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REFLECTION AND REFRACTION OF SHOCK WAVES IN NEO-HOOKEAN MATERIAL

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The semi-inverse method is used to examine the reflection-refraction problem for a finite amplitude oblique shock wave propagating in an unbounded medium consisting of two joined half-spaces, filled with two different neo-Hookean elastic materials. The incident shock is assumed to be a transverse, horisontally polarised, plane shock. In such a case the solution pattern can be assumed in a form of single reflected and refracted shock wave, which relative strengths follow the similar pattern and values as that of infinitesimal SH waves. The complex values of the amplitude in the linear theory (total reflection, grazing incidence) give occasion to existence of Stoneley interfacial waves but in nonlinear theory the likeness does not exist, and such solutions suggest rather the change of solution pattern.

1. Introduction

Wright in his paper [1] on reflection of an oblique finite elastic shock wave at a plane boundary of a nonlinear elastic solid presented a semi-inverse method, based on strictly mechanical considerations, of finding the reflected waves. In this method a reflection pattern is assumed: the reflected waves form a family of plane simple waves (or shocks) centered on a moving line of contact between the incident wave and the boundary. Each reflected wave connects a fixed state ahead of the wave with a one parameter family of states behind the wave. In anisotropic solids there are three possible families of reflected waves so that a sequence of such waves connects the state behind the incident shock with a three parameter family of states adjacent to the boundary. In general, there are three independent boundary conditions from which the parameter specifying the reflected waves can be determined. The assumed pattern reduces the reflection problem to an initial-boundary value problem for a system of ordinary differential equation governing the variation of the deformation gradient and velocity fields in the regions of simple waves. Its solution determines the wedge shaped regions of simple waves and the distribution and strengths of the wavelets within each wave. If the assumed reflection pattern fails the admissibility test, it is modified to include shocks as well; for shocks, the reflection problem is then reduced to solving a system of algebraic equations for the direction of propagation and strength of the reflected shocks.

The reflection-refraction problem for a shock incident on a plane interface joining two elastic solids is exactly analogous, but now there will be three reflected waves and three refracted waves, so there will be six parameters to be found from six continuity conditions for the displacements and stresses at the interface. In some cases of a particular material property there may be less than six reflected and refracted waves.

We will apply the semi-inverse method to examine the reflection-refraction problem for a plane shock wave propagating in an unbounded medium consisting of two joined half-spaces, filled with two different nonlinear elastic materials. The medium initially is unstrained and at rest. Bearing in mind the complexity of the analysis of finite amplitude waves in nonlinear solids we select the simplest nontrivial case for which a detailed discussions is conducted. We assumethat both materials are homogeneous, incompressible and isotropic, and both are characterised by the neo-Hookean strain energy function but with different material constants. Since incompressible solids transmit transverse waves only, the incident shock is assumed to be a transverse shock; we assume further that this shock is horisontally polarised. Such waves have the displacement components in one direction only. In such a case the reflection-refraction solution pattern can be assumed in a form of singlesimple reflected wave and a single simple refracted wave both centered on the line of incidence at the interface.

Section 2 contains a summary of necessary theory and derivation of the propagation condition for shocks and simple waves in incompressible materials. In sections 3 and 4 geometric and analytic description of the incident shock and the solution pattern are given. The analysis of the solution for shock reflection-refraction in neo-Hookean elastic material are presented in section 5. The results are illustrated graphically.

2. Basic equations

The motion of the continuum is given by $x_i = x_i(X_\alpha, t)$ where x_i and X_α are the Cartesian coordinates of a material particle in the present configuration B and the reference configuration B_R respectively, and the reference configuration is given by $x_i = \delta_{i\alpha} X_\alpha$. The deformation gradient $x_{i\alpha}$, , its inverse $X_{\alpha i}$ and the velocity are defined by

$$x_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}, \qquad X_{\alpha i} = \frac{\partial X_\alpha}{\partial x_i}, \qquad \dot{x}_i = u_i = \frac{\partial x_i}{\partial t}. \tag{2.1}$$

It is assumed that the material is homogeneous, elastic and incompressible. The incompressibility constraint requires

$$J = \det(x_{i\alpha}) = 1. \tag{2.2}$$

The Piola-Kirchhoff stress tensor for such material is

$$T_{i\alpha} = \rho_R \frac{\partial \sigma}{\partial x_{i\alpha}} + pX_{\alpha i}, \tag{2.3}$$

where σ denotes internal energy per unit mass in B_R, $\rho = \rho_R$ is the density and $p(X_\alpha)$ is a pressure field to be determined in each problem.

If the stress and velocity fields are differentiable, then the equations expressing balance of momentum and moment of momentum are

$$T_{i\alpha,\alpha} = \rho \dot{u}_i, \qquad x_{i\alpha} T_{j\alpha} = x_{j\alpha} T_{i\alpha}. \tag{2.4}$$

If the functions $x_i = x_i(X_\alpha, t)$ are continuous ewerywhere but have discontinuous first derivatives on some propagating surface $S(X_\alpha, t) = 0$, the equations (2.4) must be replaced by the jump conditions on this surface

$$\begin{aligned} &\text{a)} \qquad [T_{i\alpha}]N_\alpha = -\rho V[u_i], \\ &\text{b)} \quad [x_{i\alpha}] = a_i N_\alpha, \qquad [u_i] = -a_i V. \end{aligned} \tag{2.5}$$

Such a surface is called a shock wave. The vector N_α is a material unit normal to the wave, V is the speed of propagation along N_α and a_i is the amplitude vector of the jump. The double square brackets indicate the jump in the quantity enclosed across S; thus

$$[\;] = (\;)^B - (\;)^F,$$

where the letters F and B refer to the limit values taken in front and rear sides of S respectively.

