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THEORETICAL AND NUMERICAL STUDIES OF RELAXATION DIFFERENTIAL APPROACH IN VISCOELASTIC MATERIALS USING GENERALIZED VARIABLES

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The phenomenon of incrementalization in the time domain, for linear non-ageing viscoelastic materials undergoing mechanical deformation, is investigated. Analytical methods of solution are developed for linear viscoelastic behavior in two dimensions utilizing generalized variables and realistic material properties. This is accomplished by the use of time-dependent material property characterization through a Dirchilet series representation, thus the transformation of the viscoelastic continuum problem from the integral to a differential form is achieved. The behavior equations are derived from linear differential equations based on the discrete relaxation spectrum. This leads to incremental constitutive formulations using the finite difference integration, thus the difficulty of retaining the strain history in computer solutions is avoided. A complete general formulation of linear viscoelastic strain analysis is developed in terms of increments of generalized stresses and strains.

Key words: linear viscoelasticity, differential approach, incremental constitutive law, discrete relaxation spectrum, generalized variables

1. Introduction

Viscoelastic materials are characterized by possessing infinite memory. Their actual mechanical response is a function of the whole past history of stress and strain. In most cases, the behavior of any linear viscoelastic material may be represented by a hereditary approach based on the Boltzmann superposition principle (Boltzmann, 1878). This implies that stress and strain analysis of viscoelastic phenomena which can be observed in the behavior of many real materials, presents many difficulties for real problems of complex geometry. The analysis of linear viscoelastic materials is usually obtained by an application of the correspondence principle to the equations of elasticity (Birnecker, 1992; Chazal and Dubois, 2001; Christensen, 1971). This approach is restricted to problems for which it is possible to find an explicit solution to the associated equations of elasticity. In order to obtain solutions to more complicated problems, it is necessary to develop numerical rather than analytical techniques. These numerical methods avoid the retaining of the whole past history of stress and strain in the memory of a digital computer and permit to deal with complex viscoelastic structures involving complicated boundary conditions. The key to such methods is to incrementalize the hereditary integral equations by means of analytical techniques. Thus, the difficulty of computer storage requirements is avoided and the complete past history of stress and strain is represented by means of some auxiliary tensors.

The problem of finding incremental formulations of linear non-ageing viscoelastic problems has been investigated thoroughly by a number of authors (Jurkiewiez *et al.*, 1999); Chazal and Moutou Pitti, 2010a,b, 2011a,b; Moutou Pitti *et al.*, 2011). The interest for such a formulation lies in its help both in understanding theoretical aspects, especially in the mathematical treatment of the integral equations (for instance, they might turn out to be useful as a tool for deriving approximate governing equations for the behavior of viscoelastic continua) and in developing efficient numerical integration methods on a clearly stated theoretical basis.

Several recent studies have addressed the subject of incremental constitutive laws for linear non-ageing viscoelastic materials undergoing mechanical deformation (Chazal and Moutou Pitti, 2009a,b, 2010a, 2011a,b; Moutou Pitti *et al.*, 2011). Early works by Chazal and Moutou Pitti (2010b) considered the theory of linear viscoelasticity to establish incremental constitutive equations using creep or relaxation functions. By assuming a strong form of the creep or relaxation function, and by defining the behavior of the material in differential or integral approach that previously was proposed by Zocher *et al.* (1997), incremental equations in the time domain for a linear non-ageing viscoelastic material can be constructed (Chazal, 2000; Chazal and Moutou Pitti, 2009c, 2010a, 2011a,b; Moutou Pitti *et al.*, 2011). However, these formulations deal with local integration for the Volterra equations.

Chazal and Moutou Pitti (2009b) established a general method for finding the incremental formulation for any linear ageing viscoelastic problem on the basis of the choice of an integrating operator, while Chazal and Moutou Pitti (2009d), Moutou Pitti *et al.* (2010a,b,c,d) established a similar method for the specific case of the linear non-ageing problem. These methods were successfully applied to general linear material continua in deriving an extended formulation of the classical principle of Boltzmann in the static case (see Chazal 2000; Chazal and Dubois, 2001; Chazal and Moutou Pitti, 2009c; Moutou Pitti *et al.*, 2009). Bozza and Gentili (1995) used the theory of linear viscoelasticity to establish constitutive equations using relaxation functions. They sought solutions to the inversion problem of the constitutive equations. Drozdov and Dorfmann (2004) derived constitutive equations for the nonlinear viscoelastic behavior after performing tensile relaxation tests. Kim and Lee (2007) and Theocaris (1964) have proposed an incremental formulation and constitutive equations in the finite element context (see also Jurkiewiez *et al.*, 1999; Zocher *et al.*, 1997; Chazal and Poutou Pitti, 2010b, 2011a). In fracture viscoelastic mechanics, Kim and Lee (2007), Moutou Pitti *et al.* (2009, 2011), Chazal and Dubois (2001) applied the incremental formulation in order to evaluate creep crack growth process in wood. Krempl (1979) and Kujawski *et al.* (1980) performed an experimental study of creep and relaxation in steel at room temperature. However, the formulation used is based on the spectral decomposition using the generalized Maxwell model.

All the above procedures adopted in order to transform the original viscoelastic formulation into a new one with a differential or integral operator were based on the idea of local integration in the global operator of the problem with various techniques.

In this paper, a different approach is adopted taking into account that in the presence of a general viscoelastic constitutive law the behavior equations are first integrated over the thickness of the structure in terms of generalized variables. This new formulation is based on a discrete relaxation spectrum and the finite difference method using generalized integral equations in the time domain. The incremental stress and strain constitutive equations are not restricted to isotropic materials and can be used to resolve complex boundary viscoelastic problems without retaining the past history of the material in the memory of a digital computer.

In the following, a formal statement of the viscoelastic initial/boundary value problem is provided. The one dimensional linear viscoelastic behavior is used to account for three dimensional responses. After that, we present the development of generalized differential equations in terms of one dimensional stress and strain components. This is followed by a discussion of the conversion through incrementalization (essentially, a finite difference procedure) of the linear viscoelastic constitutive equations into a form suitable for implementation in a finite element formulation. Finally, the incremental viscoelastic constitutive equations of the model are established.

