = 1, ..., m$, are sufficient to influence all the n state variables x^i, $i = l, ..., n$.

This concept is nontrivial only if $m < n$; dimensionally non–controllable systems with $m \geq n$ would be artificial (redundant control inputs). It is one of the important tasks of control theory to minimize the number m of controlling inputs as far as possible without violating dimensional controllability. Obviously, dimensional controllability is a necessary, but not sufficient, condition of controllability (without adjectives). Thus, it may be used as a primary test for controllability. The point is that, for a wide class of systems, it may be decided on the basis of purely local considerations. Particularly simple are system homogeneous–multiplicative in controls:

$$\frac{dx}{dt} = \sum_a f_a(x)u^a, \quad \text{i.e.,} \qquad \frac{dx^i}{dt} = \sum_a f_a^i(x)u^a. \tag{1.2}$$

Their local controllability may be decided on the basis of the Chow's lemma.

The corresponding criterion is based on the Lie brackets. Let us remind that the Lie bracket $[X, Y]$ of two vector fields on the state space is defined as a new vector field with components:

$$[X, Y]^i = \sum_j \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right). \tag{1.3}$$

The procedure based on the Chow lemma consists in taking the Lie brackets of all vector fields f_a, $a = 1, ..., m$, then the Lie brackets of so extended system, etc. At every stage we obtain a system of vector fields f_A, the label A running over an appropriate domain. We calculate the rank of the matrix $[f_A^i]$. The rank obtained on each step exceeds or equals to the previous one. After a finite number of steps the rank attains the maximal value – p. If $p < n$, then the system is incontrollable. If $p = n$, it is dimensionally controllable and there is a chance for stronger controllability.

In practical applications we deal quite often with non–homogeneous– multiplicative controls

$$\begin{aligned} \frac{dx}{dt} &= f_0(x) + \sum_{a=1}^{m} f_a(x)u^a, \quad \text{i.e.,} \\ \frac{dx^i}{dt} &= f_0^i(x) + \sum_{a=1}^{m} f_a^i(x)u^a. \end{aligned} \tag{1.4}$$

The non-controlled term f_0 represents the background dynamics, and f_a, $a = 1, ..., m$, are the basic modes of control.

In this case we can also use the Chow lemma, apply it to the system

$$(f_\mu) = (f_0, f_1, ..., f_m), \qquad \mu = 0, 1, ..., m.$$

This means that we consider the system as a homogeneous one with an extra constraint $u^0 \equiv 1$ imposed upon the control variables. In the non-homogeneous case the criterion is weaker, because the background term f_0 may be an obstacle to the local controllability of a dimensionally controllable system.

In the case of linear control systems:

$$\frac{dx}{dt} = \mathsf{A}x + \mathsf{B}u, \tag{1.5}$$

A, B being, respectively, $n \times n$ and $n \times m$ matrices, the criterion based on the Chow lemma reduces to the classical Kalman criterion of controllability (without adjectives):

$$\text{rank}[\mathsf{B}, \mathsf{AB}, \mathsf{A}^2\mathsf{B}, ..., \mathsf{A}^{n-1}\mathsf{B}] = n. \tag{1.6}$$

Similar ideas may be useful in the theory of nonlinear control systems:

$$\frac{dx}{dt} = f(x, u). \tag{1.7}$$

Namely, we take into account all possible vector fields f_u corresponding to fixed values u of the controlling input,

$$f_u(x) := f(x, u). \tag{1.8}$$

Then, we take the linear shell of this system, i.e., the linear space of all vectorfields of the form:

$$f(x) = \sum_{k=1}^{N} c_k f(x, u_k), \tag{1.9}$$

N, c_k's, u_k's running over, respectively, the set of naturals, the set of real numbers, and the set of all possible control vectors. Taking the Poisson–bracket extension of this system and calculating the ranks of matrices just as in the case of multiplicative controls, we can verify dimensional controllability.

2. Examples

As a preliminary step towards discussion of controlled motion of homogeneously deformable bodies (finite elements) let us consider certain problems of control of dynamical systems on matrix spaces.