Eliminating the velocity jump $[u_i]$ from eqs.(2.5) we obtain the equation for the shock speed, the following propagation condition

$$V^2 = \frac{1}{\rho m^2}[T_{i\alpha}]a_i N_\alpha, \tag{2.6}$$

where $m = |\mathbf{a}|$ is the shock strength.

It is known (cf.[6]) that the constraint of incompressibility (2.2) restricts the directions of propagation to transversal directions only. This means that shocks in incompressible continua are transverse waves.

Simple waves [1] are defined to be regions of space time in which all field quantities are continuous functions of a single parameter, say $\lambda = \psi(X_\alpha, t)$. Regions of constant λ are propagating surfaces, called wavelets, with unit normal and normal velocity in B_R given by

$$N_\alpha(\lambda) = \frac{\psi_{,\alpha}}{|\nabla\psi|}, \qquad U(\lambda) = \frac{-\dot{\psi}}{|\nabla\psi|}. \tag{2.7}$$

The equation of motion (2.4) and the compatibility condition in the region of simple wave are

$$\frac{\partial T_{i\alpha}}{\partial x_{j\beta}} x'_{j\beta}\psi_{,\alpha} = \rho u'_i\dot{\psi}, \qquad x'_{j\beta}\dot{\psi} = u'_j\psi_{,\beta}, \tag{2.8}$$

where the prime indicates differentiation with respect to λ. If $\dot{\psi} \neq 0$ eqs.(2.8) can be rewritten to obtain the propagation condition for simple waves (cf.[1]) and the compatibility condition in the form

$$(Q_{ij} - \rho U^2\delta_{ij})u_j = 0, \qquad U x_{j\beta} + u_j N_\beta = 0, \tag{2.9}$$

where

$$Q_{ij} = \frac{\partial T_{i\alpha}}{\partial x_{j\beta}} N_\alpha N_\beta \tag{2.10}$$

is the acoustic tensor. For simple waves to propagate it is necessary that the eigenvalues of Q_{ij} are real monotone functions of the wave parameter (cf.[1]).

3. Incident shock

Consider an unbounded medium consisting of two elastic incompressible, isotropic half-spaces of different material properties, joined rigidly along the plane $x_2 = 0$, and initially unstrained and at rest. Suppose that a plane, horisontally polarised transverse shock wave of strength m_0 propagates in the half-space $x_2 > 0$ with speed V_0, and approaches the interface $x_2 = 0$ at an angle Θ_0 (fig.1). Thus, this propagating discontinuity surface belongs to a one-parameter family of parallel planes, with normals

$$\boldsymbol{N}_0 = (\sin\Theta_0, -\cos\Theta_0, 0), \qquad 0 < \Theta_0 < \frac{\pi}{2}. \tag{3.1}$$

Such waves have displacement components in the x_3 – direction only.

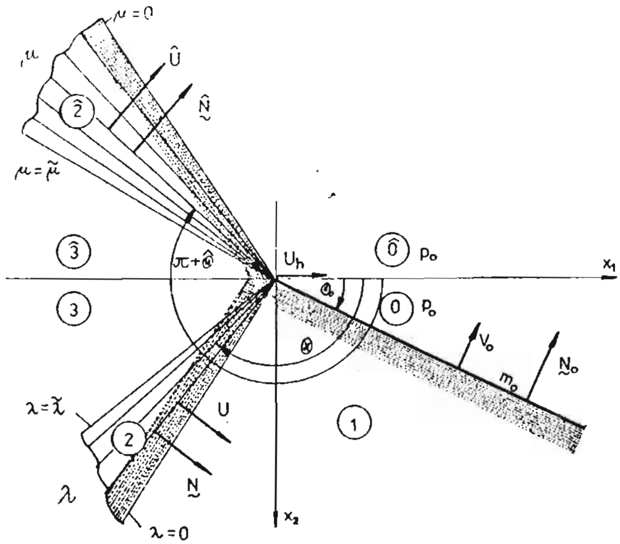

Fig. 1. Incident shock and assumed reflection and refraction pattern

Since the amplitude vector $\boldsymbol{a}$ is parallel to the x_3 axis and the medium in front of the incident shock is unstrained and at rest, the jumps (2.5) become now

$$\text{a)} \quad [x_{31}] = (x_{31})^B = m_0 \sin\Theta_0, \qquad [x_{32}] = (x_{32})^B = -m_0\cos\Theta_0, \tag{3.2}$$

$$\text{b)} \qquad [u_3] = (u_3)^B = -m_0 V_0,$$

where $m_0 = \boldsymbol{a}$ is the shock strength. Eq.(2.6) for the shock speed is

$$V_0^2 = \frac{1}{\rho m^2}([T_{31}]\sin\Theta_0 - [T_{32}]\cos\Theta_0). \tag{3.3}$$

The state behind the propagating shock (region 1 in fig.1) is now completely specified by the angle of incidence Θ_0 and the shock strength m_0. Eqs.(3.2) determine the deformation gradient and its inverse

$$(x_{i\alpha})^B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_1 & v_2 & 1 \end{bmatrix}, \qquad (X_{\alpha i}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & \\ -v_1 & -v_2 & 0 \end{bmatrix} \tag{3.4}$$

and particle velocity

$$\boldsymbol{u} = (0, 0, u) \tag{3.5}$$

in this state. We denoted here $v_1 = (x_{31})^B$, $v_2 = (x_{32})^B$, $u = (u_3)^B$.