2. Problem statement

This section concentrates on the viscoelastic response of time dependent materials at isothermal deformation with small strains. According to Christensen (1971), Mandel (1978), Salençon (1983) and Chazal and Moutou Pitti (2010b, 2011a), the components of the relaxation tensor $\mathbf{J}(t)$ can be represented in terms of exponential series

$$J_{\alpha\beta\gamma\delta}(t) = \Big\{ J^{\infty}_{\alpha\beta\gamma\delta} + \sum_{m=1}^{M} J^{m}_{\alpha\beta\gamma\delta} \mathrm{e}^{-t\lambda^{m}_{\alpha\beta\gamma\delta}} \Big\} H(t) \tag{2.1}$$

where $\lambda^{m}_{\alpha\beta\gamma\delta}$, $m = 1, \ldots, M$, are strictly positive scalars and repeated indices do not imply summation convention. $J^{\infty}_{\alpha\beta\gamma\delta}$ and $J^{m}_{\alpha\beta\gamma\delta}$ represent the equilibrium and the differed part of the relaxation tensor respectively, and $H(t)$ is the Heaviside unit step function. According to Boltzmann's superposition principle in linear non-ageing viscoelasticity (Boltzmann, 1878), the constitutive equations between the components $\sigma_{\alpha\beta}(t)$ of the stress tensor and the components of the strain tensor $e_{\alpha\beta}(t)$ for non-ageing linear viscoelastic materials can be expressed in the time domain by hereditary Volterra's integral equation

$$\sigma_{\alpha\beta}(t) = \sum_{\gamma} \sum_{\delta} \int\limits_{-\infty}^{t} J_{\alpha\beta\gamma\delta}(t-\tau) \frac{\partial e_{\gamma\delta}(\tau)}{\partial \tau}\, d\tau \tag{2.2}$$

We introduce stresses and strains in generalized variables according to Love's first-order shell theory. The strain at any point of the thin structure may be given as

$$e_{\alpha\beta}(t) = \varepsilon_{\alpha\beta}(t) + \zeta \chi_{\alpha\beta}(t) \qquad \alpha, \beta = 1, 2 \tag{2.3}$$

where $\varepsilon_{\alpha\beta}(t)$ and $\chi_{\alpha\beta}(t)$ are the middle surface extensional strain and curvature, respectively. If we consider a plane stress state, the non-vanishing resultant of stresses is then defined by

$$N_{\alpha\beta}(t) = \int\limits_{-h/2}^{+h/2} \sigma_{\alpha\beta}(\zeta, t)\, d\zeta \qquad M_{\alpha\beta}(t) = \int\limits_{-h/2}^{+h/2} \zeta \sigma_{\alpha\beta}(\zeta, t)\, d\zeta \tag{2.4}$$

$N_{\alpha\beta}(t)$ and $M_{\alpha\beta}(t)$ are the generalized stresses and h is the thickness of the structure assumed to be constant. Note that the radii of the curvature for the middle surface do not enter into equation (2.4) because of the thin shell assumption. In order to determine the constitutive equation in terms of generalized stresses and strains, we introduce generalized strains, given by equation (2.3), into Volterra integral equation (2.2). One finds

$$\sigma^N_{\alpha\beta}(t) + \sigma^M_{\alpha\beta}(t) = \sum_\gamma \sum_\delta \int_{-\infty}^{t} J_{\alpha\beta\gamma\delta}(t-\tau) \frac{\partial}{\partial \tau}[\varepsilon_{\gamma\delta}(\tau) + \zeta \chi_{\gamma\delta}(\tau)]\, d\tau \tag{2.5}$$

Note that the total stress $\sigma_{\alpha\beta}(t)$ is separated into two parts: normal stress $\sigma^N_{\alpha\beta}(t)$ due to extensional strain and bending stress $\sigma^M_{\alpha\beta}(t)$ due to curvature. The constitutive equations in generalized variables can now be obtained from behavior equation (2.5).

Using equation (2.4) and integrating equation (2.5) over the thickness, we find

$$\begin{aligned} N_{\alpha\beta}(t) &= \int_{-h/2}^{+h/2} \sigma^N_{\alpha\beta}(\zeta, t)\, d\zeta = \sum_\gamma \sum_\delta h \int_{-\infty}^{t} J_{\alpha\beta\gamma\delta}(t-\tau) \frac{\partial}{\partial \tau} \varepsilon_{\gamma\delta}(\tau)\, d\tau \\ M_{\alpha\beta}(t) &= \int_{-h/2}^{+h/2} \zeta \sigma^M_{\alpha\beta}(\zeta, t)\, d\zeta = \sum_\gamma \sum_\delta \frac{h^3}{12} \int_{-\infty}^{t} J_{\varepsilon\beta\gamma\delta}(t-\tau) \frac{\partial}{\partial \tau} \chi_{\gamma\delta}(\tau)\, d\tau \end{aligned} \tag{2.6}$$

Let us consider the two pseudo fourth order tensors $\overline{\mathbf{N}}$ and $\overline{\mathbf{M}}$ of components $\overline{N}_{\alpha\beta\gamma\delta}(t)$ and $\overline{M}_{\alpha\beta\gamma\delta}(t)$ respectively. These tensors are defined by

$$\begin{Bmatrix} a_1 \overline{N}_{\alpha\beta\gamma\delta}(t) \\ a_2 \overline{M}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} = \int_{-\infty}^{t} J_{\alpha\beta\gamma\delta}(t-\tau) \frac{\partial}{\partial \tau} \begin{Bmatrix} \varepsilon_{\gamma\delta}(\tau) \\ \chi_{\gamma\delta}(\tau) \end{Bmatrix} \tag{2.7}$$

a_1 and a_2 are geometric constants and are defined by: $a_1 = 1/h$, and $a_2 = 12/h^3$. $\overline{N}_{\alpha\beta\gamma\delta}(t)$ and $\overline{M}_{\alpha\beta\gamma\delta}(t)$ are pseudo mono-dimensional stresses obtained from Volterra's integral equation as given by equation (2.7). These components can be interpreted as the contribution of the strain history $\{e_{\gamma\delta}(\tau), \tau \leqslant t\}$ of the components $e_{\gamma\delta}(t)$ of the strain tensor to the stress components $\sigma_{\alpha\beta}(t)$. Introducing equation (2.7) into equations (2.6), it can be shown that the pseudo mono-dimensional stress components satisfy the following equations

$$\begin{Bmatrix} N_{\alpha\beta}(t) \\ M_{\alpha\beta}(t) \end{Bmatrix} = \sum_{\gamma=1}^{3}\sum_{\delta=1}^{3} \begin{Bmatrix} \overline{N}_{\alpha\beta\gamma\delta}(t) \\ \overline{M}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} = \sum_{\gamma=1}^{3}\sum_{\delta=1}^{3} \int\limits_{-\infty}^{t} J_{\alpha\beta\gamma\delta}(t-\tau)\frac{\partial}{\partial\tau} \begin{Bmatrix} \dfrac{1}{a_1}\varepsilon_{\gamma\delta}(\tau) \\ \dfrac{1}{a_2}\chi_{\gamma\delta}(\tau) \end{Bmatrix} \tag{2.8}$$

Each equation of relation (2.8) represents a one-dimensional non-ageing linear viscoelastic material defined by its relaxation function $J(t)$ given by equation (2.1).