Thus, we use the group $GL^{+}(n)$ of all positive–determinant matrices as a manifold of states and consider the following models with multiplicative controls:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) + \mathbf{U}\mathbf{X}, \qquad \frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) + \mathbf{X}\mathbf{U}. \tag{2.1}$$

States of controlling inputs are represented by matrices $\mathbf{U}$; controllability properties of the system depend on the admissible range of them.

Let us quote a few examples relevant for our purpeses.

Example 1

As a set C of admissible controls $\mathbf{U}$ we take the space of symmetric matrices $L_{sym}(n) \subset L(n)$.

The criterion based on the Chow's lemma tells us that the above systems are dimensionally controllable. In the homogeneous case, when $\mathbf{F} = \mathbf{0}$, they are simply controllable. This is due to the fact that symmetric matrices do not form a Lie algebra. Moreover, taking all their commutators we can generate the $sl(n)$–Lie algebra of traceless matrices. Thus, the extended system coincides with $L(n)$ – the total algebra of all $n \times n$ real matrices. In other words, the manifold of nonsingular symmetric matrices is not a subgroup of $GL(n)$. It generates the total $GL(n)$, i.e., composing linear transformations corresponding to symmetric matrices we can obtain all posible linear transformations. Roughly speaking, homogeneous deformations can produce rotations.

Example 2

As a set C of admissible controls we take $so(n) \subset L(n)$ – the space of skew–symmetric matrices. If $\mathbf{F} = \mathbf{0}$ the systems (2.1) are evidently non–controllable. The reason is that the space $so(n)$ is closed under the commutator operation, i.e., it is a Lie algebra. Every possible motion holds within the subgroup $SO(n) \subset GL(n)$ of proper orthogonal matrices or within its cosets (subsets of the form $LSO(n)$ or $SO(n)L$; L being an element of $GL(n)$. On these submanifolds the system is controllable. Therefore, there are $\frac{1}{2}n(n+1)$ non–controllable parameters corresponding to deformations, i.e., to symmetric matrices. Nevertheless, for a generic non–vanishing $\mathbf{F}$ the system may be controllable because it may happen quite easily that the Lie–bracket extension of the system of vectorfields $\mathbf{F}$, $\mathbf{F}_{\mathbf{U}}$ (where

$\mathsf{F_U} = \mathsf{UX}$ or XU and $\mathsf{U} \in so(n)$), has the rank n^2, i.e., equals to the dimension of the state space.

Example 3

As a set $\mathcal{C}$ of admissible controls we take $sl(n) \subset L(n)$ – the space of traceless matrices. If $\mathsf{F} = \mathbf{0}$, the systems (2.1) are non–controllable. The mechanism of this non–controllability is the same as in example 2. Namely, the Lie–algebraic structure of $sl(n)$ implies that the Lie–bracket extension of the space of vector fields $\mathsf{F_U}$ (where $\mathsf{F_U} = \mathsf{UX}$ or XU, $\mathsf{U} \in sl(n)$) coincides with the same space. The system moves within the group $SL(n)$ of unimodular matrices or its cosets $LSL(n)$, $SL(n)L$, where $L \in GL(n)$. On these (n^2-1)–dimensional surfaces the motion is controllable. There is exactly one completely incontrollable parameter corresponding to dilatations. Just as in example 2 this parameter may be made controllable by switching on an appropriate non–homogeneous term F. These examples illustrate in a very convincing way the mechanism of controllability of N state variables with the help of $m < N$ control inputs.

An interesting question is how to reduce the number of control parameters in models (2.1) as far possible. The crucial problem is the minimization of inputs for homogeneous models

$$\frac{d\mathsf{X}}{dt} = \mathsf{UX}, \qquad \frac{d\mathsf{X}}{dt} = \mathsf{XU}, \tag{2.2}$$

because in general the non–controllable background term F is a favourable factor for controllability.

