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THE VIBRATIONS OF SELF-EXCITED SYSTEM WITH PARAMETRIC EXCITATION AND NON-SYMMETRIC ELASTICITY CHARACTERISTIC

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1. Introduction

In the mechanical vibrations, we can distinguish a rather broad class of the self-excited systems with simultaneous parametric excitation [3,6,8,12]. In non-linear systems there exists u mutual interaction between those two types of vibrations, which explains the lack of the superposition of "component" vibrations. Numerous scientific monographs are concerned with the problems of synchronization (entrainment of frequency) in the self-excited systems with external excitation. The cases of parametric and self-excited vibrations are considered in very few papers [1 $\div$ 3, 7 $\div$ 12]. A review of the papers in this field is contained in the monograph [6], which presents the investigations of the interactions in the vibrating systems with the sources of energy performing – so called – limited excitations. As in the case of vibrations of the system with the extarnal excitation so in the case of parametric excitation – under certain conditions – there exists a phenomenon of the drifting frequency [7].

The present paper presents an analysis of the vibrations of the self-excited and parametric model of non-linear characteristic of elasticity of the quadratic type without the external excitation. In the real systems, this type of vibrations, taking into account two mechanisms of excitation and the form of restitution force, can occur in some automotive vehicles. It concern the vibrations with feedback of a physical model of the car in which self-excted vibrations of the wheel [15] rotating on the smooth surface and the periodically changing radial stiffness of the tyre [13] and the pneumatic supension of the sprung mass are taken into account. In this case the characteristic of elasticity of the suspension can be approximated by a function of the second degree [13].

Although technical applications require the analysis of real systems models with many degrees of freedom, yet parametric self-excited systems vibrations ought to be examined also in the basic variant. The aim of this work is the analysis of the effects of interaction between self-excited and parametric vibrations of the main resonance. The development of investigations in this field is determined by the specific kind of excitation of parametric vibrations in comparison with the external excitation, and the numerous mathematical models describing real mechanical systems. Analytical, numerical and analogue methods have been applied. The applications of different methods of investigation have not only given a wider scope of information, but have also enabled the verification of some results.

Let us consider the vibrations described by the differential equation containing Van der Pol and Mathieu differential equations terms, and non-linear restitution force of the quadratic type:

$$m\ddot{x}_1 - (a - bx^2)\dot{x}_1 + (c - c_0 \cos 2\omega t)x_1 + c_1 x_1^2 = 0, \tag{1.1}$$

where: m – mass, a and b – damping Van der Pol's coefficients, c – average stiffness, c_0 – amplitude of stiffness modulation, c_1 – non-linearity coefficient, ω – parametric excitations circular frequency. Introducing non-dimensional time:

$$\tau = \omega t,$$

and the following constants:

$$p^2 = \frac{c}{m}, \qquad \mu = \frac{c_0}{c}, \qquad x_0 = \frac{mg}{c},$$

$$\alpha = \frac{a}{mp}, \qquad \beta = \frac{bx_0^2}{mp}, \qquad \gamma = \frac{c_1 x_0}{c},$$

$$x = \frac{x_1}{x_0}, \qquad \lambda = \frac{p}{\omega}, \qquad \overset{|}{x} = \frac{dx}{d\tau}.$$

we obtain the non-dimensional form of the equation (1.1):

$$\overset{||}{x} - \lambda(\alpha - \beta x^2)\overset{|}{x} + \lambda^2[(1 - \mu\cos 2\tau)x + \gamma x^2] = 0. \tag{1.2}$$

In the ffollowing considerations it will be assumed that the parameters $\alpha, \beta, \mu, \gamma$ are small enough and positive.

2. Analytical investigations

If in the equation (1.1) we put $a = b = 0$, its solutions will describe parametric non-damped vibrations, whereas for $c_0 = 0$ the solutions will concern only self-excited vibrations. As the result of the co-existence of parametric excitation and self-excitation components in differential equation (1.1) we anticipate tha vibrations of these kinds will appear together and we seek for an approximate solution of equation (1.2) in the following form:

$$x(\tau) = A(\tau)\cos\Omega\tau + B_1(\tau)\cos\tau + B_2(\tau)\sin\tau + D(\tau), \tag{2.1}$$

where: A – self-excited vibrations amplitude, Ω – self-excited vibraions frequency, B_1, B_2 – component amplitudes of parametric vibrations, D – translation of vibrations center. We assume also that functions $A(\tau), B_1(\tau), B_2(\tau), D(\tau)$ are changing slowly. The amplitude of parametric vibrations and phase difference are:

$$R = \sqrt{B_1^2 + B_2^2}, \qquad \varphi = \text{arc tg}\frac{B_2}{B_1}.$$

Putting the solution (2.1) into the equation (1.2), ignoring second derivatives of slowly changing functions, and terms including the products of the first derivatives of those functions we get:

$$A[-\Omega^2 + \lambda^2(1 + 2\gamma D)] = 0, \tag{2.2}$$

$$\begin{aligned}
\frac{dA}{d\tau} &= \frac{1}{2}A\lambda\Omega[\alpha - \beta D^2 - \frac{1}{2}\beta(\frac{1}{2}A^2 + R^2)], && (2.3)\\
\frac{dB_1}{d\tau} &= \frac{1}{2}\{\lambda[\alpha - \beta D^2 - \frac{1}{2}\beta(A^2 + \frac{1}{2}R^2)]B_1 + \\
&\quad +[-1 + \lambda^2(1 + \frac{1}{2}\mu + 2\gamma D)]B_2\}, && (2.4)\\
\frac{dB_2}{d\tau} &= -\frac{1}{2}\{[-1 + \lambda^2(1 - \frac{1}{2}\mu + 2\gamma D)]B_1 + \\
&\quad +\lambda[-\alpha + \beta D^2 + \frac{1}{2}\beta(A^2 + \frac{1}{2}R^2)]B_2\}, && (2.5)
\end{aligned}$$

$$D(D\gamma + 1) + \frac{1}{2}\gamma(A^2 + R^2) = 0. \tag{2.6}$$

In equation (2.6) the term containing the first derivative of the vibrations centre is neglected, such simplification can be found in analyses of vibrating systems with non-symmetric characteristics [5]. When the above mentioned term is taken into account, instead of equation (2.6) we obtain the following one:

$$\frac{dD}{d\tau} = -\frac{\lambda}{\beta}[\gamma + \frac{2D(1 + \gamma D)}{A^2 + R^2}]. \tag{2.7}$$

Let us investigate steady-state vibrations for which:

$$\frac{dA}{d\tau} = X_0(A, B_1, B_2, D) = 0, \qquad \frac{dB_1}{d\tau} = X_1(A, B_1, B_2, D) = 0,$$

$$\frac{dB_2}{d\tau} = X_2(A, B_1, B_2, D) = 0, \qquad \frac{dD}{d\tau} = X_3(A, B_1, B_2, D) = 0.$$

From the differential equation (2.7) follows the relation obtained before (2.6).