For isotropic incompressible materials the internal energy σ is a function of I_1, I_2 and S, where

$$I_1 = B_{ii}, \qquad I_2 = \frac{1}{2}(B_{ii}B_{jj} - B_{ij}B_{ij})$$

are the invariants of the left Cauchy-Green strain tensor B_{ij}, and S is the entropy (per unit volume in B_R).

The components of the stress tensor $T_{i\alpha}$ required in this paper, are now

$$\begin{aligned} T_{11} &= 2\rho\{\sigma_1 + \sigma_2(2 + v_2^2)\} + p, & T_{13} &= -2\rho\sigma_2 v_1 - pv_1, \\ T_{22} &= 2\rho\{\sigma_1 + \sigma_2(2 + v_1^2)\} + p, & T_{31} &= 2\rho(\sigma_1 + \sigma_2)v_1, \\ T_{33} &= 2\rho\{\sigma_1 + 2\sigma_2\} + p, & T_{23} &= -2\rho\sigma_2 v_2 - pv_2, \\ T_{12} &= T_{21} = -2\rho\sigma_2 v_1 v_2, & T_{32} &= 2\rho(\sigma_1 + \sigma_2)v_2, \end{aligned} \tag{3.6}$$

where

$$\sigma_1 = \frac{\partial\sigma}{\partial I_1}, \qquad \sigma_2 = \frac{\partial\sigma}{\partial I_2}, \qquad I_1 = I_2 = 3 + v_1^2 + v_2^2.$$

4. Reflection-refraction pattern

When the incident shock wave strikes the interface $x_2 = 0$, part of it is reflected and part transmitted across the interface, in a form of plane reflected and refracted waves centered on the line of contact with the boundary $x_2 = 0$ (point Q in fig.1), and propagating away from the interface. The point Q moves along the boundary (x_1 – axis) with speed

$$U_h = \frac{V_0}{\sin\Theta_0}. \tag{4.1}$$

It is also assumed that both the reflected wave (region 2 in fig.1) and the refracted wave (region $\hat{2}$) are simple waves connecting the front and rear regions of constant state. Since all waves are centered on Q, we have for the reflected wave

$$\begin{aligned} &\text{a)} \quad \boldsymbol{N}(\lambda) = (\sin\Theta(\lambda),\ -\cos\Theta(\lambda),\ 0), \qquad \tfrac{\pi}{2} < \Theta < \pi, \\ &\text{b)} \quad U(\lambda) = U_h \sin\Theta(\lambda), \end{aligned} \tag{4.2}$$

and for the refracted wave

$$\begin{aligned} &\text{a)} \quad \boldsymbol{N}(\mu) = (\sin\hat{\Theta}(\mu),\ -\cos\hat{\Theta}(\mu),\ 0), \qquad 0 < \hat{\Theta} < \tfrac{\pi}{2}, \\ &\text{b)} \quad U(\mu) = \frac{U_h}{\sin\hat{\Theta}(\mu)}, \end{aligned} \tag{4.3}$$

where Θ and $\hat{\Theta}$ denote the angle of reflection and refraction (ref. fig.1), and U, λ, $\hat{U}$, μ are the normal speed and parameter of the reflected and refracted wave, respectively. The material region ahead of the refracted wave (region $\hat{0}$) is unstrained and at rest. The deformation gradient $x_{i\alpha}(\mu)$ and the velocity $u(\mu)$ in region $\hat{2}$) assume an analogous to (3.4) and (3.5) form.

It is required that the displacement vector and the stress vector are continuous at the interface $x_2 = 0$. This means that the system of the incident, reflected and refracted waves must satisfy four continuity conditions for the velocity and stress fields in both media at $x_2 = 0$,

$$u = \hat{u}, \qquad T_{i2} = \hat{T}_{i2}, \quad i = 1, 2, 3. \tag{4.4}$$

Since regions 0 and $\hat{0}$ are unstrained and at rest, conditions (4.4) at the interface joining these two regions are, for the suitably chosen pressures p_0 and $\hat{p}_0$, satisfied identically.

The constant values of region 3 and $\hat{3}$ adjacent to the remaining part of the interface are the final values of region 2 and $\hat{2}$ corresponding to the final values $\tilde{\lambda}$ and $\tilde{\mu}$ of the wave parameters λ and μ. Substitution into (4.4) gives a system of four independent equations involving two unknown terminal values of the wave parameters. In general system (4.4) has no solution in $\tilde{\lambda}$ and $\tilde{\mu}$. A solution may exist, however, if some additional restrictions on σ or on the incident shock are imposed.