The problem of minimizing the manifold $\mathcal{C}$ of controls U in models (2.2) belongs to the realm of Lie algebras theory. Indeed, in purely geometric terms it may be formulated as follows: find a minimal subspace $\mathcal{C}$ of the space $L(n)$ of $n \times n$ matrices, generating the total $L(n)$–space in the sense of commutator. By generating in the sense of commutator we mean that taking $\mathcal{C}$, the space $[\mathcal{C},\mathcal{C}]$ of commutators of elements from $\mathcal{C}$, next iterating this procedure, and at the end taking the linear shell of resulting subspaces, we obtain the space $L(n)$. In more precise terms: we construct the sequence $\mathcal{C}_i$ of linear subspaces of $L(n)$ such that:

$$\begin{aligned} \mathcal{C}_0 &:= \mathcal{C}, \\ \mathcal{C}_{i+1} &:= \left[\mathcal{C}_i, \bigoplus_{j=0}^{i} \mathcal{C}_j\right] = \bigoplus_{j=0}^{i} [\mathcal{C}_i,\mathcal{C}_j]. \end{aligned} \tag{2.3}$$

$L(n)$ is generated in the sense of commutator by $\mathcal{C}$ if there exists a natural number p such that $L(n) = \mathcal{C}_p$.

The commutation Lie algebra $L(n)$ is the direct sum of $sl(n)$, the semisimple Lie algebra of traceless matrices, and the one–dimensional centre spanned by real multiples of the identity matrix I. This reflects the fact that the proper linear group

$GL^+(n)$ (the group of real matrices with positive determinants) is the direct sum of the one–dimensional centre $R^+\mathbf{I}$ (dilatations, positive multiples of the identity matrix) and the semisimple Lie group $SL(n)$ of matrices with determinats equal to unity (pure shears and rotations).

As it follows from the bove examples, the subspace of symmetric matrices generates the total $L(n)$ in the commutator sense.

This is an infinitesimal counterpart of the fact that the manifold of symmetric matrices with positive determinants generates the proper linear group $GL^+(n, R)$. We are particularly interested in special cases: $n = 2$, $n = 3$, corresponding to planar an spatial finite elements, respectively.

If $n = 2$, then the space of symmetric matrices is a minimal space generating $L(n)$ in the commutator sense. This subspace may be spanned on the following basic matrices:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{a} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.4}$$

$\mathbf{I}$ generates dilatations, and $\mathbf{a}$, $\mathbf{b}$ are infinitesimal basic shears.

$\mathbf{a}$ generates elongations along the x axis accompanied by contractions along the y axis (so that the two–dimensional area is preserved), and $\mathbf{b}$ generates the two–dimensional Lorentz transformations (hyperbolic rotations) in the (x, y)–plane.

Rigid rotations are generated by the skew–symmetric matrix

$$\mathbf{c} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.5}$$

It is clear that

$$[\mathbf{a}, \mathbf{b}] = -2\mathbf{c},$$

thus the space $sl(2)$ is commutator–generated by the 2–dimensional space spanned by $\mathbf{a}$, $\mathbf{b}$.

The total $L(2)$ is generated by: $\mathbf{I}$, $\mathbf{a}$, $\mathbf{b}$.

Let us mention that one can also use another system of generators, e.g., $\mathbf{I}$, $\mathbf{a}$, $\mathbf{c}$, because $\mathbf{b}$ may be produced by commuting $\mathbf{a}$ and $\mathbf{c}$.

However, the further reduction is impossible; one must always use three control inputs for models (2.2) if they are to be controllable.

There is no input gain if we are interested in controlling deformations. Three deformational state variables must be controlled by three inputs if the dynamics is ruled by equations (2.2).

Let us now consider the really physical dimension $n = 3$.

Obviously, as for any dimension n, the one–dimensional dilatation centre is separated from shears and rotations. It is generated by the 3–dimensional unit matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One can show that the semisimple unimodular part $sl(3, R)$ may be generated by the 3–dimensional space spanned on the following matrices:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \multicolumn{2}{c}{\mathbf{a}} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{B} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \multicolumn{2}{c}{\mathbf{b}} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{C} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\mathbf{c}} \\ 0 & & \end{bmatrix}, \end{aligned} \tag{2.6}$$

where symbols **a**, **b**, **c** denote the basic 2×2 matrices introduced above. Thus, in three dimensions the group $SL(3, R)$ of unimodular matrices may be generated by two basic shears in a fixed two–dimensional subspace and by one planar rotation of another two–dimensional subspace (non–identical with the previous one).