The solution of equation system (2.2) – (2.6) for the steady-state case is called trivial when $A = R = D = 0$, and non-trivial when $A \neq 0$, $R \neq 0$, $D \neq 0$. The solutions for $A = 0$, $R \neq 0$, $D \neq 0$ and $A \neq 0$, $R = 0$, $D \neq 0$ is called semi-trivial.

2.1. Semi-trivial solutions for $R = 0$, $A \neq 0$, $D \neq 0$

In this case we get from the equations (2.2), (2.3) and (2.6):

$$-\Omega^2 + \lambda^2(1 + 2\gamma D) = 0, \tag{2.8}$$

$$\alpha - \beta D^2 - \frac{1}{4}\beta A^2 = 0, \tag{2.9}$$

$$\gamma D^2 + D + \frac{1}{2}\gamma A^2 = 0. \tag{2.10}$$

For self-excited vvibrations amplitude in the real domain, from equations (2.9), (2.10) we get the condition:

$$D + \frac{\alpha}{\beta}\gamma < 0,$$

giving the negative value of the vibrations centre translation.

Taking it into account we obtain:

$$D = \frac{1}{2}\gamma(1 - \sqrt{1 + 8\frac{\alpha}{\beta}\gamma^2}), \tag{2.11}$$

$$A^2 = 4\frac{\alpha}{\beta} - \frac{1}{\gamma^2}(\sqrt{1 - 1 + 8\frac{\alpha}{\beta}\gamma^2}). \tag{2.12}$$

From the equation (2.8) we find the square of self-excited vibrations frequency:

$$\Omega^2 = \lambda^2(2 - \sqrt{1 + 8\frac{\alpha}{\beta}\gamma^2}), \tag{2.13}$$

and for real (dimensional) time:

$$\Omega_r^2 = p^2(2 - \sqrt{1 + 8\frac{\alpha}{\beta}\gamma^2}). \tag{2.14}$$

2.2. Semi-trivial solution for $A = 0,\ R \neq 0,\ D \neq 0$

In this case the equations (2.4), (2.5) and (2.6) assume the form:

$$\begin{aligned}
\frac{dB_1}{d\tau} &= \frac{1}{2}\left\{\lambda[\alpha - \beta D^2 - \frac{1}{4}\beta R^2]B_1 + [-1 + \lambda^2(1 + \frac{1}{2}\mu + 2\gamma D)]B_2\right\} = \\
&= Y_1(B_1, B_2, D) = 0, \qquad (2.15)\\
\frac{dB_2}{d\tau} &= -\frac{1}{2}\left\{[-1 + \lambda^2(1 - \frac{1}{2}\mu + 2\gamma D)]B_1 + \lambda[-\alpha + \beta D^2 + \frac{1}{4}\beta R^2]B_2\right\} = \\
&= Y_2(B_1, B_2, D) = 0, \qquad (2.16)
\end{aligned}$$

$$\gamma D^2 + D + \frac{1}{2}\gamma R^2 = Y_3(B_1, B_2, D) = 0, \qquad (2.17)$$

from which we get:

$$[4\gamma D(1 + \gamma D) - \frac{1}{4}\mu^2 + 1]\lambda^4 + \left\{[\alpha + \frac{1}{2}\beta D(\frac{1}{\gamma} - D)]^2 - 2(1 + 2\gamma D)\right\}\lambda^2 + 1 = 0.$$

The latter has the following roots:

$$\lambda_{1,2}^2 = \frac{2(1 + 2\gamma D) - [\alpha + \frac{1}{2}\beta D(\frac{1}{\gamma} - D)] \pm \Delta}{2[4\gamma D(1 + \gamma D) - \frac{1}{4}\mu^2 + 1]}, \qquad (2.18)$$

where:

$$\Delta = \left\{[\alpha + \frac{1}{2}\beta D(\frac{1}{\gamma} - D)]^2 - 2(1 + 2\gamma D)\right\}^2 - 4[4\gamma D(1 + \gamma D) - \frac{1}{4}\mu^2 + 1].$$

Let us consider solutions only in real range. Equation (2.17) implies that (except the case $R = 0, D = 0$) for $R \neq 0$ and $D < 0$ such a solution correspond only to one root:

$$D = \frac{-1 + \sqrt{1 - 2\gamma^2 R^2}}{2\gamma}. \qquad (2.19)$$

Putting (2.19) in (2.20), $\lambda^2(R)$ can ne found.

2.3. Non-trivial solution for $A \neq 0,\ R \neq 0,\ D \neq 0$

From equations (2.3) – (2.5) we get:

$$\alpha - \beta D^2 - \frac{1}{2}\beta(\frac{1}{2}A^2 + R^2) = 0, \qquad (2.20)$$

$$\lambda[\alpha - \beta D^2 - \frac{1}{2}\beta(A^2 + \frac{1}{2}R^2)]B_1 + [-1 + \lambda^2(1 + \frac{1}{2}\mu + 2\gamma D)]B_2 = 0, \qquad (2.21)$$

$$[-1+\lambda^2(1-\frac{1}{2}\mu+2\gamma D)]B_1+\lambda[-\alpha+\beta D^2+\frac{1}{2}\beta(A^2+\frac{1}{2}R^2)]B_2=0. \quad (2.22)$$

Equations (2.2) and (2.6) are still valid.
From terms (2.6) and (2.20) we find:

$$A^2=-4(\frac{\alpha}{\beta}+\frac{D}{\gamma}). \quad (2.23)$$

Hence it appears that self-excited vibrations disappear for:

$$D\geq-\frac{\alpha}{\beta}\gamma,$$

but they appear when:

$$D<-\frac{\alpha}{\beta}\gamma.$$

This condition is fulfilled for:

$$D^2+\frac{1}{2}(A^2+R^2)>\frac{\alpha}{\beta}.$$

Taking into account equations (2.2) and (2.23) we get the self-excited vibrations angular frequency:

$$\Omega=\lambda\sqrt{1-2\gamma^2(\frac{\alpha}{\beta}+\frac{1}{4}A^2)}. \quad (2.24)$$

Putting into equation (2.20) expression $D(A)$ we find:

$$R^2=2\frac{\alpha}{\beta}(1-\frac{\alpha}{\beta}\gamma^2)-[(\frac{1}{2}+\frac{\alpha}{\beta}\gamma^2)A^2+\frac{1}{8}\gamma^2A^4].$$