3. Analysis of the proposed model

When we apply the mechanical strain defined by the strain history $\{e_{\alpha\beta}(\tau), \tau \in \Re\}$, the response of the material is then given by the history of stresses $\{\overline{N}_{\alpha\beta\gamma\delta}(t), \overline{M}_{\alpha\beta\gamma\delta}(t), t \in \Re\}$ defined by the behavior equation (2.7) in which the relaxation function is given by equation (2.1).

If the generalized strain $\{\varepsilon_{\gamma\delta}(t), \chi_{\gamma\delta}(t)\}$ is applied to the material at time t, then the response in stresses can be obtained using the finite relaxation spectrum representation given by equation (2.1). This leads to

$$\begin{Bmatrix} a_1\overline{N}_{\alpha\beta\gamma\delta}(t) \\ a_2\overline{M}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} = \int\limits_{-\infty}^{t} \Big[J_{\alpha\beta\gamma\delta}^{\infty} + \sum_{m=1}^{M} J_{\alpha\beta\gamma\delta}^{m} \mathrm{e}^{-\lambda_{\alpha\beta\gamma\delta}^{m}(t-\tau)}\Big]\frac{\partial}{\partial\tau} \begin{Bmatrix} \varepsilon_{\gamma\delta}(\tau) \\ \chi_{\gamma\delta}(\tau) \end{Bmatrix} \tag{3.1}$$

Thus the pseudo mono-dimensional stresses given by the last equation, and written as a function of equilibrium and a differed part of the relaxation spectrum, can be rewritten in the following form

$$\begin{Bmatrix} \overline{N}_{\alpha\beta\gamma\delta}(t) \\ \overline{M}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} = \begin{Bmatrix} \overline{N}_{\alpha\beta\gamma\delta}^{\infty}(t) + \sum_{m=1}^{M} \overline{N}_{\alpha\beta\gamma\delta}^{m}(t) \\ \overline{M}_{\alpha\beta\gamma\delta}^{\infty}(t) + \sum_{m=1}^{M} \overline{M}_{\alpha\beta\gamma\delta}^{m}(t) \end{Bmatrix} \tag{3.2}$$

with

$$\begin{aligned}
\overline{N}_{\alpha\beta\gamma\delta}^{\infty}(t) &= \frac{1}{a_1}\int\limits_{-\infty}^{t} J_{\alpha\beta\gamma\delta}^{\infty}\frac{\partial\varepsilon_{\gamma\delta}(\tau)}{\partial\tau}\,d\tau = \frac{1}{a_1}J_{\alpha\beta\gamma\delta}^{\infty}\varepsilon_{\gamma\delta}(t) \\
\overline{N}_{\alpha\beta\gamma\delta}^{m}(t) &= \frac{1}{a_1}\int\limits_{-\infty}^{t} J_{\alpha\beta\gamma\delta}^{m}\mathrm{e}^{-\lambda_{\alpha\beta\gamma\delta}^{m}(t-\tau)}\frac{\partial\varepsilon_{\gamma\delta}(\tau)}{\partial\tau}\,d\tau \\
\overline{M}_{\alpha\beta\gamma\delta}^{\infty}(t) &= \frac{1}{a_2}\int\limits_{-\infty}^{t} J_{\alpha\beta\gamma\delta}^{\infty}\frac{\partial\chi_{\gamma\delta}(\tau)}{\partial\tau}\,d\tau = \frac{1}{a_2}J_{\alpha\beta\gamma\delta}^{\infty}\chi_{\gamma\delta}(t)
\end{aligned} \tag{3.3}$$

$$\overline{M}^{m}_{\alpha\beta\gamma\delta}(t) = \frac{1}{a_2}\int\limits_{-\infty}^{t} J^{m}_{\alpha\beta\gamma\delta}\mathrm{e}^{-\lambda^{m}_{\alpha\beta\gamma\delta}(t-\tau)}\frac{\partial\chi_{\gamma\delta}(\tau)}{\partial\tau}\,d\tau$$

It should be noted that $\overline{N}^{\infty}_{\alpha\beta\gamma\delta}(t)$ and $\overline{M}^{\infty}_{\alpha\beta\gamma\delta}(t)$ represent the equilibrium part of the pseudo mono-dimensional stress of the material while $\overline{N}^{m}_{\alpha\beta\gamma\delta}(t)$ and $\overline{M}^{m}_{\alpha\beta\gamma\delta}(t)$ represent the differed part of the same pseudo mono-dimensional stress.

As we mentioned in the above section, a differential approach is used in order to establish the differential equations of the mechanical model. Thus, we need to express the viscoelastic response of the material as a function of stress and strain derivatives. For this reason, let us use equation (2.8), the rate of the total stress is determined by

$$\begin{aligned}\frac{\partial}{\partial t}\begin{Bmatrix} N_{\alpha\beta}(t) \\ M_{\alpha\beta}(t)\end{Bmatrix} &= \sum_{\gamma=1}^{3}\sum_{\delta=1}^{3}\frac{\partial}{\partial t}\begin{Bmatrix} \overline{N}_{\alpha\beta\gamma\delta}(t) \\ \overline{M}_{\alpha\beta\gamma\delta}(t)\end{Bmatrix} \\ &= \frac{\partial}{\partial t}\sum_{\gamma=1}^{3}\sum_{\delta=1}^{3}\begin{Bmatrix} \overline{N}^{\infty}_{\alpha\beta\gamma\delta}(t) + \sum\limits_{m=1}^{M}\overline{N}^{m}_{\alpha\beta\gamma\delta}(t) \\ \overline{M}^{\infty}_{\alpha\beta\gamma\delta}(t) + \sum\limits_{m=1}^{M}\overline{M}^{m}_{\alpha\beta\gamma\delta}(t)\end{Bmatrix}\end{aligned} \tag{3.4}$$

The rate of the equilibrium part of the pseudo one-dimensional stress $\overline{N}^{\infty}_{\alpha\beta\gamma\delta}(t)$ and $\overline{M}^{\infty}_{\alpha\beta\gamma\delta}(t)$ is easy to be evaluated. According to equations $(3.3)_1$ and $(3.3)_3$, and after applying a time derivative operator, one find

$$a_1\frac{\partial\overline{N}^{\infty}_{\alpha\beta\gamma\delta}(t)}{\partial t} = J^{\infty}_{\alpha\beta\gamma\delta}\frac{\partial\varepsilon_{\gamma\delta}(t)}{\partial t} \qquad a_2\frac{\partial\overline{M}^{\infty}_{\alpha\beta\gamma\delta}(t)}{\partial t} = J^{\infty}_{\alpha\beta\gamma\delta}\frac{\partial\chi_{\gamma\delta}(t)}{\partial t} \tag{3.5}$$