Thus, we conclude that the systems

$$\frac{d\mathbf{X}}{dt} = \mathbf{U}\mathbf{X}, \qquad \frac{d\mathbf{X}}{dt} = \mathbf{X}\mathbf{U}, \tag{2.7}$$

are controllable in the 9–dimensional state space $GL(3)$ of nonsingular 3×3 matrices if the controlling matrices **U** run over the 4–dimensional space spanned on matrices **I**, **A**, **B**, **C**. The systems (2.7) are controllable in the 8–dimensional space $SL(3)$ of unimodular matrices if **U** runs over the 3–dimensional space spanned on matrices **A**, **B**, **C**. Therefore, 8 degrees of freedom of an incompressible finite element may be controlled by two purely deformative inputs, namely, the basic planar shears and by one rotational parameter (provided that the rotation vector is not orthogonal to the plane of shears).

The situation is pictured in the following figure 1.

The coordinate axis z is perpendicular to the figure plane.

The matrix **A** generates the one–parameter group which does not affect the z variable and gives rise to elongations along the x axis accompanied by the same ratio contractions along y.

The matrix **B** also does not affect the z coordinate and generates the oneparameter group of hyperbolic rotations in the (x, y)–plane. The "light cone" of the corresponding "Lorentz transformation" is denoted by dotted straight lines.

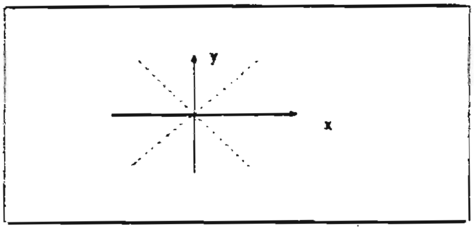

Fig. 1.

The matrix **C** generates the group of one–parameter rotations on the x axis, thus, the x variable is nonaffected, and the (y, z)–variables are subject to planar rotations.

The controls spanned on matrices **A**, **B**, **C** are sufficient for controlling an incompressible finite element.

I ensures the control of uniform dilatations; we must include the control input represented by matrices of the form

$$u\mathbf{I},$$

where u denotes a time–dependent scalar factor.

3. Control of homogeneous deformations

Let us turn now to the proper description of dynamics.

In the aforementioned considerations we were dealing with dynamical systems, i.e., first order ordinary differential equations in matrix spaces. This was a merely preliminary step, because the dynamics of homogeneous elements is based on the second–order ordinary differential equations in the matrix group $GL(n)$ (or $SL(n)$ in the case of incompressible finite elements).

Let us remind briefly equations of motion for finite elements and rewrite them to a few alternative forms convenient for controllability analysis.

Let the matrix **X** denote the deformation gradient (placement) of a homogeneous element. Thus, spatial and material coordinates: x^i and a^K, are related to each other through the following relations:

$$x^i = X^i_K a^K. \tag{3.1}$$

If we consider also translational motion of the centre of mass, this formula is to be replaced by

$$x^i = X^i_K a^K + b^i. \tag{3.2}$$

However, from now on we consider only rotational and deformative motion of finite elements, thus we put $b^i = 0$.

Equations of the rotational–deformative motion have the form:

$$\mathsf{X}\mathsf{J}\frac{d^2\mathsf{X}^{\mathsf{T}}}{dt^2} = \mathbf{N}, \tag{3.3}$$

where J is a constant material matrix describing inertial properties of the finite element with respect to the rotational and deformative degrees of freedom, and the dynamical term $\mathbf{N}$ is an asymmetric moment of forces acting on the element:

$$\begin{aligned} J^{KL} &:= \int a^K a^L dM(a), \\ N^{ij} &:= \int x^i F^j(x) dm(x), \end{aligned} \tag{3.4}$$

dM denotes the element of mass in material representation (Lagrangian variables), dm denotes the element of mass in spatial representation (Eulerian variables), and F^j is the spatial density of forces per unit mass.