Monotonic character of the non-linear function $R(A)$ is defined by the derivative $\frac{dR}{dA}<0$. Solving equation (2.29) in respect to A^2 we get two roots, one of which having the form:

$$A^2=\frac{-(\frac{1}{2}+\frac{\alpha}{\beta}\gamma^2)+\sqrt{\frac{1}{4}+\gamma^2(2\frac{\alpha}{\beta}-\frac{1}{2}R^2)}}{\frac{1}{4}\gamma^2}.$$

fulfills the condition of existence of self-excited vibrations (of real solution) with:

$$R^2<2\frac{\alpha}{\beta}(1-\frac{\alpha}{\beta}\gamma^2).$$

Self-excited vibrations disappear when:

$$R^2\geq2\frac{\alpha}{\beta}(1-\frac{\alpha}{\beta}\gamma^2).$$

The equality sign corresponds to the value of parametric vibrations of amplitude $A = 0$. Self-excited vibrations amplitude for $R = 0$ is given by the formula:

$$A^2 = \frac{-(\frac{1}{2} + \frac{\alpha}{\beta}\gamma^2) + \sqrt{\frac{1}{4} + 2\frac{\alpha}{\beta}\gamma^2}}{\frac{1}{4}\gamma^2}.$$

Putting (2.23) and:

$$\frac{1}{2}R^2 = 2\frac{\alpha}{\beta} + \frac{D}{\gamma} - D^2, \tag{2.25}$$

into equations (2.21) and (2.22) we get for $B_1 \neq 0$, and $B_2 \neq 0$:

$$[1 - \frac{1}{4}\mu^2 + 4\gamma D(1 + \gamma D)]\lambda^4 + \left\{[2\alpha + \frac{1}{2}\beta D(\frac{3}{\gamma} - D)]^2 - 2(1 + 2\gamma D)\right\}\lambda^2 + 1 = 0,$$

hence we obtain:

$$\lambda_{1,2}^2 = \frac{-\left\{[2\alpha + \frac{1}{2}\beta D(\frac{3}{\gamma} - D)]^2 - 2(1 + 2\gamma D)\right\} \pm \sqrt{\Delta}}{2[1 - \frac{1}{4}\mu^2 + 4\gamma D(1 + \gamma D)]}, \tag{2.26}$$

where:

$$\Delta = \left\{[2\alpha + \frac{1}{2}\beta D(\frac{3}{\gamma} - D)]^2 - 2(1 + 2\gamma D)\right\}^2 - 4[1 - \frac{1}{4}\mu^2 + 4\gamma D(1 + \gamma D)].$$

Using the solutions in the real range we get from the equation (2.25) one of the values of the vibrations centre translation:

$$D = \frac{1 - \sqrt{1 + 4\gamma^2(2\frac{\alpha}{\beta} - \frac{1}{2}R^2)}}{2}. \tag{2.27}$$

From equations (2.17) and (2.25) we conclude that for semi-trivial solution $R \neq 0$ and $A = 0$ and non-trivial $R \neq 0, A \neq 0$ we get equal values of translation of the vibrations centre:

$$D = -\frac{\alpha}{\beta}\gamma,$$

to which corresponds also the same values of parametric vibrations amplitude:

$$R = \sqrt{2\frac{\alpha}{\beta}(1 - \frac{\alpha}{\beta}\gamma^2)}.$$

Introducing (2.27) into (2.26) we find $\lambda^2(R_{-})$. Then putting in this equation:

$$D = -\gamma(\frac{\alpha}{\beta} + \frac{1}{4}A^2), \tag{2.28}$$

we get $\lambda^2(A)$.

2.4. The stability of the solutions

Let us investigate by the analytical method the behaviour of approximated solutions after small perturbations. According to Lapunow's theory, the stability condition will be satisfied when all of the roots of the characteristic equation for the first approximation system have negative real parts. In the case of the semi-trivial solution for $A = 0$, $R \neq 0$, $D \neq 0$, from equations (2.15) – (2.17) we get a linear variational equations system leading to the characteristic equation:

$$a_0\rho^2 + a_1\rho + a_2 = 0. \tag{2.29}$$

The stability conditions for solutions at $a_0 > 0$ have the form:

$$a_1 > 0, \quad a_2 > 0,$$

where:

$$a_1 = \left[\frac{\partial Y_1}{\partial D}\frac{\partial Y_3}{\partial B_1} + \frac{\partial Y_2}{\partial D}\frac{\partial Y_3}{\partial B_2} - \frac{\partial Y_3}{\partial D}\left(\frac{\partial Y_1}{\partial B_1} + \frac{\partial Y_2}{\partial B_2}\right)\right]_0,$$

$$a_2 = \left[\frac{\partial Y_1}{\partial B_1}\left(\frac{\partial Y_2}{\partial B_2}\frac{\partial Y_3}{\partial D} - \frac{\partial Y_2}{\partial D}\frac{\partial Y_3}{\partial B_2}\right) + \right. \tag{2.30}$$

$$\left. + \frac{\partial Y_1}{\partial B_2}\left(\frac{\partial Y_2}{\partial D}\frac{\partial Y_3}{\partial B_1} - \frac{\partial Y_2}{\partial B_1}\frac{\partial Y_3}{\partial D}\right) + \frac{\partial Y_1}{\partial D}\left(\frac{\partial Y_2}{\partial B_1}\frac{\partial Y_3}{\partial B_2} - \frac{\partial Y_3}{\partial B_1}\frac{\partial Y_2}{\partial B_2}\right)\right]_0.$$

In these equations index 0 means that values of all partial derivatives are determined for B_1, B_2, D in the state of equilibrium. Introducing corresponding derivatives to (2.29), the stability condition $a_1 > 0$ can be written in the form of:

$$R^2 > 2\left(\frac{\alpha}{\beta} - D^2\right)(1 + 2\gamma D). \tag{2.31}$$

In the stability examinations the influence of the first derivative of the coordinate of the vibrations centre displacement will be taken into account. From equation (2.7) we get:

$$\frac{dD}{d\tau} = -\frac{\lambda}{\beta}\left[\gamma + \frac{2D(1+\gamma D)}{R^2}\right].$$