5. Reflection and refraction in neo-Hooken material

For rubber-like materials under moderate strain, the strain energy function W can be approximated by [7,8]

$$W(I_1, I_2) = \rho\sigma(I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(I_1^2 - 9). \tag{5.1}$$

The components (3.6) of the stress tensor $T_{i\alpha}$ are now

$$\begin{aligned} T_{12} &= 2C_2 v_1 v_2, \\ T_{32} &= c^2(1 + \eta(v_1^2 + v_2^2))v_2, \\ T_{22} &= c^2(1 + \eta(v_1^2 + v_2^2)) + 2C_2 v_1^2 + p, \end{aligned} \tag{5.2}$$

where

$$c^2 = 2(C_1 + C_2 + 6C_3), \qquad \eta = \frac{4C_3}{c^2}.$$

The other field quantity required here the static pressure $p(X_\alpha)$, is given in the region of simple wave by (cf.[9])

$$p(\lambda) = -4C_2[v_1^2(\lambda) + v_2^2(\lambda)] + p_o. \tag{5.3}$$

The requirement that u is continuous on $x_2 = 0$ implies continuity of v_1 as well. Indeed, substituting (4.2),(4.3) into the compatibility condition (2.9) and integrating the first equation with boundary conditions (3.2) and conditions for region $\hat{0}$ we obtain

$$U_h v_1 + u = 0, \qquad U_h \hat{v}_1 + \hat{u} = 0,$$

substracting these equations for the final values of the simple wave parameters $\tilde{\lambda}$, $\tilde{\mu}$ (regions 3,$\hat{3}$), $U_h[v_1 - \hat{v}_1] + [u - u] = 0$ on $x_2 = 0$, where U_h is the apparent speed of the line of contact of the waves with the interface (fig.1). It is obvious that if $u = \hat{u}$ then also

$$v_1 = \hat{v}_1 \quad \text{on} \quad x_2 = 0. \tag{5.4}$$

Substituting (5.2) and (5.3) into (4.4), and using (5.4) and the relation $p_0 - \hat{p}_0 = \hat{c}^2 - c^2$ connecting the constant pressures of region 0 and $\hat{0}$, we obtain the conditions of continuity in the following form

$$\begin{gathered} u = \hat{u}, \qquad (v_1 = \hat{v}_1), \\ (C_2 v_2 - \hat{C}_2 \hat{v}_2) v_1 = 0, \\ c^2[1 + \eta(v_1^2 + v_2^2)] v_2 = \hat{c}^2[1 + \hat{\eta}(\hat{v}_1^2 + \hat{v}_2^2)] \hat{v}_2, \\ (C_2 - \hat{C}_2) v_1^2 = 0. \end{gathered} \tag{5.5}$$

It is evident that system (5.5) reduces to two independent nontrivial equations for $v_1 = 0$, i.e. for shocks travelling in the direction normal to the interface $x_2 = 0$ (cf.[10,11]). It also reduces to two equations if the medium is characterised by the neo-Hookean strain-energy function, i.e. when the constants C_2 and C_3 in (5.1) are zero (we take $C_1 = C$).

Let us assume that the strain-energy function is

$$W = \rho\sigma(I_1) = C(I_1 - 3). \tag{5.6}$$

In such materials the shock speed (3.3) is independent of the direction of propagation (3.1) and of the deformation gradient (3.4)

$$V_0^2 = \frac{2}{\rho} C \tag{5.7}$$

and the transverse shocks propagate with constant speed, without change in the form. Hence,the propagation of transverse shocks waves in a neo-Hookean material follows the same pattern as that of small amplitude waves in a homogeneous linear isotropic elastic solid.

We also note that the components $\partial T_{i\alpha}/\partial x_{j\beta}$ in the acoustic tensor (2.10) are constant. Since the characteristic equation for the propagation condition (2.9a) for simple waves can be written in the form

$$\det\left(\frac{\partial T_{i\alpha}}{\partial x_{j\beta}} \check{N}_\alpha \check{N}_\beta - U_h^2 \delta_{ij}\right) = 0, \tag{5.8}$$

where
$$\check{N} = \frac{N}{\sin\Theta(\lambda)} = (1,\ -\tau,\ 0), \quad \tau = \cot\Theta(\lambda), \quad U_h = \frac{U}{\sin\Theta_0}$$
and the speed U_h of the point Q (ref.4.1) is constant, we conclude that the characteristic roots of (5.8) that define speed $U(\lambda)$ of the reflected simple waves (and by a similar argument speed $\hat{U}(\mu)$ of the refracted simple waves) are independent of the wave parameter and that the wave speed is constant across the wave; hence according to the admissibility criterion (cf.[1]), such solutions do not represent simple waves.

We modify the solution pattern assuming now that region 2 is a plane shock wave centered on Q, with strength m and direction of propagation $N = (sin\Theta,\ -cos\Theta,\ 0)$. An analogous assumption applies to region $\hat{2}$. Eqs. of motion (2.4) are now replaced by the jump conditions (2.5) connecting the corresponding quantities in regions 1 and 3 across the wave. From (3.3) we calculate the reflected shock speed
$$V^2 = \frac{2C}{\rho}, \tag{5.9}$$
which is exactly the speed of the incident shock.
Since by (4.2b) $V = U_h \sin\Theta = V_0 \sin\Theta / \sin\Theta_0$, the equality $V = V_0$ can be satisfied only if
$$\Theta = \pi - \Theta_0. \tag{5.10}$$

The constant state of region 1 is defined by equations (3.2) (3.4) and (3.5). Applying (5.10) in the jump conditions (2.5) across region 2 we obtain the constant values of region 3
$$\begin{aligned} v_1^B &= (m_0 + \varepsilon m)\sin\Theta_0, \\ v_2^B &= (-m_0 + \varepsilon m)\cos\Theta_0, \\ u^B &= -(m_0 + \varepsilon m)V_0, \end{aligned} \tag{5.11}$$
where ε is $+1$ or -1, depending on the orientation of the reflected shock polarisation vector a with respect to the x_3 – axis.