In other words, the equilibrium part of the pseudo one-dimensional stress is directly proportional to the total strain at time t. It is completely defined by the history of the applied strain. However, the rate of the differed part of the pseudo one-dimensional stresses $\overline{N}^{m}_{\alpha\beta\gamma\delta}(t)$ and $\overline{M}^{m}_{\alpha\beta\gamma\delta}(t)$ is more difficult to be determined. Using equations $(3.3)_2$ and $(3.3)_4$, and applying a time derivative operator, we can write

$$\begin{aligned}a_1\frac{\partial}{\partial t}\overline{N}^{m}_{\alpha\beta\gamma\delta}(t) &= J^{m}_{\alpha\beta\gamma\delta}\mathrm{e}^{-\lambda^{m}_{\alpha\beta\gamma\delta}(t-t)}\frac{\partial}{\partial t}\varepsilon_{\gamma\delta}(t) \\ &\quad - \int\limits_{-\infty}^{t} J^{m}_{\alpha\beta\gamma\delta}\lambda^{m}_{\alpha\beta\gamma\delta}\mathrm{e}^{-\lambda^{m}_{\alpha\beta\gamma\delta}(t-\tau)}\frac{\partial}{\partial\tau}\varepsilon_{\gamma\delta}(\tau)\end{aligned} \tag{3.6}$$

$$a_2 \frac{\partial}{\partial t} \overline{M}^m_{\alpha\beta\gamma\delta}(t) = J^m_{\alpha\beta\gamma\delta} \mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}(t-t)} \frac{\partial}{\partial t} \chi_{\gamma\delta}(t) - \int\limits_{-\infty}^{t} J^m_{\alpha\beta\gamma\delta} \lambda^m_{\alpha\beta\gamma\delta} \mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}(t-\tau)} \frac{\partial}{\partial \tau} \chi_{\gamma\delta}(\tau)$$

These integral equations give the total rate of the differed part of the pseudo mono-dimensional stresses.

The main purpose of our development in this section is to establish differential equations between the total rate of the pseudo mono-dimensional stresses and the rate of the total strain. For this reason, we will transform the last equations in a differential type.

Let us introduce behavior equations $(3.3)_2$ and $(3.3)_4$ in integral equations (3.6). This leads to linear differential equations with constant coefficients and can be integrated analytically

$$\begin{aligned} \frac{\partial}{\partial t} \overline{N}^m_{\alpha\beta\gamma\delta}(t) + \lambda^m_{\alpha\beta\gamma\delta} \overline{N}^m_{\alpha\beta\gamma\delta}(t) &= \frac{1}{a_1} J^m_{\alpha\beta\gamma\delta} \frac{\partial}{\partial t} \varepsilon_{\gamma\delta}(t) \\ \frac{\partial}{\partial t} \overline{M}^m_{\alpha\beta\gamma\delta}(t) + \lambda^m_{\alpha\beta\gamma\delta} \overline{M}^m_{\alpha\beta\gamma\delta}(t) &= \frac{1}{a_2} J^m_{\alpha\beta\gamma\delta} \frac{\partial}{\partial t} \varepsilon_{\gamma\delta}(t) \end{aligned} \tag{3.7}$$

The solution to these linear differential equations gives the rate of the pseudo one-dimensional stresses $\overline{N}^m_{\alpha\beta\gamma\delta}(t)$ and $\overline{M}^m_{\alpha\beta\gamma\delta}(t)$.

Finally, the general differential equations governing the non-ageing linear viscoelastic behavior can be obtained from equation (3.4) after summation on γ and δ indices. One finds

$$\frac{\partial}{\partial t} \begin{Bmatrix} N_{\alpha\beta}(t) \\ M_{\alpha\beta}(t) \end{Bmatrix} = \sum_{m=1}^{M} \frac{\partial}{\partial t} \begin{Bmatrix} \varGamma^m_{\alpha\beta}(t) \\ \varPsi^m_{\alpha\beta}(t) \end{Bmatrix} + \sum_{\gamma=1}^{3} \sum_{\delta=1}^{3} J^\infty_{\alpha\beta\gamma\delta} \frac{\partial}{\partial t} \begin{Bmatrix} \dfrac{1}{a_1} \varepsilon_{\gamma\delta}(t) \\ \dfrac{1}{a_2} \chi_{\gamma\delta}(t) \end{Bmatrix} \tag{3.8}$$

where $\varGamma^m_{\alpha\beta}(t)$ and $\varPsi^m_{\alpha\beta}(t)$, $\alpha, \beta \in \{1,2,3\}$, $m \in \{1,\dots,M\}$ are the solutions to the following equations

$$\begin{Bmatrix} \varGamma^m_{\alpha\beta}(t) \\ \varPsi^m_{\alpha\beta}(t) \end{Bmatrix} = \sum_{\gamma=1}^{3} \sum_{\delta=1}^{3} \begin{Bmatrix} \overline{N}^m_{\alpha\beta\gamma\delta}(t) \\ \overline{M}^m_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} \tag{3.9}$$

Note that $\varGamma^m_{\alpha\beta}(t)$ and $\varPsi^m_{\alpha\beta}(t)$ can be interpreted as pseudo stresses and represent the influence of the past history of strain on the material behavior. They are given by the solution to linear differential equations (3.7). It also should be mentioned that the non-ageing linear viscoelastic behavior is completely defined by differential equations (3.8). We note that this formulation, written in terms of generalized stresses and strains rates, is easily adapted to temporal discretization methods such as the finite difference method.