This expression performs the role (for $A = 0$) of equation (2.6), and in the state of equilibrium is reduced to equation (2.17). For this case the characteristic equation has the form:

$$\left|\begin{matrix} \frac{\partial Y_1}{\partial B_1} - \rho & \frac{\partial Y_1}{\partial B_2} & \frac{\partial Y_1}{\partial D} \\ \frac{\partial Y_2}{\partial B_1} & \frac{\partial Y_2}{\partial B_2} - \rho & \frac{\partial Y_2}{\partial D} \\ \frac{\partial Y_3}{\partial B_1} & \frac{\partial Y_3}{\partial B_2} & \frac{\partial Y_3}{\partial D} - \rho \end{matrix}\right|_0 = 0,$$

or after the expansion we obtain:

$$a_0\rho^3 + a_1\rho^2 + a_2\rho + a_3 = 0. \tag{2.32}$$

Solutions stability conditions (at $a_0 > 0$) express the following inequalities:

$$a_1 > 0, \qquad a_3 > 0, \qquad a_1a_2 - a_0a_3 > 0.$$

Let us examine the stability of non-trivial solutions describing the nearly periodic vibrations. For this kind of vibration according to [16], we will determine the roots of characteristic equation established for the variational system. In the result we obtain:

$$\left|\begin{array}{cccc}
\frac{\partial X_0}{\partial A} - \rho & \frac{\partial X_0}{\partial B_1} & \frac{\partial X_0}{\partial B_2} & \frac{\partial X_0}{\partial D} \\
\frac{\partial X_1}{\partial A} & \frac{\partial X_1}{\partial B_1} - \rho & \frac{\partial X_1}{\partial B_2} & \frac{\partial X_1}{\partial D} \\
\frac{\partial X_2}{\partial A} & \frac{\partial X_2}{\partial B_1} & \frac{\partial X_2}{\partial B_2} - \rho & \frac{\partial X_2}{\partial D} \\
\frac{\partial X_3}{\partial A} & \frac{\partial X_3}{\partial B_1} & \frac{\partial X_3}{\partial B_2} & \frac{\partial X_3}{\partial D} - \rho
\end{array}\right|_0 = 0,$$

and the characteristic equation of the form:

$$a_0\rho^4 + a_1\rho^3 + a_2\rho^2 + a_3\rho + a_4 = 0. \tag{2.33}$$

Partial derivatives are expressed as follows:

$$\frac{\partial X_0}{\partial A} = \frac{1}{2}\lambda[\alpha - \beta D^2 - \frac{1}{2}\beta(\frac{3}{2}A^2 + R^2)],$$

$$\frac{\partial X_0}{\partial B_1} = -\frac{1}{2}\lambda\beta AB_1, \qquad \frac{\partial X_0}{\partial B_2} = -\frac{1}{2}\lambda\beta AB_2,$$

$$\frac{\partial X_0}{\partial D} = -\lambda\beta AD, \qquad \frac{\partial X_1}{\partial A} = -\frac{1}{2}\lambda\beta AB_1,$$

$$\begin{aligned}
\frac{\partial X_1}{\partial B_1} &= \frac{1}{2}\lambda[\alpha - \beta D^2 - \frac{1}{2}\beta A^2 - \frac{1}{4}\beta(3B_1^2 + B_2^2)], \\
\frac{\partial X_1}{\partial B_2} &= \frac{1}{2}[-1 + \lambda^2(1 + \frac{1}{2}\mu + 2\gamma D) - \frac{1}{2}\beta\lambda B_1B_2], \\
\frac{\partial X_1}{\partial D} &= \lambda(\gamma\lambda B_2 - \beta B_1 D), \qquad \frac{\partial X_2}{\partial A} = -\frac{1}{2}\beta\lambda AB_2, \\
\frac{\partial X_2}{\partial B_1} &= -\frac{1}{2}\lambda[-1 + \lambda^2(1 - \frac{1}{2}\mu + 2\gamma D) + \frac{1}{2}\beta\lambda B_1B_2],
\end{aligned}$$

$$\frac{\partial X_2}{\partial B_2} = -\frac{1}{2}\lambda[-\alpha + \beta D^2 + \frac{1}{2}\beta A^2 + \frac{1}{4}\beta(B_1^2 + 3B_2^2)],$$

$$\frac{\partial X_2}{\partial D} = -\lambda(\gamma\lambda B_1 + \beta D B_2),$$

$$\frac{\partial X_3}{\partial A} = \frac{4\lambda A D(1+\gamma D)}{\beta(A^2+R^2)^2}, \qquad \frac{\partial X_3}{\partial B_1} = \frac{4\lambda B_1 D(1+\gamma D)}{\beta(A^2+R^2)^2},$$

$$\frac{\partial X_3}{\partial B_2} = \frac{4\lambda B_2 D(1+\gamma D)}{\beta(A^2+R^2)^2}, \qquad \frac{\partial X_3}{\partial D} = \frac{-2\lambda(1+2\gamma D)}{\beta(A^2+R^2)}.$$

In considering the non-trivial solutions stability the equation (2.7) was taken into account. Let us now perform the simplified investigation of the stability of the above-mentioned solutions replacing the equation (2.7) by the approximate equation (2.6). Instead of the equation (2.33) we obtain now the characteristic equation having the form (2.32), and the presented above three Routh-Hurvitz stability conditions. In the last verse of the determinant related to equation (2.33) the terms ρ vanish and the partial derivatives of X_3 and Y_3 are expressed as follows:

$$\begin{aligned} \frac{\partial Y_3}{\partial A} &= \gamma\lambda^2 A, & \frac{\partial Y_3}{\partial B_1} &= \gamma\lambda^2 B_1, \\ \frac{\partial Y_3}{\partial D} &= \lambda^2(1+2\gamma D), & \frac{\partial Y_3}{\partial B_2} &= \gamma\lambda^2 B_2. \end{aligned} \tag{2.34}$$

3. Investigation by the methods of numerical analysis and analogue simulation; numerical examples for certain values

In the stability examinations the numerical analysis and analogue modelling methods were also used. The values of the coefficients of the equation (1.2) were assumed according to the applications [3,4,8]. Six alternative sets of numerical data were created, presented in table 1.

To obtain the diagrams of functions established previously by analytical methods a microcomputer equipped with a plotter was used. Figure 1[1] shows diagrams of $R(\lambda^2)$ and of $D(\lambda^2)$ for semi-trivial solutions and for all possible sets of values of the parameters. Figure 2 illustrates the value of phase displacement $\varphi, (\lambda^2)$ for the variant I.

The diagrams corresponding to the respective functions show the influence of different parameters on the shape of the amplitude curves and the value of the vibrations centre translation. Especially the influence of the values of following

[1] All figures at the and of paper

Table 1.

Variant	α	β	γ	μ
I	0.06	0.02	0.05	0.2
II	0.06	0.02	0.01	0.2
III	0.06	0.02	0.05	0.15
IV	0.06	0.03	0.05	0.2
V	0.04	0.02	0.05	0.2
VI	0.01	0.05	0.01	0.1

parameters is remarkable: of the parameter μ – on the width of resonance region, of non-linearity coefficient γ – on deflection of amplitude curves and the centre of vibration translation value from the perpendicular, of the quotient $\alpha\beta$ – maximum values of parametric vibrations amplitudes. For the solutions of this kind the self-excited vibrations amplitude $A = 0$. Beynod the resonance regions there exists another semi-trivial solution, describing self-excited vibrations. The parameters of those vibrations depend only on the constant coeffcients of the differential equations (2.11) – (2.14). In the figure 1a self-excited vibrations amplitudes for variant I are shown by horizontal lines. For non-trivial solutions and numerical values of variant I, figure 3 shows amplitudes A, R and vibrations centre translation D. In the figure the fragments of amplitude curves and vibrations centre translation for semi-trivial solution (fig.1) marked by R_0 and D_0 are presented.