The speed of the refracted shock is $\hat{V}^2 = 2\hat{C}/\rho$.
Since by (4.3b) $\hat{V} = U_h \sin\hat{\Theta} = V_0 \sin\hat{\Theta} / \sin\Theta_0$, we find that
$$\sin\hat{\Theta} = \frac{\hat{V}}{V_0}\sin\Theta_0. \tag{5.12}$$

Applying (5.12) in the jump conditions (2.5) across region $\hat{2}$ we obtain the constant values of region $\hat{3}$
$$\hat{v}_1^B = \hat{\varepsilon}\hat{m}\frac{\hat{V}}{V_0}\sin\Theta_0,$$

$$\hat{v}_2^B = -\hat{\varepsilon}\hat{m}\sqrt{1-\left(\frac{\hat{V}}{V_0}\right)^2\sin^2\Theta_0}, \qquad (5.13)$$
$$\hat{u}^B = -\hat{\varepsilon}\hat{m}\hat{V},$$

where $\hat{\varepsilon} = \pm 1$.

The requirement that the solution of equation (5.12) for $\hat{\Theta}$ must be real imposes on Θ_0 the condition

$$\sin\Theta_0 \leq \frac{V_0}{\hat{V}}, \qquad (5.14)$$

which may restrict, depending on the material constants in both semi-spaces, the interval of variation of the angle of incidence.

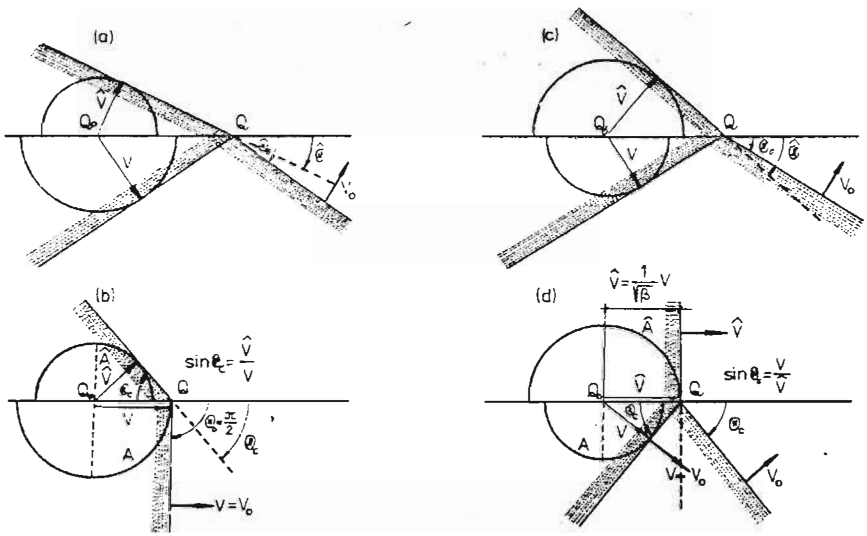

Fig. 2. Incident, reflected and refracted shocks a) for $\hat{V} < V_0$ and $0 < \Theta_0 < \frac{\pi}{2}$; b) for $\hat{V} < V_0$ and $\Theta_0 = \frac{\pi}{2}$; c) for $\hat{V} > V_0$ and $0 < \Theta_0 < \Theta_c$; d) for $\hat{V} > V_0$ and $\Theta_0 = \Theta_c$

Relations (5.10),(5.12) and (5.14) have a simple geometrical interpretation. Though the speeds $\hat{V}$ and V $(= V_0)$ depend on the material constants only, the corresponding directions of propagation depend on the angle of incidence as well. Let Q_0 and Q identify the position of point of contact of the incident, reflected and refracted wave on the x_1 – axis at the instant $t = 0$ and $t = 1$, respectively; the distance from Q to Q_0 is then equal U_h (ref.(4.1)). Let $\hat{K}$ and K be two semicircles with centre at Q_0, and radius $\hat{r} = \hat{V}$, $r = V$, and U_h be large enough for Q to be outside of $\hat{K}$ and K (fig.2). The tangents to $\hat{K}$ and K issuing from Q form with the x_1 – axis the angles Θ and Θ_0 (fig.2a) that satisfy eqs.(5.10) and

(5.12). The position of point Q changes with angle of incidence Θ_0, thus forming two one-parameter families of tangents which represent the families of reflected and refracted waves at unit time after passing through Q_0. The semicircles $\hat{K}$ and K are envelopes of these two families; they are called wave curves.

If $\hat{V} < V_0$, the point Q moves toward K as Θ_0 increases to $\frac{\pi}{2}$. For $\Theta_0 = \frac{\pi}{2}$ (grazing incidence) the incident and reflected wave coincide, and the refracted wave assumes its extreme position, its angle Θ_c given by the equation $\sin\Theta_c = \hat{V}/V_0$ (fig.2b).

If $\hat{V} > V_0$, the angle of refraction $\hat{\Theta}$ is greater than the angle of incidence Θ_0 (fig.2c). The point Q moves toward $\hat{K}$ with increasing Θ_0 and meets K when $\Theta_0 = \Theta_c$; we have then a "grazing refraction" (fig.2d). The critical value Θ_c of the angle of incidence is given by the eq. $\sin\Theta_c = V_0/\hat{V}$ (ref.5.14).