4. Conversion to incremental equations

Here we will describe the solution process of a step-by-step nature in which loads are applied stepwise at various time intervals. Let us consider the time step $\Delta t_n = t_{n+1} - t_n$. The subscript n and $n+1$ refer to the values at the beginning and end of the time step, respectively. This technique is successfully used by Chazal and Dubois (2001) in the case of viscoelastic structures. We assume that the time derivative during each time increment is constant and is expressed by

$$\frac{\partial \zeta_{ij}}{\partial t} = \frac{\zeta_{ij}(t_{n+1}) - \zeta_{ij}(t_n)}{\Delta t_n} = \frac{\Delta(\zeta_{ij})_n}{\Delta t_n} \tag{4.1}$$

where ζ_{ij} represents generalized strains or stresses. The following expressions can then be written for the rate of pseudo stresses at the beginning of the time step

$$\frac{\partial}{\partial t}\begin{Bmatrix} \Gamma^m_{\alpha\beta}(t_n) \\ \Psi^m_{\alpha\beta}(t_n) \end{Bmatrix} = \frac{1}{\Delta t_n}\begin{Bmatrix} \Gamma^m_{\alpha\beta}(t_{n+1}) - \Gamma^m_{\alpha\beta}(t_n) \\ \Psi^m_{\alpha\beta}(t_{n+1}) - \Psi^m_{\alpha\beta}(t_n) \end{Bmatrix} = \frac{1}{\Delta t_n}\begin{Bmatrix} \Delta\Gamma^m_{\alpha\beta}(t_n) \\ \Delta\Psi^m_{\alpha\beta}(t_n) \end{Bmatrix} \tag{4.2}$$

A linear approximation is used for strains, and is expressed by

$$\begin{Bmatrix} \varepsilon_{\gamma\delta}(\tau) \\ \chi_{\gamma\delta}(\tau) \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{\gamma\delta}(t_n) \\ \chi_{\gamma\delta}(t_n) \end{Bmatrix} + \frac{\tau - t_n}{\Delta t_n} H(\tau - t_n)\begin{Bmatrix} \varepsilon_{\gamma\delta}(t_{n+1}) - \varepsilon_{\gamma\delta}(t_n) \\ \chi_{\gamma\delta}(t_{n+1}) - \chi_{\gamma\delta}(t_n) \end{Bmatrix} \tag{4.3}$$

This linear approximation leads to very accurate results in finite element discretization as it is shown by Chazal and Dubois (2001). Thus we do not need higher approximations for the strain during a finite increment of the time load. This leads to a constant rate during each time increment:

$$\frac{\partial}{\partial t}\begin{Bmatrix} \varepsilon_{\alpha\beta}(t_n) \\ \chi_{\alpha\beta}(t_n) \end{Bmatrix} = \frac{1}{\Delta t_n}\begin{Bmatrix} \varepsilon_{\alpha\beta}(t_{n+1}) - \varepsilon_{\alpha\beta}(t_n) \\ \chi_{\alpha\beta}(t_{n+1}) - \chi_{\alpha\beta}(t_n) \end{Bmatrix} = \frac{1}{\Delta t_n}\begin{Bmatrix} \Delta\varepsilon_{\alpha\beta}(t_n) \\ \Delta\chi_{\alpha\beta}(t_n) \end{Bmatrix} \tag{4.4}$$

By integrating equation (3.8) between t_n and t_{n+1}, it can be written in the following form

$$\begin{Bmatrix} \Delta N_{\alpha\beta}(t_n) \\ \Delta M_{\alpha\beta}(t_n) \end{Bmatrix} = \sum_{m=1}^{M}\begin{Bmatrix} \Delta\Gamma^m_{\alpha\beta}(t_n) \\ \Delta\Psi^m_{\alpha\beta}(t_n) \end{Bmatrix} + \sum_{\gamma=1}^{3}\sum_{\delta=1}^{3} J^{\infty}_{\alpha\beta\gamma\delta}\begin{Bmatrix} \dfrac{1}{a_1}\Delta\varepsilon_{\gamma\delta}(t_n) \\ \dfrac{1}{a_2}\Delta\chi_{\gamma\delta}(t_n) \end{Bmatrix} \tag{4.5}$$

In order to determine the generalized stress increments from this equation, we have to determine the generalized pseudo stress increments $\Delta\Gamma^m_{\alpha\beta}(t_n)$ and

$\Delta\Psi^m_{\alpha\beta}(t_n)$ during the time step Δt_n. First, let us consider differential equation (3.7). The analytical solution to this differential equation can be expressed as

$$
\begin{aligned}
&\overline{N}^m_{\alpha\beta\gamma\delta}(t_{n+1}) - \overline{N}^m_{\alpha\beta\gamma\delta}(t_n) = \left(\mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n} - 1\right)\overline{N}^m_{\alpha\beta\gamma\delta}(t_n) \\
&\qquad + \frac{1}{a_1} J^m_{\alpha\beta\gamma\delta} \frac{\Delta\varepsilon_{\gamma\delta}(t_n)}{\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\left(1 - \mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\right) \\
&\overline{M}^m_{\alpha\beta\gamma\delta}(t_{n+1}) - \overline{M}^m_{\alpha\beta\gamma\delta}(t_n) = \left(\mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n} - 1\right)\overline{M}^m_{\alpha\beta\gamma\delta}(t_n) \\
&\qquad + \frac{1}{a_2} J^m_{\alpha\beta\gamma\delta} \frac{\Delta\chi_{\gamma\delta}(t_n)}{\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\left(1 - \mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\right)
\end{aligned}
\tag{4.6}
$$

Consequently, when we substitute equations (4.6) into equation (3.9), we obtain the generalized pseudo stress increments $\Delta\Gamma^m_{\alpha\beta}(t_n)$ and $\Delta\Psi^m_{\alpha\beta}(t_n)$

$$
\left\{\begin{array}{c} \displaystyle\sum_{m=1}^{M} \Delta\Gamma^m_{\alpha\beta}(t_n) \\ \displaystyle\sum_{m=1}^{M} \Delta\Psi^m_{\alpha\beta}(t_n) \end{array}\right\} = \sum_{m=1}^{M}\sum_{\gamma=1}^{3}\sum_{\delta=1}^{3} \left\{\begin{array}{c} \overline{N}^m_{\alpha\beta\gamma\delta}(t_{n+1}) - \overline{N}^m_{\alpha\beta\gamma\delta}(t_n) \\ \overline{M}^m_{\alpha\beta\gamma\delta}(t_{n+1}) - \overline{M}^m_{\alpha\beta\gamma\delta}(t_n) \end{array}\right\}
\tag{4.7}
$$

or

$$
\begin{aligned}
&\sum_{m=1}^{M} \Delta\Gamma^m_{\alpha\beta}(t_n) = \sum_{m=1}^{M}\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \left(\mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n} - 1\right)\overline{N}^m_{\alpha\beta\gamma\delta}(t_n) \\
&\qquad + \frac{1}{a_1} \frac{J^m_{\alpha\beta\gamma\delta}\Delta\varepsilon_{\gamma\delta}(t_n)}{\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\left(1 - \mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\right) \\
&\sum_{m=1}^{M} \Delta\Psi^m_{\alpha\beta}(t_n) = \sum_{m=1}^{M}\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \left(\mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n} - 1\right)\overline{M}^m_{\alpha\beta\gamma\delta}(t_n) \\
&\qquad + \frac{1}{a_2} \frac{J^m_{\alpha\beta\gamma\delta}\Delta\varepsilon_{\gamma\delta}(t_n)}{\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\left(1 - \mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\right)
\end{aligned}
\tag{4.8}
$$