Non-trivial solution first approximation exists for small frequency ranges of parametric excitation $\Delta(\lambda^2) \approx 10^{-3}$ in which there exist unique and ambiguous solutions. Outside the points of intersection of amplitude curves A and R, inside the range of ambiguous solutions, to amplitudes defined by the lower branch of the curve $R(\lambda^2)$ there correspond points of the upper part of the curve $A(\lambda^2)$ and inversely.

Differential equation (1.2) was examined with the analogue computer MEDA 43H. Analogue simulation was performed for values of variants I and VI. At the automatically controlled change of λ^2 parameter, the limit values of $[X]$ were registered; for one-frequency vibrations the smooth curve was sketched forming the amplitude curve, and for two-frequency vibrations it was a plane figure. During the sketching λ^2 was alternately slowly increased and descreased continuously. Figure 4 shows results of variant I investigation, and figure 5 of variant VI. In both cases extreme deflexions diagrams are situated asymmetrically with respect to the λ^2 axis. It implies that the locus of vibrations centres is situated below the axis of abscissa. The parameter γ is five times greater in variant I than in variant VI, it is particularly visible in figure 4.

In the resonance region (fig.5) some points illustating results of semi-trivial solutions are marked. Comparative analysis in the frequency entraiment range shows good agreement of R and D vibrations parameters obtained by analytical and analogue methods. This applies also to the width of frequency ranges where self-excited vibrations do not exist. The results from analogue simulation show that beyond the entrainment region, two-frequency steady vvibrations exist, and this is a proof of the existence of the interaction between self-excited vibrations and parametric vibrations, whose effects are noted beyond the resonance region. As we move away from that region in either direction, the depth of amplitude modulation for self-excited vibrations decreases. Practically, only self-excited vibrations with the amplitude defined by (2.13) can be observed. Such a case is equivalent to the assumption of the value of parameter $\mu = 0$ in equation (1.2). Time functions registered by analogue computer for variant I data are shown in figure 6. Investigations were performed for six values of λ^2 parameter. Figure 6a shows the appearance of beats as a phenomenon of two-frequency vibrations near the synchronisation region. Figures 6b,e,f illustrate periodic vibrations with amplitudes corresponding to semi-trivial stable solutions inside the resonance region. It must be underlined that initial conditions of function $x(\tau)$ for $\lambda^2 = 0.94$ and $\lambda^2 = 1.11$ are computed from non-trivial solutions. Different curves (fig.6c,d) were obtained for $\lambda^2 \approx 0.93$. Depending on the method of approaching that value (with increase or decrease of λ^2 – signed by arrows) almost – periodic and harmonic vibrations were registered. That indicates ambiguity of stable solutions. Hysteresis seen on figure 4a (ten times magnified scale of λ^2 comparing to basical sketch) exists in an extremely small region of λ^2 values placed on the boundary of the entrainment frequency region. Approximated value of parameter $\lambda^2 \approx 0.93$ results from limited accuracy of analogue modelling and the very small range of existence of the *two* kinds of vibrations. Figure 7 presents the course of vibrations in time obtained by IBM PC/AT computer. All courses concern variant I data and three values of λ^2 parameter. The first of them (fig.7a) was obtained from equation (2.1) for steady-state vibrations. The values of vibrations parameters calculated from analytical terms for non-trivial solutions were used. Next the numerical integration of differential equation (1.2) by Runge-Kutt's method was performed. Figures 7b,c show the functions $x(\tau)$ as a result of numerical integrration with initial conditions related to non-trivial solutions. In both cases almost periodic vibrations were not obtained, as it follows from non-trivial solutions, but vibrations with amplitudes consistent with steady-state semi-trivial solutions in the resonance ranges. Numerical integration provided more accurate determina-

Table 2.

Lp	λ^2	R	$\rho_{1,2,3}$	Type of singularity
1	0.93	3.0774	$\rho_{1,2} = -0.0184 \pm 0.03481$	stable focus
			$\rho_3 = -9.9365$	stable focus-node
2	1.0	5.0464	$\rho_{1,2} = -0.1106 \pm 0.05691$	stable focus
			$\rho_3 = -3.6523$	stable focus-node
3	1.11	5.4916	$\rho_1 = -0.2097,\ \rho_2 = -0.0872$	stable node
			$\rho_3 = -3.1897$	stable node
4	1.11	1.8033	$\rho_1 = -0.0296,\ \rho_2 = 0.0582$	saddle
			$\rho_3 = -3.3200$	saddle of the first order

tion of the value of λ^2, for which two steady-state solutions were established. Figures 7d,e show the time-courses of corresponding vibrations for appropriately determined initial conditions and $\lambda^2 = 0.924$. It is worth underling the similarity of the almost-periodic vibrations obtained by analogue simulation and numerical integration. In the fig.1a,b the vibrations center translations are marked by broken lines related to unstable solutions obtained for variant I. The boundaries of the stability region are determined by the term (2.31) and by the principle of vertical tangent taken from Routh-Hurvitz's condition. The paratemtric vibrations stability investigation included also the determination of the character of singular points. To achieve this the roots of characteristic equations (2.29) and (2.32) were calculated. The results of this investigation are given in table 2. In the last column the first kinds of singularities correspond to the roots $\rho_{1,2}$ and the second types of singularities correspond to three roots of the characteristic equation. The results of stability investigation of non-trivial solutions for $\lambda^2 < 1$ are shown in the fig.8a and for $\lambda^2 > 1$ in the fig.8b. On the diagrams $R(A)$, the labels show the type of four roots of characteristic equation (2.33). The digits means: 1 – two comples coupled roots with positive real parts, two real roots less than zero; 2 – two complex coupled roots with negative real parts, two real roots – one negative, one positive; 3 – all roots real, two positive and two negative; 4 – all roots real, three negative, one positive. That implies that at least one real term of the calculated roots for the whole range of that class off solutions is positive. That means that first order approximate non-trivial solutions are unstable. In simplified examinations of the stability of non-trivial solutions stability for $\lambda^2 > 1$, the roots of characteristic equation of the third order were determined. As in [5] the first derivative of vibrations centre translation was ignored. In this case for points "5" placed in e – f range of the $R(A)$ curve (fig.8b) all three roots are real and less than zero. For the points of the upper parrt of the curve, the instability is characterized by the saddle – the focus of second order [5]. The results of investigations point out the instability of solutions in e – f range of A and R amplitudes and translation D; they are drawn in fig. 3c with continuous lines. The same kind of results of stability examination of non-trivial solutions were obtained in [11], where, by the same analytical method, identical parametric self-excited vibrations with cubic elasticity characteristic were considered. What is different in comparison with quadratic characteristic is that in cubic case the vibrations centre translation does not occur, so in [11] there did not exist the problem of stability investigation. The stability examinations for non-linear solutions of differential equations system (2.3) – (2.5) and (2.7) were conducted also by numerical integration method.