The remaining two unknown quantities, the strengths of the reflected and refracted shock m and $\hat{m}$, will be found from the boundary conditions (5.5) connecting the constant states of region 3 and $\hat{3}$ across the interface. Analysing the jump conditions (5.11),(5.13) and the relation (5.12) it is easy to see that if $\hat{u} = u$ then also $\hat{v}_1 = v_1$, as for the pattern in the form of the simple wave. Conditions (5.5) are now reduced to two nontrivial equations involving the components of the particle velocity and deformation gradient in both half-spaces as function of m and $\hat{m}$ (ref. also (5.4))

$$u = \hat{u} \Rightarrow (v_1 = \hat{v}_1), \qquad Cv_2 = \hat{C}\hat{v}_2 \text{ at } x_2 = 0, \tag{5.15}$$

with an additional equation $p_0 - \hat{p}_0 = \hat{C} - C$ relating the constant pressures of region 0 and $\hat{0}$. Solving eqs.(5.15), with the aid of (5.11) and (5.13), for m and $\hat{m}$

$$\begin{aligned} \varepsilon m &= \frac{\beta\cos\Theta_0 - \sqrt{\beta - \sin\Theta_0}}{\beta\cos\Theta_0 + \sqrt{\beta - \sin^2\Theta_0}}\, m_0, \\ \hat{\varepsilon}\hat{m} &= \frac{2\beta\sqrt{\beta}\cos\Theta_0}{\beta\cos\Theta_0 + \sqrt{\beta - \sin^2\Theta_0}}\, m_0, \end{aligned} \tag{5.16}$$

completes the reflection-refraction solution in the assumed form. Since speeds V and $\hat{V}$ are constant throughout the medium, the solution satisfies Lax's stability criterion (cf.[3]) for shocks; henced, the .eflected and the refracted wave are shocks, and they are represented by the families of planes (ref.(4.2)) defined by

$$\sin\Theta_0\, x_1 - \cos\Theta_0\, x_2 - V_0 t = 0 \tag{5.17}$$

and by

$$\sin\Theta_0\, x_1 - \sqrt{(\frac{V_0}{\hat{V}})^2 - \sin^2\Theta_0}\, x_2 - V_0 t = 0 \tag{5.18}$$

respectively.

We note here that eqs.(5.16) agree with the expressions obtained for the amplitude rations in the reflection-refraction problem for SH waves in linear theory (cf.[5],page 184).

Inspection of eqs. (5.16) leads to the following observations.

1 – The incident shock is completely refracted if $m = 0$. The corresponding equation

$$\beta^2 \cos^2 \Theta_0 - \beta(\frac{\hat{\rho}}{\rho}) + \sin^2 \Theta_0 = 0, \qquad \beta = \frac{C}{\hat{C}}, \tag{5.19}$$

shows that a combination of material properties and angle of incidence is possible for which there is no reflected wave. If $\rho = \hat{\rho}$ and $C \neq \hat{C}$, such a combination is particularly simple

$$\beta = \frac{C}{\hat{C}} = \tan^2 \Theta_0, \tag{5.20}$$

2 – The strength m of the reflected shock is a monotone function of Θ_0, $0 < \Theta_0 < \Theta_c$, so εm changes the sign in the neighbourhood of its zero (fig.3), thus changing the direction of polarisation. 3 – The expression for $\hat{\varepsilon}\hat{m}$ is positive for

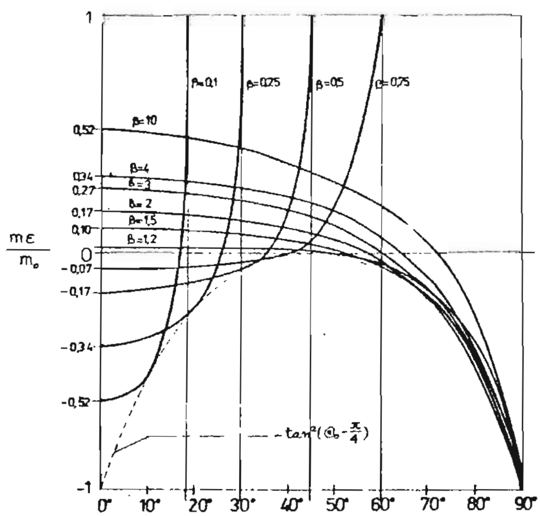

Fig. 3. Relation between the reflected shock strength and incident angle. For the interface characterised by $\beta > 1$ and $\beta_1(1/\beta)$ the strength is the same for the incident angle Θ_2', $\Theta_2''(\Theta_1)$

all $\Theta_0 \in (0, \Theta_c)$, hence $\hat{\varepsilon} = 1$; it also shows that reflection without refraction is not possible.

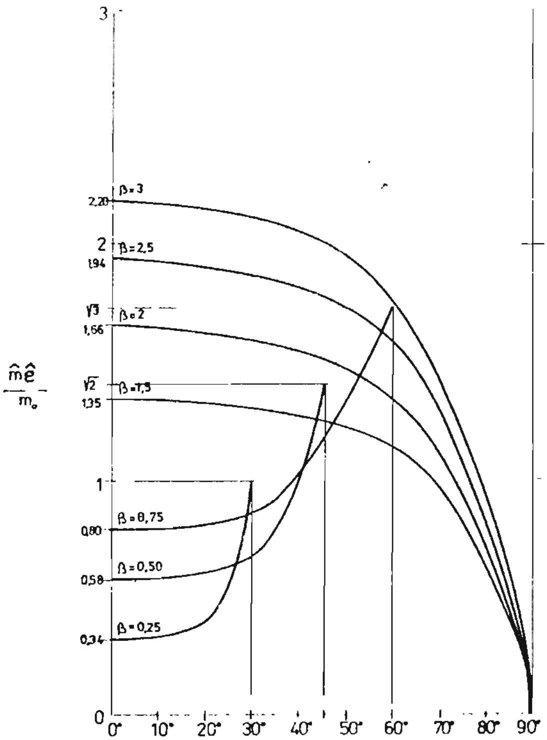

Fig. 4. Relation between the refracted shock strength and incident angle for some values of the parameter β $(\rho = \hat{\rho})$