The incremental constitutive equations can now be obtained from constitutive equation (4.5). Substituting equations (4.8) into (4.5), we find

$$
\left\{\begin{array}{c} \Delta N_{\alpha\beta}(t_n) \\ \Delta M_{\alpha\beta}(t_n) \end{array}\right\} = \sum_{\gamma=1}^{3}\sum_{\delta=1}^{3} \begin{bmatrix} \Pi_{\alpha\beta\gamma\delta}(t_n) & 0 \\ 0 & \Xi_{\alpha\beta\gamma\delta}(t_n) \end{bmatrix} \left\{\begin{array}{c} \Delta\varepsilon_{\gamma\delta}(t_n) \\ \Delta\chi_{\gamma\delta}(t_n) \end{array}\right\} - \left\{\begin{array}{c} \widetilde{N}_{\alpha\beta}(t_n) \\ \widetilde{M}_{\alpha\beta}(t_n) \end{array}\right\}
\tag{4.9}
$$

where $\Pi_{\alpha\beta\gamma\delta}(t_n)$ and $\Xi_{\alpha\beta\gamma\delta}(t_n)$ are fourth-order tensors which can be interpreted as rigidity tensors in the extensional and bending state respectively, they are given by

$$\begin{aligned}\Pi_{\alpha\beta\gamma\delta}(t_n) &= J^{\infty}_{\alpha\beta\gamma\delta} + \frac{1}{a_1}\sum_{m=1}^{M} \frac{J^m_{\alpha\beta\gamma\delta}\Delta\varepsilon_{\gamma\delta}(t_n)}{\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\Big(1 - \mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\Big)\\ \Xi_{\alpha\beta\gamma\delta}(t_n) &= J^{\infty}_{\alpha\beta\gamma\delta} + \frac{1}{a_2}\sum_{m=1}^{M} \frac{J^m_{\alpha\beta\gamma\delta}\Delta\chi_{\gamma\delta}(t_n)}{\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\Big(1 - \mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\Big)\end{aligned} \tag{4.10}$$

$\widetilde{N}_{\alpha\beta}(t_n)$ and $\widetilde{M}_{\alpha\beta}(t_n)$ are pseudo generalized stresses which represent the influence of the complete past history of extensional and bending generalized stresses. They are given by

$$\left\{\begin{matrix}\widetilde{N}_{\alpha\beta}(t_n)\\ \widetilde{M}_{\alpha\beta}(t_n)\end{matrix}\right\} = \sum_{\gamma=1}^{3}\sum_{\delta=1}^{3}\sum_{m=1}^{M}\Big(1 - \mathrm{e}^{-\lambda^m_{\alpha\beta\gamma\delta}\Delta t_n}\Big)\left\{\begin{matrix}\overline{N}^m_{\alpha\beta\gamma\delta}(t_n)\\ \overline{M}^m_{\alpha\beta\gamma\delta}(t_n)\end{matrix}\right\} \tag{4.11}$$

Finally, the incremental constitutive law given by equation (4.9) can now be inverted to obtain

$$\left\{\begin{matrix}\Delta\varepsilon_{\gamma\delta}(t_n)\\ \Delta\chi_{\gamma\delta}(t_n)\end{matrix}\right\} = \sum_{\gamma=1}^{3}\sum_{\delta=1}^{3}\begin{bmatrix}\Theta_{\alpha\beta\gamma\delta}(t_n) & 0\\ 0 & \Lambda_{\alpha\beta\gamma\delta}(t_n)\end{bmatrix}\left\{\begin{matrix}\Delta N_{\gamma\delta}(t_n)\\ \Delta M_{\gamma\delta}(t_n)\end{matrix}\right\} + \left\{\begin{matrix}\widetilde{\varepsilon}_{\alpha\beta}(t_n)\\ \widetilde{\chi}_{\alpha\beta}(t_n)\end{matrix}\right\} \tag{4.12}$$

where $\Theta_{\alpha\beta\gamma\delta}(t_n)$ and $\Lambda_{\alpha\beta\gamma\delta}(t_n)$ are compliance fourth-order tensors corresponding to extensional and bending state of deformation respectively, they are given by the inverse of the rigidity matrix

$$\begin{bmatrix}\Theta_{\alpha\beta\gamma\delta}(t_n) & 0\\ 0 & \Lambda_{\alpha\beta\gamma\delta}(t_n)\end{bmatrix} = \begin{bmatrix}\Pi_{\alpha\beta\gamma\delta}(t_n) & 0\\ 0 & \Xi_{\alpha\beta\gamma\delta}(t_n)\end{bmatrix}^{-1} \tag{4.13}$$

$\widetilde{\varepsilon}_{\alpha\beta}(t_n)$ and $\widetilde{\chi}_{\alpha\beta}(t_n)$ are pseudo strains tensors which represent the influence of the complete past history of strain. They are given by

$$\left\{\begin{matrix}\widetilde{\varepsilon}_{\alpha\beta}(t_n)\\ \widetilde{\chi}_{\alpha\beta}(t_n)\end{matrix}\right\} = \sum_{\gamma=1}^{3}\sum_{\delta=1}^{3}\begin{bmatrix}\Theta_{\alpha\beta\gamma\delta}(t_n) & 0\\ 0 & \Lambda_{\alpha\beta\gamma\delta}(t_n)\end{bmatrix}\left\{\begin{matrix}\widetilde{N}_{\gamma\delta}(t_n)\\ \widetilde{M}_{\gamma\delta}(t_n)\end{matrix}\right\} \tag{4.14}$$

The incremental constitutive law represented by equation (4.9) can be introduced in a finite element discretization in order to obtain solutions to complex viscoelastic problems.

Finally, in order to use the incremental viscoelastic formulation presented in this paper, we need to identify the relaxation components of the relaxation tensor. The experimental identification of viscoelastic properties is treated in details by Jäger and Lackner (2007) and Müllner and Jäger (2008). The viscoelastic solution is obtained by the application of the method of functional equations to the elastic solution to the indentation problem and by means of torsional rheometry.