The integral curves are in space (R, A, D, τ).

Figure 9 shows the projections of phase trajectories into the $R - A$ plane. The results were registered by the plotter. Numerical integration for small perturbations of initial conditions corresponding to points in the phase plane lying inside and outside certain spheres was performed. The examinations were done for $\lambda^2 = 1.13$, because for that value non-linear solutions exist; one of which is situated inside the stable solutions region, which had been suggested by simplified examinations. Steady-state non-trivial solutions are shown by coordinates of the middle-point of spheres. All phase trajectories move away from their centres to the point $A = 0$, $R = 5.4$, which corresponds to resonance value of parametric vibrations amplitude. The results of examinations confirm also the instability of non-trivial solutions. The investigations included numerical integration of equation (1.2) using the phase-trajectories method on stroboscope plane [5,7]. Fig.10 shows stable limit cycles – C_s for $\lambda^2 = 0.9$ and digital data for variant I (fig.10a) and in the special case, when $\mu = 0$ (fig.10b). The different shape of limit cycles shows the interaction effect between parametric vibrations and self-excited vibrations in the neighbourhood of the synchronisation region. The type of singularity points is differ in the two cases, related to trivial solutions. The equilibrium points are unstable because we are dealing with the soft self-excitation. For parametric self-excitation (fig.10a) the singular point turns to be unstable node W_n, and without excitation – unstable focus 0_n (fig.10b).

Figure 11 presents phase-trajectories for $\lambda^2 = 1.11$. Situation of singular points W_s – stable nodes, shows the translation of the centre of vibrations and, what is more, the existence of the saddle – S. The results of these examinations agree with analytical methods (table 2). The results for $\lambda^2 = 1.13$ (fig.12) prove the non-existence off non-trivial, stable solutions. Qualitatively the results for both values of λ^2 parameter are identiccal. Figure 13 shows the results of examination of the equation (1.2) for $\lambda^2 = 0.924$. For that value, two stable solutions were found. One of them is related to limit cycle – C_s , the second to singular point – stable focus 0_s . Its englarged neighbourhood is presented in the fig.13a.

The binding region of this solution is very small and is determined by separatrice – S_p . The equilibrium point is an unstable node. The value $\lambda^2 = 0.924$ can be assumed as bifurcation value of the parameter λ^2, hence at small detuning of frequency there exists qualitative change in topological structure of phase trajectories. For $\lambda^2 = 0.922$ (fig.14) we get only stable limit cycle and for $\lambda^2 = 0.926$ (fig.15) limit cycle disappears. Besides stable focuses, two saddles and one unstable node exist characteristing non-trivial solutions. Further evolution on the frequency entrainment boundary leads to the alternation of equilibrium points singularity. Unstable node alternates with saddle (fig.16) for $\lambda^2 = 0.93$.

Analogical examinations were done for variant VI data and for three values of parameter $\lambda^2 = \lambda_0^2 \pm \varepsilon$, where bifurcation valuee $\lambda_0^2 = 0.9523$, and $\varepsilon = 3\times10^{-4}$. For λ_0^2 (fig.17) two compound singular points like saddle-node (SW) were obtained, for $\lambda_0^2 = 0.952$ – stable limit cycle (fig.18), and for $\lambda^2 = 0.9526$ the character off singularities is represented by the saddle and stable node (fig.19). Fig.20 shows the stable limit cycle for variant I and $\lambda^2 = 1.24$. And in contradistinction to the limit cycles at $\lambda^2 < 1$ (fig.10a), the equilibrium point plays the role of unstable focus and trajectories are evolving into cycle in the clockwise direction.

4. Conclusions

The effects of interaction between parametric vibrations and self-excited vibrations in the considered system, with non-symmetric characteristic of elasticity are characterised by periodic and nearly-periodic vibrations and by the vibrations centre translation. In the main resonance region self-excited vibrations synchronize with parametric vibrations. Resonance amplitudes, however, depend on parameters of Van der Pol's model. This influence, seen in analytical relations of semi-trivial solutions, was verified by analogue simulation.

The influence of systems parameters on the shape of amplitude curves and vibrations centre, as well as the width of synchronisation regions was investigated. Outside the frequency entrainment region there exist nearly periodic vibrations with depths decreasing amplitude modulation, when we move away from the resonance region almost-periodic vibrations, to which correspond stable limit-cycles point out the influence of parametric excitation on the self-excited system also in two regions adjoining the frequency entrainment regions. The analytical solution was approximated by the sum of three functions presenting self-excited behaviour, parametric vibrations and vibrations centre translation. Basing on different methods it was settled that two-frequency non-trivial solutions are unstable and they exist in the very narrow spectrum of frequency. From the simplified examination of the stability for the solutions above, ignoring the first derivate of vibrations centre, similar stable solutions were obtained.

The situation of non-trivial solutions with respect to beats existing on the boundaries of frequency entrainment region proves that non-trivial solutions are not entirely adequate as a model for two-frequency vibrations. This conclusion is in agreement with [11], where by the same analytical method vibrations of the same type of parametric – self-excited system with cubic non-linearity were investigated.

It is worth mentioning that for vibrations of the system with non-symmetric elasticity characteristic the discrepancy between non-trivial solution and the results of simulation built on the basic mathematical model is even greater. Applied analytical method was successfully used in vibrations analysis off self-excited systems with external excitation [4]. Therefore, the results of stability investigation of non-trivial solutions can be useful in the analysis for such interactions existing in the systems with non-symmetric elasticity characteristic. It is worth emphasizing that in the vibrations stability examinations the numerical analysis coupled with stroboscope method of phase plane were particularly useful. It concers first of all the determination of bifurcation points in the transition regions between entrainment vibrations and almost-periodic vibrations.