4 – The strengths of the incident, reflected and refracted shocks are connected by the formula

$$(m_0 + \varepsilon m)V_0 = \hat{m}\hat{V}. \tag{5.21}$$

For "grazing incidence" (fig.2b) we have $m_0 = -\varepsilon m$ and the superposed incident and reflected shocks produce zero displacement the reflected wave also disappears. For "grazing refraction" (fig.2d), when $\Theta_0 = \Theta_c$, we have $m = m_0$, hence $\hat{m} = 2m_0(V_0/\hat{V})$.

6. Concluding remarks

The geometry of the reflected shock wave is given by (5.17); the shock travels through the half-space $x_2 > 0$ with a constant speed V $(= V_0)$ in the direction

$\Theta = \pi - \Theta_0$. The direction of polarisation and strength of this shock depend on the material constants C and $\hat{C}$ and the incident shock parameters Θ_0 and m_0. Equation (5.17) (for $\rho = \hat{\rho}$) determines the incident angle $\bar{\Theta}_0$ for which there is no reflection. Since $(V_0/\hat{V})^2 = C/\hat{C} = \tan^2 \bar{\Theta}_0$, such an angle is within the admissibility interval $(0, \Theta_c)$ defined by (5.12). If $C > \hat{C}$, the reflection ratios $R = \varepsilon m/m_0$ is a decreasing function of Θ_0, and it is positive for $\Theta_0 \in (\bar{\Theta}_0, \Theta_c)$. Hence,the reflected wave is a shock propagating into a deformed body while:
(a) – increasing the strain level if $\Theta_0 \in (0, \bar{\Theta}_0)$,
(b) – decreasing the strain level if $\Theta_0 \in (\bar{\Theta}_0, \Theta_c)$. The situation is reversed for $C < \hat{C}$.

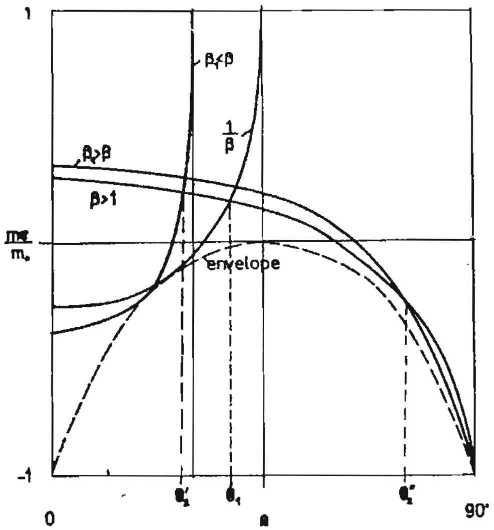

Fig. 5. Relations between the reflected shock strength, critical angle and the parameter β

The family $\mathcal{F}$ of curves representing the reflection ratio $R = \varepsilon m/m_0$ as a function of Θ_0 for various values of the parameter $\beta = (C/\hat{C})$ is shown in figure 3. It can be seen that the family curves intersect at some points, thus indicating that a given incident shock combined with two different composite materials may produce the same reflected shock. Indeed, if a family curve β passes through a point (Θ_0, R) then the curve β_1 defined by

$$\beta_1 = \frac{\beta \sin^2 \Theta_0}{\beta - \sin^2 \Theta_0}, \tag{6.1}$$

also passes through this point; since $\beta_1 - \sin^2 \Theta_0 = \sin^4 \Theta_0/(\beta - \sin^2 \Theta_0) > 0$, condition (5.14) is satisfied, thus confirming that the curve β_1 belongs to family $\mathcal{F}$. In the trivial case of $\beta = 1$ the ratio $R = \varepsilon m/m_0$ is zero for arbitrary Θ_0; from

(6.1) we find the parameter β_1 defining the associated curve to be $\beta_1 = \tan^2 \Theta_0$, and the point of intersection is $(\Theta_0, 0)$ (ref.(5.17)).

Suppose now that $\beta_1 = 1/\beta$. From (6.1) it follows that the curves defined by β and $\beta_1 = 1/\beta$ intersect at (Θ_0, R) if and only if

$$\sin^2 \Theta_0 = \frac{\beta}{1+\beta^2}, \tag{6.2}$$

eq.(6.2) has a unique solution Θ_0 for arbitrary $\beta > 0$. Since in this case $\sin \Theta_0 \leq 1/\sqrt{2}$, Θ_0 is restricted to the interval $(0, \pi/4]$. Hence for an arbitrary incident angle $\Theta_0 \in (0, \pi/4]$ a medium characterised by the ratio $\beta = C/\hat{C}$ can be found such that the reflected shock wave in this medium and in the medium composed of the same materials but in the reverse order are the same (figure 5).