5. Finite element discretization

The governing equations of the discretized system, using the finite element model, are derived from the principle of virtual displacements. Let us consider a linear quasi-static non-ageing viscoelastic structure. The principle of virtual displacements implies that the increment in external virtual work is equal to the increment in internal virtual work

$$\int\limits_{{}^eA} \langle \delta_t \varDelta\varepsilon_{\alpha\beta})_n, \delta_t \varDelta(\chi_{\alpha\beta})_n \rangle \begin{Bmatrix} \varDelta(N_{\alpha\beta})_n \\ \varDelta(M_{\alpha\beta})_n \end{Bmatrix} d^eA = \int\limits_{{}^eV} \varDelta(f_i^v)_n \delta_t \varDelta(u_i)_n \, d^eV \tag{5.1}$$

where $\varDelta(u_i)_n$ is the incremental displacement field between t_n and t_{n+1}, $\varDelta(f_i^v)_n$ is the incremental body forces per unit volume, ${}^eA$ and ${}^eV$ are the area and the volume of the element, and δ_t is the variation symbol. For the sake of simplicity, the surface traction term is omitted in the last equation. Assuming small displacements, strains are derived from shape functions using a standard manner in the context of the finite element method. Using a matrix notation, the strain increment can be written as

$$\begin{Bmatrix} \varDelta\varepsilon_{\alpha\beta})_n \\ \varDelta(\chi_{\alpha\beta})_n \end{Bmatrix} = [B_L]\{\varDelta(U^e)_n\} \tag{5.2}$$

where $\varDelta(U^e)_n$ is the local element displacement increment and $[B_L]$ is the strain-displacement transformation matrix. Introducing incremental viscoelastic constitutive equations (4.9) into equilibrium equations (5.1) and using finite element approximation (5.2), the equilibrium equations for linear viscoelastic behavior can be rewritten as

$$[K_T]_n\{\varDelta(U^e)\}_n = \{\varDelta F^{ext}\}_n + \{\varDelta F^{vis}\}_n \tag{5.3}$$

where

$$[K_T]_n = \int\limits_{{}^eA} [B_L]^\top [\varOmega_n][B_L] \, d^eA \tag{5.4}$$

and

$$\{\varDelta F^{vis}\}_n = \int\limits_{^eV} [B_L]^\top \left\{ \begin{array}{c} \dfrac{1}{a_1}(\widetilde{N}_{\alpha\beta})_n \\ \dfrac{1}{a_2}(\widetilde{M}_{\alpha\beta})_n \end{array} \right\} d^eA \tag{5.5}$$

$\{\varDelta F^{ext}\}_n = \{F^{ext}\}_{n+1} - \{F^{ext}\}_n$ is the external load vector increment, $\{\varDelta F^{vis}\}_n$ is the viscous load vector increment corresponding to the complete past history and $[\varOmega_n]$ is the viscoelastic constitutive matrix.

The formulation is introduced in the software Cast3m used by the French Energy Atomic Agency. The software can be employed for linear viscoelasticity structures using triangular elements. The global incremental procedure for the relaxation differential approach is described as:

1. At time t_n, compute the tangent moduli $\varPi_{\alpha\beta\gamma\delta}(t_n)$ and $\varXi_{\alpha\beta\gamma\delta}(t_n)$ from equations (4.10)
2. Compute the viscoelastic constitutive matrix $[\varOmega_n]$

$$[\varOmega_n] = \begin{bmatrix} \varPi_{\alpha\beta\gamma\delta}(t_n) & 0 \\ 0 & \varXi_{\alpha\beta\gamma\delta}(t_n) \end{bmatrix}$$

3. Compute the pseudo generalized stresses $\{\widetilde{N}_{\alpha\beta}(t_n)\}$ and $\{\widetilde{M}_{\alpha\beta}(t_n)\}$ from equation (4.11)
4. Determine the increment of the viscous load vector $\{\varDelta F^{vis}\}_n$ from equation (5.5)
5. Update the viscoelastic stiffness matrix $[K_T]_n$ from equation (5.4)
6. Assemble and solve viscoelastic equilibrium equations (5.3) in order to obtain the displacement increment vector $\{\varDelta(U)\}_n$
7. Compute the generalized strain increment $\{\varDelta\varepsilon_{\alpha\beta}(t_n)\}$ and $\{\varDelta\chi_{\alpha\beta}(t_n)\}$ from equation (5.2)
8. Compute the generalized stress increment $\{\varDelta N_{\alpha\beta}(t_n)\}$ and $\{\varDelta M_{\alpha\beta}(t_n)\}$ from equation (4.9)
9. Using the results of step 3, compute the generalized pseudo-strains $\{\widetilde{\varepsilon}_{\alpha\beta}(t_{n+1})\}$ and $\{\widetilde{\chi}_{\alpha\beta}(t_{n+1})\}$ from equation (4.14)
10. Update the state

$$\begin{aligned}
&\{U\}_{n+1} = \{U\}_n + \{\varDelta(U)\}_n \\
&\{N_{\alpha\beta}\}_{n+1} = \{N_{\alpha\beta}\}_n + \{\varDelta(N_{\alpha\beta})\}_n \\
&\{M_{\alpha\beta}\}_{n+1} = \{M_{\alpha\beta}\}_n + \{\varDelta(M_{\alpha\beta})\}_n \\
&\{\varepsilon_{\alpha\beta}\}_{n+1} = \{\varepsilon_{\alpha\beta}\}_n + \{\varDelta(\varepsilon_{\alpha\beta})\}_n \\
&\{\chi_{\alpha\beta}\}_{n+1} = \{\chi_{\alpha\beta}\}_n + \{\varDelta(\chi_{\alpha\beta})\}_n
\end{aligned}$$

11. Go to step 1

6. Numerical example

This example will be illustrated by a viscoelastic circular cylindrical shell fixed at one end and loaded at the free end. The applied load is a unit radial loading at the free end while the other end is built on. The geometrical and loading details are given in Fig. 1. It should be noted that the mesh of the cylinder is graded so that there are more elements near the loaded point, since in this region the stresses and deflections change most rapidly. The material properties used for the cylinder are given in Table 1.