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Streszczenie

W pracy poddano analizie drgania układu samowzbudnego z wymuszeniem parametrycznym i nieliniową charakterystyką sprężystości typu kwadratowego. Badania przeprowadzono przy zastosowaniu jednej z metod analitycznych, symulacji analogowej i analizy numerycznej. Oceniono przydatność zastosowanej metody analitycznej w badaniach drgań tego typu układów. Rozpatrzono zagadnienie stateczności rozwiązań, ustalając przy tym wartości parametru częstości, dla których następuje jakościowa zmiana strruktury topologicznej trajektorii fazowych.

Резюме

В работе проанализовано колебания самовозбуждающей системы с параметрическим возбуждением и нелинейной упругой характеристикой квадратного типа. Исследования проведено применяя один из аналитических методов, аналогового моделирования и численного анализа. Оценивается пригодность применяемого аналитического метода в исследованиях колебаний систем этого типа. Рассмотрено проблему устойчивости решений, установив при этом значения параметра частоты, которых наступает изменение качественной картины фазовых траекторий .

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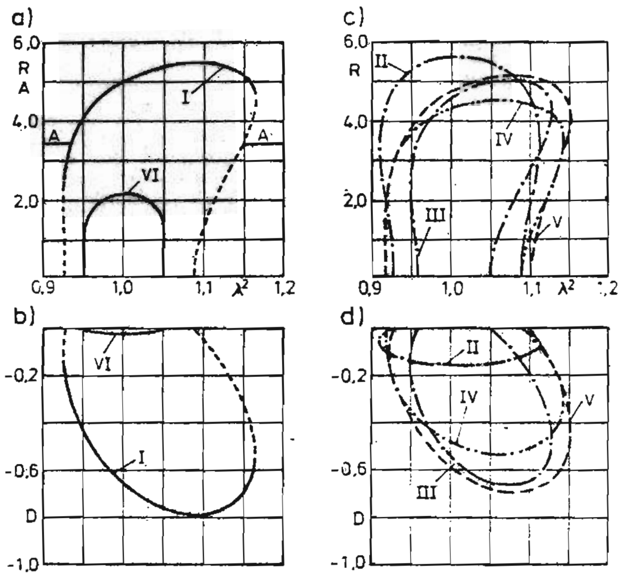
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Fig. 1.

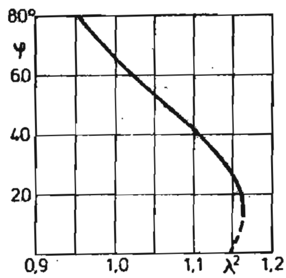
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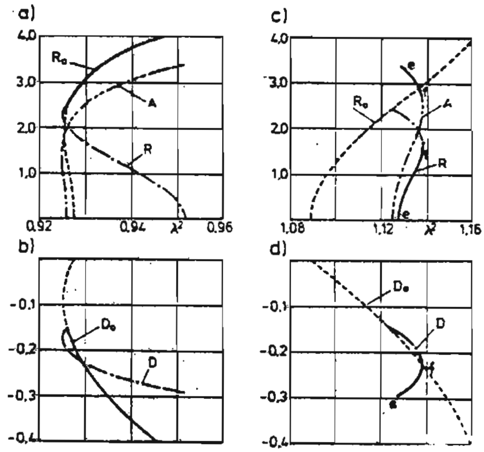
Fig. 2.



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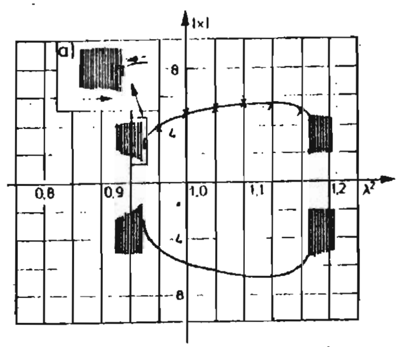

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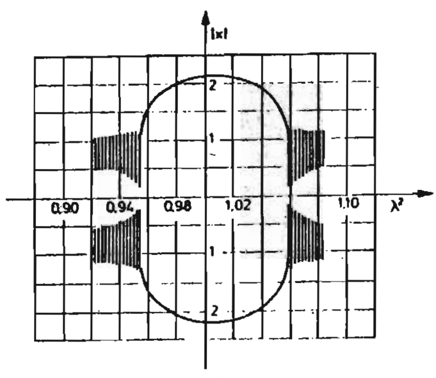

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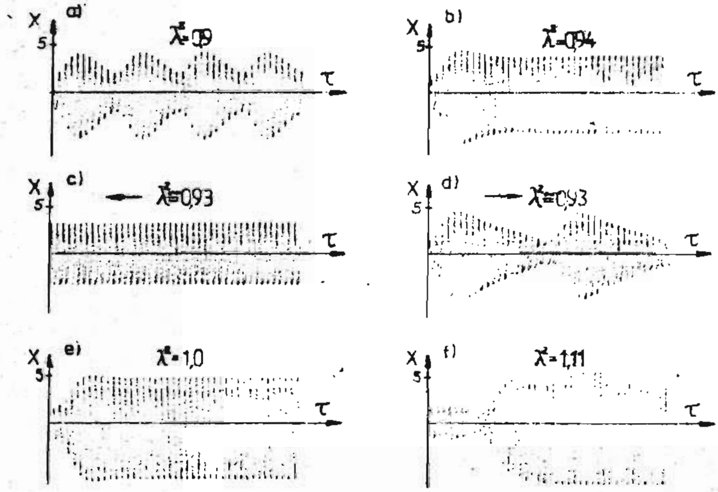

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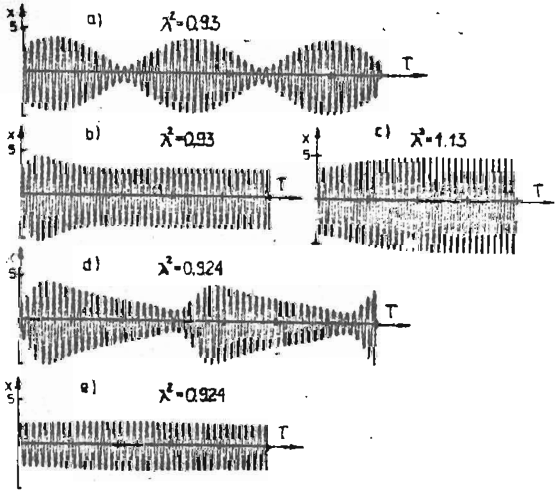

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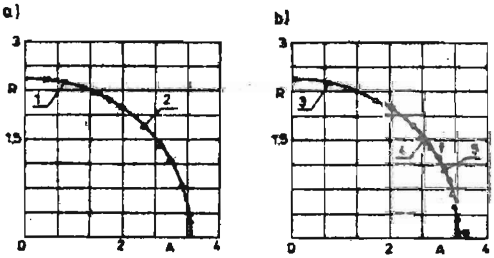

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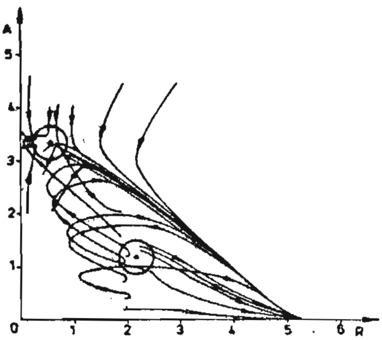
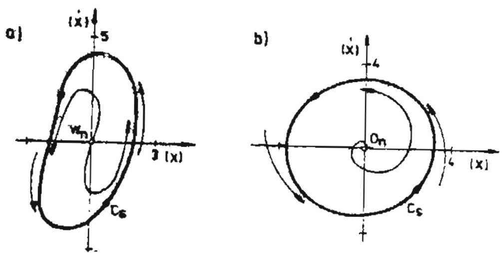


Fig. 9.