The quantity J is symmetric and equivalent to the usual inertial tensor I:

$$I^{KL} = \delta_{CD} J^{CD} \delta^{KL} - J^{KL}. \tag{3.5}$$

The doubled antisymmetric part of $\mathbf{N}$ equals the usual moment of forces, responsible for rotational dynamics.

The dynamics of deformative motion is ruled by the symmetric part of $\mathbf{N}$.

In the case of internal contact forces we have:

$$N^{ij} = -\int \sigma^{ij} d^3x, \tag{3.6}$$

where $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor.

Let us stress an important point that equations of motion (3.3) involve only forces being applied and are explicitly free of reactions responsible for constraints of approximately homogeneous deformability. The point is that although reactions $\mathcal{R}$ are nontrivial, their asymmetric moment $N^{ij}_{\mathcal{R}}$ vanishes due to the d'Alembert principle.

To discuss controllability properties we should rewrite the system of $n^2 = 9$ second–order ordinary differential equations to a dynamical system, i.e., to a system of $2n^2 = 18$ first–order equations.

As usual, the simplest possibility is to introduce formally a new variable coinciding with generalized velocity:

$$\mathsf{V} := \frac{d\mathsf{X}}{dt}. \tag{3.7}$$

Then equations of motion have the form:

$$\frac{d\mathsf{X}}{dt} = \mathsf{V}, \qquad \frac{d\mathsf{V}}{dt} = \mathsf{N}^{\mathsf{T}}\tilde{\mathsf{X}}^{\mathsf{T}}\tilde{\mathsf{J}}, \tag{3.8}$$

where tilde denotes the operation of taking the reciprocal matrix:

$$\tilde{\mathsf{J}}\mathsf{J} = \mathsf{I}, \qquad \tilde{\mathsf{F}}\mathsf{F} = \mathsf{I}. \tag{3.9}$$

Iy may be also convenient to use canonical momenta instead of generalized velocities.

One can show that expressing the kinetic energy through variables X, V, we have

$$T = \frac{1}{2}\mathrm{Tr}(\mathsf{V}^{\mathsf{T}}\mathsf{V}\mathsf{J}). \tag{3.10}$$

Thus, for systems ruled by classical Lagrangians

$$L = T - W(\mathsf{X}), \tag{3.11}$$

we have the following expression for canonical momentum P:

$$\mathsf{P} = \mathsf{J}\mathsf{V}^{\mathsf{T}}. \tag{3.12}$$

Thus, we have the following set of equations of motion:

$$\frac{d\mathsf{X}}{dt} = \mathsf{P}^{\mathsf{T}}\tilde{\mathsf{J}}, \qquad \frac{d\mathsf{P}}{dt} = \tilde{\mathsf{F}}\mathsf{N}. \tag{3.13}$$

For models $L = T - W(\mathsf{X})$ we have: $\mathsf{N} = -\mathsf{X}\left(\frac{\partial W}{\partial \mathsf{X}}\right)^{\mathsf{T}}$.

For certain reasons it may be convenient to use another, equivalent description. Namely, instaed of the state variables (X,V) or (X,P) we can use the alternative representations (X,E), $(\mathsf{X},\hat{\mathsf{E}})$, (X,K), $(\mathsf{X},\hat{\mathsf{K}})$, where $\mathsf{E} = \mathsf{V}\mathsf{X}^{-1}$ denotes the quantity called by Eringen–gyration, and by others – affine velocity, K is called affine momentum, and $\hat{\mathsf{E}}$, $\hat{\mathsf{K}}$ are the co–moving representations of these quantities, thus

$$\hat{\mathsf{E}} = \mathsf{X}^{-1}\mathsf{E}\mathsf{X} = \mathsf{X}^{-1}\mathsf{V}, \qquad \mathsf{K} = \mathsf{X}\hat{\mathsf{K}}\mathsf{X}^{-1}. \tag{3.14}$$

The mechanical meaning of E is that it represents a gradient of the Eulerian velocity field within the homogeneous element,

$$v^i(x) = E^i_j x^j. \tag{3.15}$$

The quantity K represent an asymmetric part of linear momentum

$$K^{ij} = \int x^i v^j(x) dm(x), \tag{3.16}$$

the doubled skew–symmetric part of K equals the angular momentum.