The locus of points at which two "neighbouring" family curves β and $\beta_1 = \beta + d\beta$, $d\beta \to 0$, intersects forms an envelope of $\mathcal{F}$

$$R = -\tan^2(\Theta_0 - \frac{\pi}{4}), \tag{6.3}$$

with the family parameter β constrained by

$$\beta = 2\sin^2 \Theta_0. \tag{6.4}$$

This curve has a specific property that to each of its points there corresponds one and only one combination of incident shock and material properties. Moreover, since on this curve the derivative $\partial R/\partial \beta = 0$, eq.(6.3) gives the minimum values of the reflection ratio $R = \varepsilon m/m_0$ for fixed Θ_0 and varying β.

The refracted shock (5.19) travels through the half-space $x_2 < 0$ with a constant speed $\hat{V}$, in the direction $\hat{\Theta}$ defined by (5.12); its strength is given by eq.$(5.16)_2$, and it propagates into an unstrained medium, loading the material.

The family $\mathcal{F}$ of curves representing the refraction ratio $\hat{R} = \hat{m}/m_0$ as a function of Θ_0 for various values of β is shown in fig.4. The curves intersect at some points, thus indicating that different combinations of material properties and angle of incidence are possible for which the refracted shock is the same. The analytic expression of such combinations, however, is more complex than in the case of reflection (ref.(6.1)), as here the refraction angle $\hat{\Theta}$ depends on both the incident angle Θ_0 and the material ratio β.

On the other hand, it is easy to show that for the combinations that produce equal reflected shocks, the corresponding refracted shocks are not equal. Indeed, by (5.12), for a fixed Θ_0 the refraction angle $\hat{\Theta}$ is a monotone function of β, thus assuming different values for different values of β. This means that different material combinations lead to a given shock being refracted in different directions. A similar conclusion concerning the shock strength $\hat{m}$ can be derived from expression (5.21). Denoting by R the reflection ratio corresponding, for a fixed Θ_0, to β and

β_1 related by (6.1) and by $\hat{R}$ and $\hat{R}_1$ the associated refraction ratios, we obtain from (5.21)

$$\begin{aligned} (1+R)\sqrt{\beta} &= \hat{R}, \\ (1+R)\sqrt{\beta_1} &= \hat{R}_1, \end{aligned} \tag{6.5}$$

eliminating R we get

$$R_1 = R\sqrt{\frac{\beta_1}{\beta}}. \tag{6.6}$$

Since by assumption $\beta_1 \neq \beta$, it follows that $\hat{m}_1 \neq \hat{m}$.

In the special case of total refraction the corresponding refraction angle and refracted shock strength are $\hat{\Theta} = \pi/2 - \Theta_0$ and $\hat{m} = \sqrt{\beta}m_0$. Of course, in the trivial case of identical materials ($\beta = 1$) there is also a total (apparent) refraction: $m = 0$ for arbitrary value of Θ_0, the "refracted" planes (4.2) become extensions of the "incident" planes (3.1) and the shock travels throughout the medium unobstructed.

The other special case refers to the reflected shocks represented by the envelope (6.3). Substitution from (6.4) into (5.12) gives the refraction angle $\hat{\Theta} = \pi/4$. Hence, the refracted shocks associated with the reflected shocks given by (6.3) and (6.4) have a fixed direction of propagation $\Theta_0 = \pi/4$; strength $\hat{m}$ of these shocks is given by

$$\hat{m} = \frac{\sqrt{2}m_0 \sin\Theta_0 \sin 2\Theta_0}{\cos^2(\Theta_0 - \frac{\pi}{4})}. \tag{6.7}$$

The method of solution used in this paper is based on an assumption that solutions must be real. For this reason it is required that the incident angle Θ_0 satisfies the condition: $0 < \Theta_0 < \Theta_c$ where Θ_c is a certain critical angle determined by the material properties of the composite medium. We have $\Theta_c < \pi/2$ for $\beta < 1$ and $\Theta_c = \pi/2$ for $\beta > 1$. The case of total reflection when Θ_0 exceeds its critical angle leads to a complex solution and is beyond applicability of the present method. In linear theory the solution for this case is composed of infinitesimal waves and Stoneley waves. For $\beta > 1$, the case of "grazing incidence" is illustrated in fig.2b; the corresponding reflection and refraction shock strengths are $m = m_0$ and $\hat{m} = 0$, and the case represents a zero motion. The technique used in [12] to derive a special solution for this case through a limit procedure is not applicable here, since, the solution waves have finite amplitudes. To include these particular cases in the problem considered here it would be necessary to modify the assumed reflection-refraction pattern.

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Streszczenie

W pracy zastosowano metodę pólodwrotną do badania zjawiska odbicia-zalamania ukośnej fali uderzeniowej o skończonej amplitudzie propagującej się w ośrodku złożonym z dwóch sztywno połączonych pólprzestrzeni sprężystych. Pólprzestrzenie wypełnione są neo-Hookeanem o różnych własnościach sprężystych. Założono, że fala padająca jest poprzeczną poziomo spolaryzowaną falą uderzeniową. W tym przypadku rozwiązanie sklada się z pojedyńczej odbitej i zalamanej fali uderzeniowej. Względne amplitudy tych fal pozostają w podobnym stosunku jak w przypadku infinitezymalnych fal typu SH w liniowej teorii. Zespolone wartości względnych amplitud w liniowej teorii prowadzą do występowania fal Stoneleya. Rozwiązania zespolone w teorii nieliniowej sugerują raczej inną niż założoną konfigurację fal odbitych i zalamanych.

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