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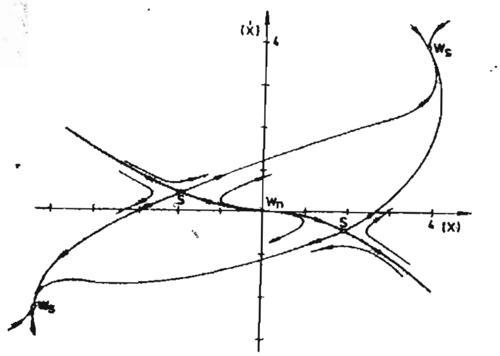

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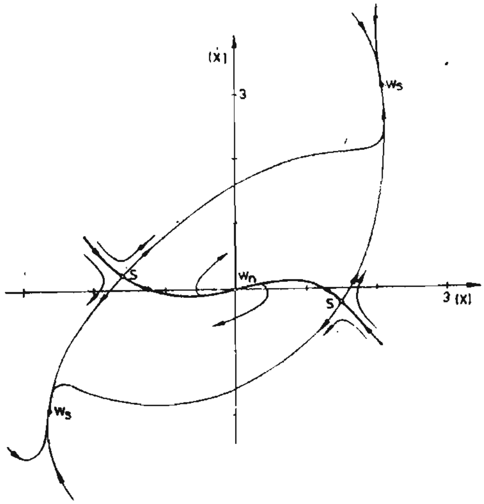

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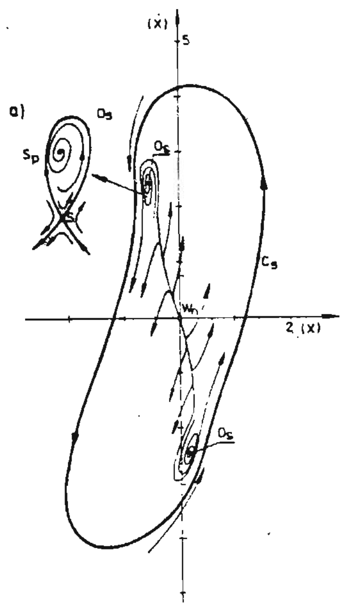

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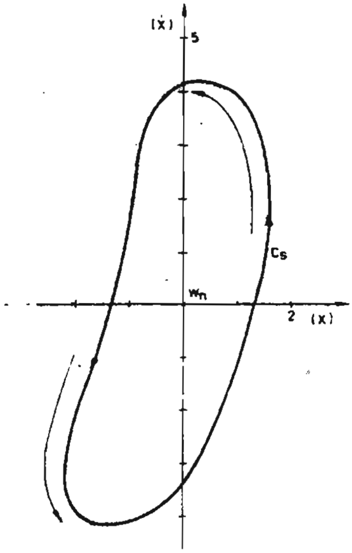

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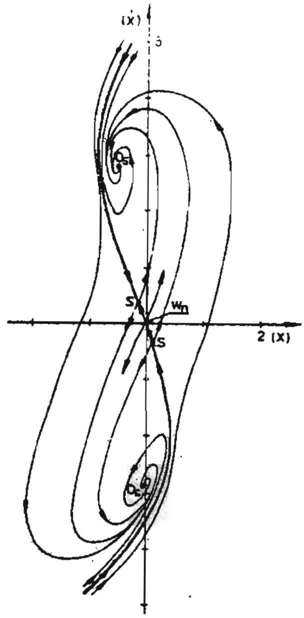

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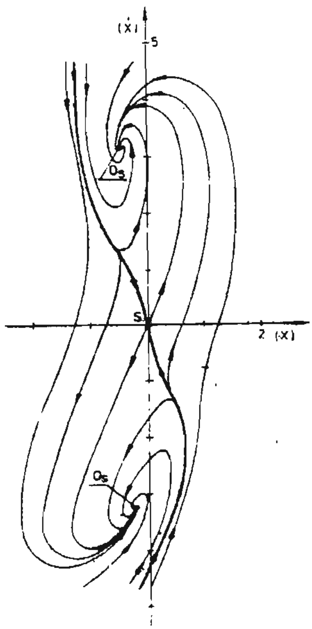

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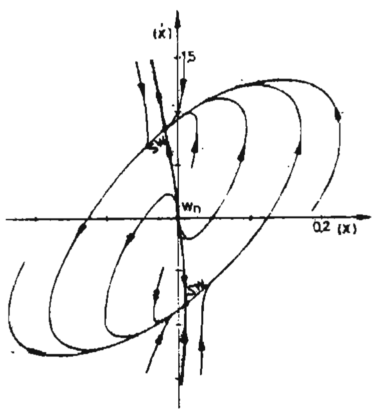

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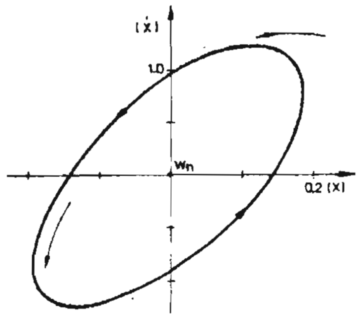

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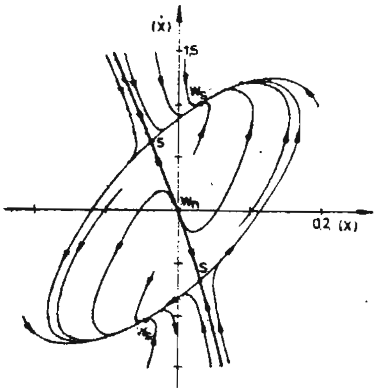

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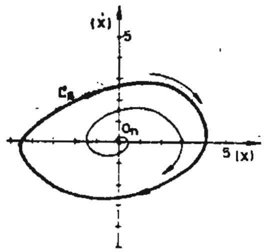

Fig. 20.