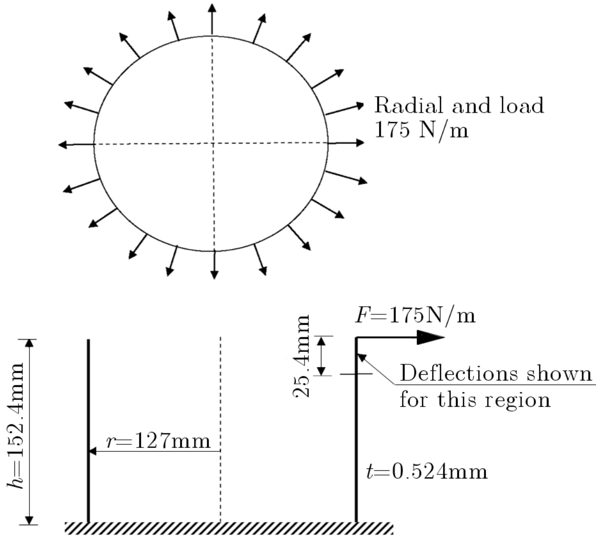

Fig. 1. Circular cylindrical shell using axisymetric shell elements

Table 1. Constants used for material properties

J_0	J_1	J_2	λ_1	λ_2
$1.45 \cdot 10^{-5}$	$20 \cdot 10^{-5}$	$33.33 \cdot 10^{-5}$	0.001	0.01

The results of the numerical process are shown in Figs. 2-5. In Figure 2, the numerical results for the radial displacement are plotted versus the axial position measured from the free end, while in Fig. 3, the meridional moment is plotted for 20-element idealization. Both the radial displacement and the meridional moment are compared with the theoretical solution given in Timoshenko *et al.* (1959). A very good agreement with the theoretical results can be observed.

The results of the viscoelastic analysis are given in Figs. 4 and 5. Figure 4 shows how the radial deflection varies versus time, while in Fig. 5, we plotted the variation of the meridional strain versus time. It can be shown that strains keep on building up leading to the strain failure.

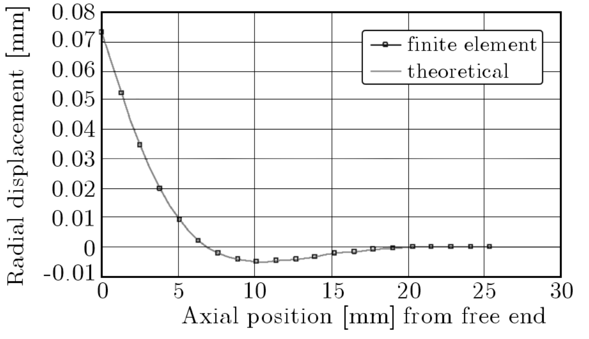

Fig. 2. Variation in the radial deflection of the shell undergoing radial load

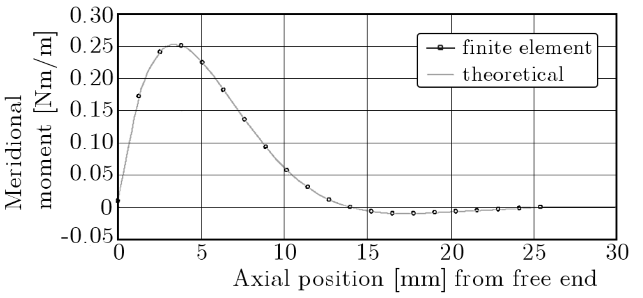

Fig. 3. Variation in the meridional moment in the shell undergoing radial end load

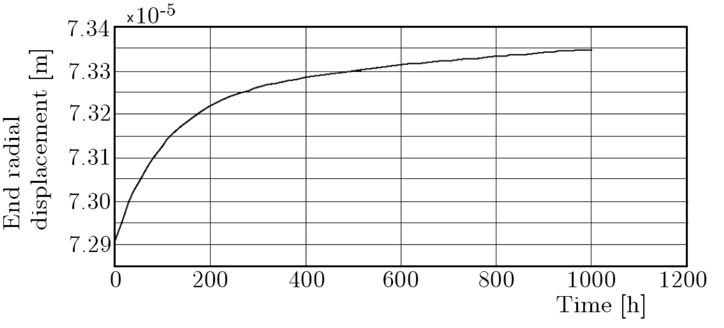

Fig. 4. Free end radial displacements versus time in the shell with radial load applied

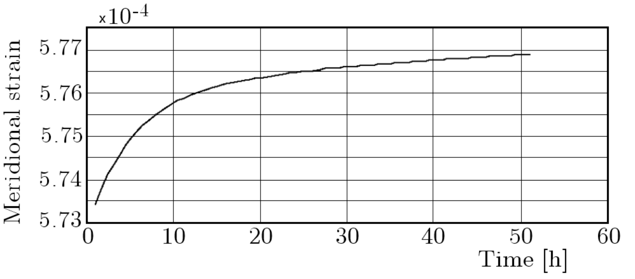

Fig. 5. Free end meridional strains versus time in the shell loaded radially

7. Conclusions

The transformation in differential terms of the integral formulation of the viscoelastic continuum problem has been successfully achieved through the introduction of a discrete spectrum representation of the relaxation tensor. This leads to a new linear incremental formulation in the time domain for non-ageing viscoelastic materials undergoing mechanical deformation. The formulation is based on a differential approach using the discrete spectrum representation for the relaxation components. The governing equations are then obtained using a discretized form of the Boltzmann superposition principle (Boltzmann, 1878). The analytical solution to the differential equations is obtained using finite difference discretization in the time domain. In this way, the incremental constitutive equations of the linear viscoelastic material use a pseudo fourth order rigidity tensor and the influence of the whole past history on the behavior of the material at time t is given by a pseudo second order tensor. The generality allowed by this approach has been established by finding an incremental formulation through simple choices of the tensor relaxation components. This approach appears to open a wide horizon(to explore) of new incremental formulations according to particular relaxation components. Remarkable incremental constitutive laws, for which the above technique is applied, are given.

Among the numerous applications of the incremental formulations presented in this paper, there is numerical implementation in finite element software, thus the behavior of complex boundary viscoelastic problems can be obtained.

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Teoretyczna i numeryczna analiza lepko-sprężystych właściwości materiałów za pomocą różniczkowej metody relaksacji opartej na zmiennych uogólnionych

Streszczenie

W pracy zbadano zagadnienie inkrementalizacji czasowej zjawisk zachodzących w liniowych, niestarzejących się materiałach lepko-sprężystych poddanych mechanicznej deformacji. Zastosowano metody analityczne do określenia lepko-sprężystego zachowania się materiału w dwuwymiarowej przestrzeni, używając zmiennych uogólnionych i realistycznych parametrów określających właściwości próbki. Badania przeprowadzono poprzez zdefiniowanie tych właściwości w postaci szeregu Dirichleta umożliwiającego transformację całkowej reprezentacji badanego kontinuum w formę różniczkową. Równania stanu zaczerpnięto z liniowego modelu opisanego równaniami różniczkowymi bazującymi na zdyskretyzowanym widmie relaksacji. Pozwoliło to uzyskać konstytutywne wyrażenia przyrostowe poprzez całkowanie różnic skończonych, co z kolei wyeliminowało konieczność zachowywania historii odkształceń w pamięci komputera. Pełna analiza przebiegu deformacji liniowo lepko-sprężystego materiału została przeprowadzona w dziedzinie przyrostów uogólnionych naprężeń i odkształceń.

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