One can show that:

$$\begin{aligned} \mathsf{K} &= \mathsf{XJV}^\mathsf{T} = \mathsf{XJX}^\mathsf{T}\mathsf{E}^\mathsf{T} = \mathsf{XP}, \\ \hat{\mathsf{K}} &= \mathsf{J}\hat{\mathsf{E}}^\mathsf{T}. \end{aligned} \tag{3.17}$$

K describes a non–holonomic canonical momentum conjugate to non–holonomic velocity E,

$$\mathrm{Tr}(\mathsf{KE}) = \mathrm{Tr}(\mathsf{PV}) = \mathrm{Tr}(\hat{\mathsf{K}}\hat{\mathsf{E}}). \tag{3.18}$$

The formula for the power of forces has the following form:

$$P = \mathrm{Tr}(\mathsf{NE}) = \mathrm{Tr}(\hat{\mathsf{N}}\hat{\mathsf{E}}), \tag{3.19}$$

thus N denotes the generalized force conjugate to non–holonomic velocity E and, as we shall see, responsible for the balance of K. According to our notational conventions $\hat{\mathsf{N}}$ in the last formula is defined by: $\hat{\mathsf{N}} = \tilde{\mathsf{X}}\mathsf{NX}$.

Quantities K, $\hat{\mathsf{K}}$ denote respectively, Hamiltonian generators of transformations

$$\begin{aligned} \mathsf{X} &\longmapsto \mathsf{LX}, \\ \mathsf{X} &\longmapsto \mathsf{XL}, \end{aligned} \tag{3.20}$$

acting on the group $GL(n)$.

The matrix L in these formulae creates an element of $GL(n)$.

Interpretation of K, $\hat{\mathsf{K}}$ as generators of the above described transformations is important in the discussion of controllability.

Let us observe that formulae

$$\begin{aligned} \hat{\mathsf{N}} &= \tilde{\mathsf{X}}\mathsf{NX}, \\ \hat{\mathsf{K}} &= \tilde{\mathsf{X}}\mathsf{KX}, \\ \hat{\mathsf{E}} &= \tilde{\mathsf{X}}\mathsf{EX}, \text{ etc.} \end{aligned} \tag{3.21}$$

transform the matrices N, K, E, etc. according to the rule for mixed tensors.

In certain problems it may be convenient to use material tensor obtained from N and K according to the transformation rule for twice contravariant tensors

$$\begin{aligned} \bar{\mathsf{N}} &= \tilde{\mathsf{X}}\mathsf{N}\tilde{\mathsf{X}}^\mathsf{T}, \\ \bar{\mathsf{K}} &= \tilde{\mathsf{X}}\mathsf{K}\tilde{\mathsf{X}}^\mathsf{T}. \end{aligned} \tag{3.22}$$

In general $\bar{N} \neq \hat{N}$, $\bar{K} \neq \hat{K}$.

In the special case of a non-deformable (i.e., rigid) element, when $X^T X = I$, both transformation rules coincide and $\bar{N} = \hat{N}$, $\bar{K} = \hat{K}$.

However, we are interested in deformative processes, thus $\bar{N}$, $\bar{K}$ must be carefully distinguished from $\hat{N}$, $\hat{K}$.

Let us quote a few formulae interrelating the above quantities:

$$\begin{aligned} P &= JV^T, \\ K &= XJV^T = XJX^T E^T, \\ \bar{K} &= J\hat{E}^T, \\ \hat{K} &= J\hat{E}^T G = \bar{K}G, \end{aligned} \tag{3.23}$$

where the matrix G represents the Green deformation tensor for the finite element,

$$G = X^T X.$$

When rewriting our equations of motion as a dynamical system, we can use the following equivalent forms differing in the state variables:

$$\begin{aligned}
&(a) && \frac{dX}{dt} = V, \quad \frac{dV}{dt} = N^T \tilde{X}^T \tilde{J}, \\
&(b) && \frac{dX}{dt} = P^T \tilde{J}, \quad \frac{dP}{dt} = \tilde{X}N, \\
&(c) && \frac{dX}{dt} = EX, \quad \frac{dE}{dt} = -E^2 + N^T(\tilde{X}^T \tilde{J}\tilde{X}), \\
&(d) && \frac{dX}{dt} = K^T \tilde{X}^T \tilde{J}, \quad \frac{dK}{dt} = K^T(\tilde{X}^T \tilde{J}\tilde{X})K + N, \\
&(e) && \frac{dX}{dt} = X\hat{E}, \quad \frac{d\hat{E}}{dt} = -\hat{E}^2 + \bar{N}^{-T}\tilde{J} = -\hat{E}^2 + \tilde{G}\hat{N}^T\tilde{J},
\end{aligned} \tag{3.24}$$

$$(f)\qquad \frac{dX}{dt} = X\bar{K}^{T}\bar{J},$$
$$\frac{d\bar{K}}{dt} = -\bar{K}\bar{J}\bar{K} + \bar{N},$$

$$(g)\qquad \frac{dX}{dt} = \bar{X}^{T}\hat{K}^{T}\bar{J},$$
$$\frac{d\hat{K}}{dt} = \hat{K}\bar{G}\bar{K}^{T} + \hat{N}, \qquad (G = X^{T}X).$$

The dynamical subsystem (second subsystem) of equations (d) has the form of a balance law for affine momentum. Let us observe that even in the interaction free case, when $N = 0$, this balance fails to be a conservation law. Restricting X to orthogonal matrices, and taking the skew–symmetric part of the second subsystem of (d) or (g) we obtain equations of rotational motion of a rigid body.

It is obvious that equations (e), (f), (g) resemble the structure of the Euler equations in dynamics of a rigid body. In the special case of constraints of deformation–free motion they exactly reduce to the Euler equations.

The form (g) is the most geometric and enables one to discus controllability and reduction of inputs on the basis of Lie–algebraic considerations quoted above, especially when we combine it with (e):

$$(h)\qquad \frac{dX}{dt} = X\hat{E},$$
$$G\frac{d\hat{E}}{dt}J = -G\hat{E}^{2}J + \hat{N}^{T}.$$

The form (f) is not so geometric as (g) because the quantity $\bar{K}$ is not a Hamiltonian generator of right regular translations: $X \longmapsto XL$; nevertheless its analogy with gyroscopic Euler equations is closest in that the non–dynamical term quadratic in $\bar{K}$ is configuration–independent.

The main difference between equations for rigid bodies and finite elements is that in the latter case the non–dynamical term does not vanish even in the case of spherical inertial symmetry (when $J = aI$).

The asymmetric moments N, $\hat{N}$ in general consists of the background and control terms. As a controlling agent the inertial tensor J may be used as well.

There are two kinds of control problems in the finite–elements–dynamics: the inner problem and the outer problem.

The mathematical peculiarity of inner problems of steering is that one uses co–moving Lagrangian components of controlling moments $\bar{N}$ or $\hat{N}$ as directly manipulated quantities. Thus, for example, when $\hat{N} = \hat{N}_0 + U$ ($\hat{N}_0$ denoting the background and $\mathbf{U}$ the control term) then in the inner problems $\mathbf{U}$ depends directly only on time (not through the state variables $(\mathbf{X}, \hat{\mathbf{K}})$.

Such mathematical models describe situations where the controlling devices are frozen immovably in the material.

In out–steering problems there are spatial (Eularian) components of the controlling moment $\mathbf{N}$ that are assumed to be directly manipulated quantities. Such models describe situations where the control influences are produced by external devices like pull rods, external fields, etc.

Having in view developments of controlled flexible manipulator we are more interested in the problem of inner steering of finite elements. Indeed, the most appropriate and natural way of steering such beam–like–objects is to build up into their substance several local sources of stresses. These may be electromechanical instruments released by electric signals. Another possibility is a magnetic influence of solenoids winded around the body on ferromagnetic suspensions distributed in an appropriate way within the material. To discuss controllability of a dynamical finite element working in the regime of inner steering, we have to use equations (h) and perform carefully the analysis of dimensional controllability, using the criteria based on the Chow lemma. The above-quoted results concerning controllability of systems

$$(i) \qquad \frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) + \mathbf{U}\mathbf{X}, \qquad \frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) + \mathbf{X}\mathbf{U}, \tag{3.25}$$

are very helpful in this analysis and provide us with effective guiding hints.

After rather long and unpleasant calculations of ranks and matrices one can show that, as expected from the analysis of equations, the system may be effectively controlled with the help of a 3–dimensional input. Let $\hat{\mathbf{N}} = \hat{\mathbf{N}}_0 + \hat{\mathbf{N}}_c$, where $\hat{\mathbf{N}}_0$ denotes a background moment depending only on dynamical variables $\mathbf{X}$, $\hat{\mathbf{E}}$, and perhaps on time, and $\hat{\mathbf{N}}_c$ denotes a controlling term depending on $\mathbf{X}$, $\hat{\mathbf{E}}$ and control inputs u.

For simplicity we assume the model with multiplicative controls, i.e., $\hat{\mathbf{N}}_c$ depending on values of u in a linear homogeneous way,

$$\hat{\mathbf{N}}_c = \sum_{a=1}^{m} u^a \hat{\mathbf{N}}_a. \tag{3.26}$$

Performing the Lie–bracket analysis we can show that:

- If basic control modes $\hat{\mathbf{N}}_a$ run over the space of all symmetric tensors, then the system (h), i.e., dynamical finite element, is dimensionally controllable quite independently in the form of the background term $\mathbf{N}_0$ and the constant matrix $\mathbf{J}$.

- If basic control modes $\hat{\mathbf{N}}_a$ run over the 3–dimensional space spanned by matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ quoted above, then the system is dimensionally controllable, independently in the form of $\mathbf{N}_0$ and the inertial tensor $\mathbf{J}$.
  Let us observe in this connection that this result is stronger than the corresponding statements for systems
  $$\frac{d\mathbf{X}}{dt} = \mathbf{F} + \mathbf{U}\mathbf{X},$$
  $$\frac{d\mathbf{X}}{dt} = \mathbf{F} + \mathbf{X}\mathbf{U}.$$
  For these systems the space of controls spanned by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ was sufficient for dimensional controllability for $SL(3)$, i.e., for incompressible finite element, but not for the total $GL(3)$. The reason for this additional gain is that equations (h) are not invariant under dilatations.

- If the spectrum of $\mathbf{J}$ is non–degenerate (no coincidence of main inertial moments), then the system (h) is dimensionally controllable for any choice of $\hat{\mathbf{N}}_c$, provided that the symmetric part of $\hat{\mathbf{N}}_c$ can not be diagonalized simultaneously with $\mathbf{J}$. This is true even if we have only one control input. This statement is true for a generic shape of the background tensor $\mathbf{N}_0$. The characteristic Eulerian term quadratic in $\hat{\mathbf{E}}$ and appearing on the right hand side of (h) is responsible for this controllability gain.

It is easy to see that these results resemble certain characteristic features of controllability of a rigid body motion. Let us remind, for example, that if inertial tensor is non–degenerate, then the rigid body mae be effectively controlled with the help of only one control input like a single moment of forces produced by a corrective motor, or by a single rotor, provided that the produced moment is not collinear with any principal axias of inertia.

Our results quoted above have the same origin, namely, structural properties of the group $GL(3)$ uderlying degrees of freedom of the finite element.

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Streszczenie

W pracy rozważa się zagadnienie sterowania ruchem elementu skończonego. Jest to wstęp do dalszych badań dotyczących sterowania deformacjami. Używając metod geometrii różniczkowej, opartych na pojęciach nawiasu i algebry Liego, sformulowano pewne stwierdzenie dotyczące sterowalności i minimalizacji liczby wejść.

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